Chapter 2

2.1 Sample spaces and events

2.2 Interpretations and axioms of probability

2.3 Addition rules

2.4 Conditional probability

2.5 Multiplication and total probability rules

2.6 independence

2.7 Bayes' theorem

2.8 Random variables

2.1 Sample spaces and events

- Random experiments
  - Definition--- An experiment that can result in different outcomes, even though it is repeated in the same manner every time, is called a **random experiment**.
- Sample spaces
  - Definition--- The set of all possible outcomes of a random experiment is called the **sample space** of the experiment. The sample space is denoted as S.

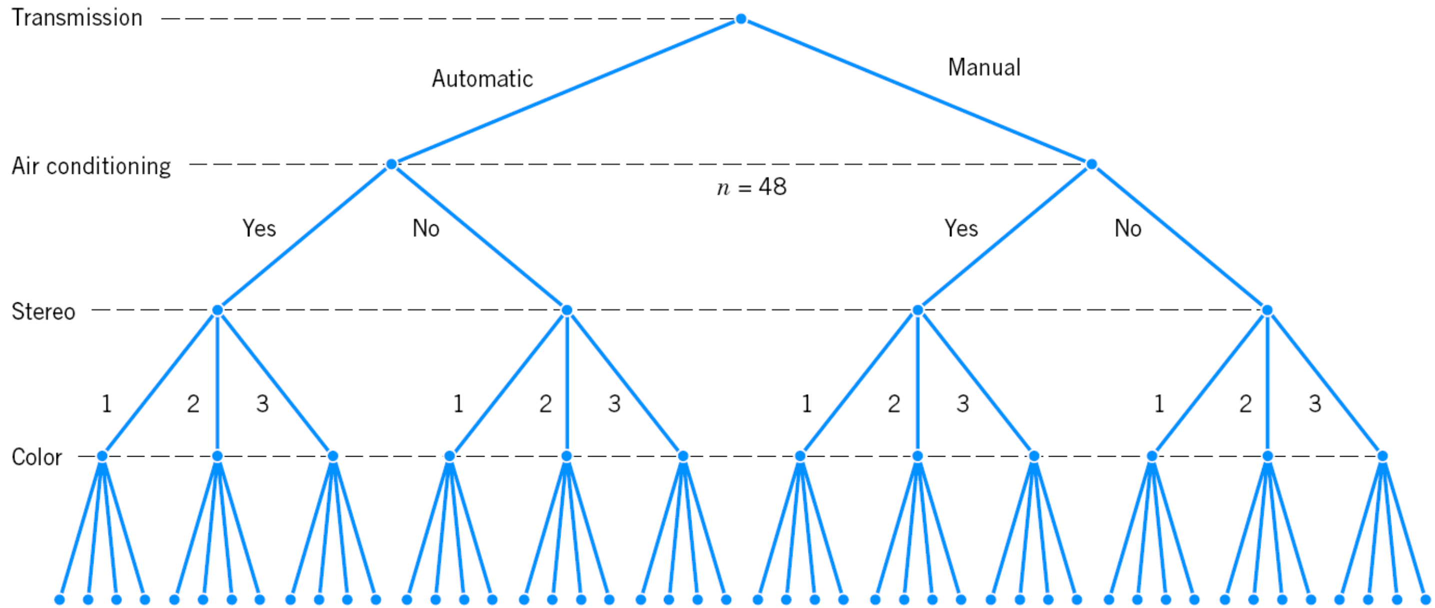
☞Example 2-1

$S=R^{+}=\{x|\ x>0\}$, $S=\{x|\ 10<x<11\}$, $S=\{\text{yes, no}\}$

- Definition---A sample space is **discrete** if it consists of a finite or countable infinite set of outcomes. A sample space is **continuous** if it contains an interval (either finite or infinite) of real numbers.
  - ☞Example 2-2 If the ordered pair yn indicates that the first connector conforms and the second does not, the sample space can be represented by the four outcomes:

$$S=\{yy, yn, ny, nn\}$$

  - ☞Example 2-4

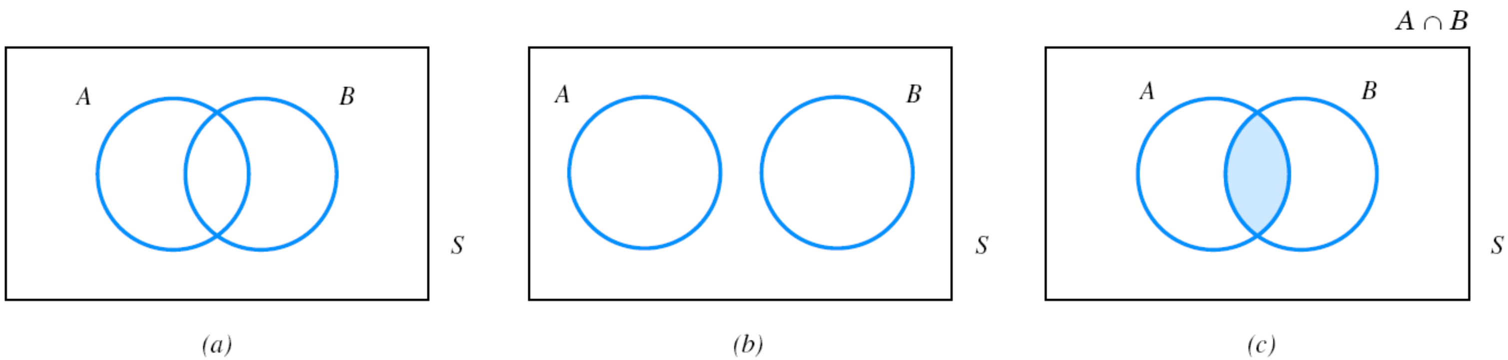


- Events
  - Definition--- An **event** is a subset of the sample space of a random experiment.
  - The **union** of two events is the event that consists of all outcomes that are contained in either of the two events. We denote the union as $E_1 \cup E_2$.
  - The **intersection** of two events is the event that consists of all outcomes that are contained in both of the two events. We denote the intersection as $E_1 \cap E_2$.
  - The **complement** of an event in a sample space is the set of outcomes in the sample space that are not in the event. We denote the component of the event E as E'. The notation E^C is also used in other literature to denote the complement.

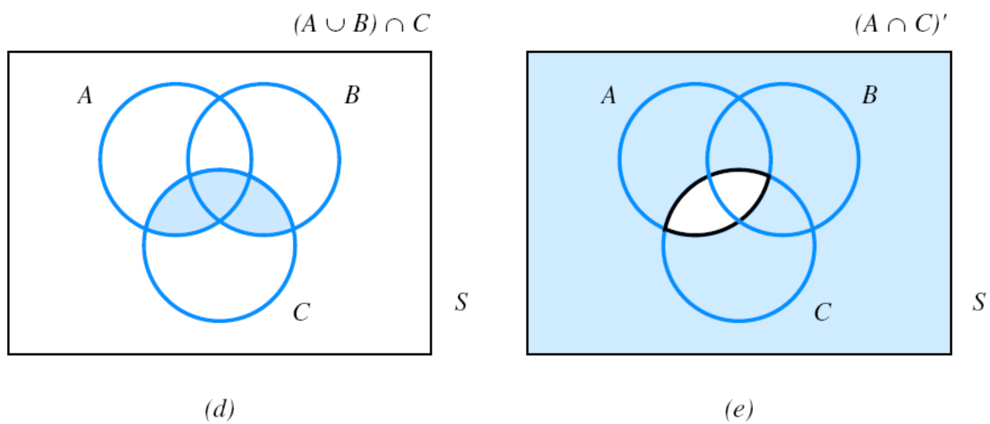
☞ Example 2-7 Measurements of the thickness of a plastic connector might be modeled with the sample space $S=R^+$, the set of positive real numbers. Let $E_1=\{x|10\leq x<12\}$ and $E_2=\{x|11<x<15\}$

- Then $E_1 \cup E_2 = \{x|10\leq x<15\}$ and $E_1 \cap E_2 = \{x|11<x<12\}$
- $E'_1=\{x|\ 0<x<10 \text{ or } 12\leq x\}$ and $E'_1 \cap E_2 = \{x|\ 12\leq x<15\}$

☞



Sample space S with events A and B



- Definition---Two events, denoted as E_1 and E_2, such that

$$E_1 \cap E_2 = \varnothing$$

are said to be **mutually exclusive**.

- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$, $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
- DeMorgan's laws

$(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$

- Counting Technique (p.24)
  - Multiplication Rule (p.25)

Assume an operation can be described as a sequence of k steps, and

the number of ways of completing step 1 is n_1, and

the number of ways of completing step 2 is n_2 for each way of completing step 1, and

the number of ways of completing step 3 is n_3 for each way of completing step 2, and

so forth.

The total number of ways of completing the operation is

$$n_1 \times n_2 \times \ldots \times n_k$$

Ex. 2-4

➢Permutations (p.25)

The number of **permutations** of n different elements is $n!$ where

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 \tag{2-1}$$

Ex. $S=\{a,b,c,d\}$

➢Permutations of subset (p.26)

The number of permutations of subsets of r elements selected from a set of n different elements is

$$P_r^n = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1) = \frac{n!}{(n-r)!} \tag{2-2}$$

➢Permutations of Similar Objects (p.26)

The number of permutations of $n = n_1 + n_2 + \cdots + n_r$ objects of which n_1 are of one type, n_2 are of a second type, ..., and n_r are of an rth type is

$$\frac{n!}{n_1!\, n_2!\, n_3! \ldots n_r!} \tag{2-3}$$

➢Combinations (p.27)

The number of combinations, subsets of size r that can be selected from a set of n elements, is denoted as $\binom{n}{r}$ or C_r^n and

$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{2-4}$$

Permutation vs Combination

1. Sampling ***with replacement*** and ***with ordering*** (Permutation)

The total number of ways a subset of r items can be drawn from a set of n distinct elements is $n^r = n\, n\, n \ldots (r\ times)$

2. Sampling ***without replacement*** and ***with ordering*** (Permutation)

a. The total number of ways a subset of r items can be drawn from a set of n distinct elements is $n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$

b. The total number of ways a subset of n items can be drawn from a set of $n = n_1 + n_2 + n_3 + \cdots + n_r$ is $\frac{n!}{n_1! n_2! n_3! \ldots n_r!}$ where n_1 are of one type, n_2 of a seconde type, and so on

3. Sampling ***with replacement*** and ***without ordering*** (Combination)

The total number of ways a subset of r items can be drawn from a set of n distinct elements is $\binom{n+r-1}{r}$

4. Sampling *without replacement* and *without ordering*(Combination)

The total number of ways a subset of r items can be drawn from a set of n distinct elements is $\binom{n}{r} = \frac{n!}{(n-r)!r!} = \binom{n}{n-r}$

Example: Enumeration of possible outcomes in various types of sampling

Suppose a box contains 5 books (1, 2, 3, 4 and 5) and you need to select two books from the box?

Solution:

with replacement and with ordering	without replacement and with ordering	with replacement and without ordering	without replacement and without ordering
(1, 1) (1, 2) (1, 3) (1, 4) (1, 5) (2, 1) (2, 2) (2, 3) (2, 4) (2, 5) (3, 1) (3, 2) (3, 3) (3, 4) (3, 5) (4, 1) (4, 2) (4, 3) (4, 4) (4, 5) (5, 1) (5, 2) (5, 3) (5, 4) (5, 5)	(1, 2) (1, 3) (1, 4) (1, 5) (2, 1) (2, 3) (2, 4) (2, 5) (3, 1) (3, 2) (3, 4) (3, 5) (4, 1) (4, 2) (4, 3) (4, 5) (5, 1) (5, 2) (5, 3) (5, 4)	(1, 1) (1, 2) (1, 3) (1, 4) (1, 5) (2, 2) (2, 3) (2, 4) (2, 5) (3, 3) (3, 4) (3, 5) (4, 4) (4, 5) (5, 5)	(1, 2) (1, 3) (1, 4) (1, 5) (2, 3) (2, 4) (2, 5) (3, 4) (3, 5) (4, 5)
$n^r = n\,n = 5(5) = 25$	$n(n-1)(n-2)\ldots(n-r+1) = 5(4) = 20$ $\frac{n!}{(n-r)!} = \frac{5!}{(5-2)!} = 20$	$\binom{n+r-1}{r} = \binom{6}{2} = 15$	$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \binom{n}{n-r}$

Note:

If the order is important, it is a Permutation

If the order is not important, it is a Combination

2.2 Interpretations of probability

- In this chapter, we introduce probability for **discrete sample spaces**—those with only a finite (or countably infinite) set of outcomes.

- **Probability** is used to quantify the likelihood, or chance, that an outcome of a random experiment will occur.
  - ➢Definition---For a discrete sample space, the *probability of an event E*, denoted as $P(E)$, equals the sum of the probabilities of the outcomes in E.
  - ➢When the model of **equally likely outcomes** is assumed, the probabilities are chosen to be equal.

☞Whenever a sample space consists of N possible outcomes that are equally likely, the probability of each outcome is $1/N$.

- Axiom of probability

Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:

If S is the sample space and E is any event in a random experiment,

(1) $P(S) = 1$

(2) $0 \leq P(E) \leq 1$

(3) For two events E_1 and E_2 with $E_1 \cap E_2 = \emptyset$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

- ➢ $P(\emptyset)=0$.
- ➢ For any event E, $P(E')=1- P(E)$

2.3 Addition rules

- Addition rules
  - The preceding example (Ex. 2-19) illustrates that the probability of *A* or *B* is interpreted as $P(A \cup B)$ and that the following general **addition rule** applies.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad (2\text{-}5)$$

  - Example 2-19 $P(H \cup C) = ?$

Table 2-1 Wafers in Semiconductor Manufacturing Classified by Contamination and Location

	Location in Sputtering Tool		
Contamination	Center	Edge	Total
Low	514	68	582
High	112	246	358
Total	626	314	

☞ $P(H \cup C) = P(H) + P(C) - P(H \cap C)$

$= 358/940 + 626/940 - 112/940 = 872/940$

- Recall that two events *A* and *B* are said to be mutually exclusive if $A \cap B = \varnothing$. Then $P(A \cap B) = 0$, and the general result for the probability of $A \cup B$ simplifies to the third axiom of probability.
  - ☞If A and B is mutually exclusive events,

$$P(A \cup B) = P(A) + P(B). \qquad (2\text{-}6)$$

- Three or more events

$$P(A \cup B \cup C) = P[(A \cup B) \cup C] = P(A \cup B) + P(C) - P[(A \cup B) \cap C]$$

$$= P(A) + P(B) - P(A \cap B) + P(C) - P[(A \cap C) \cup (B \cap C)]$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

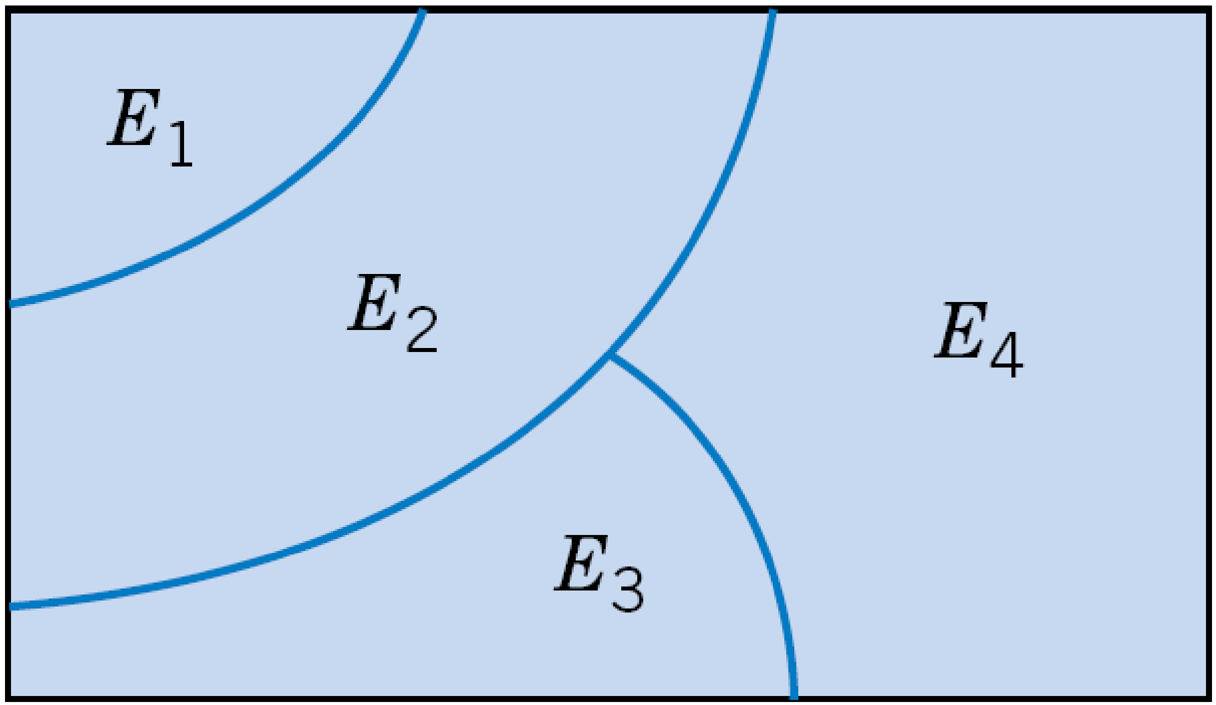
$$(2\text{-}7)$$

A collection of events, $E_1, E_2, \ldots, E_k$, is said to be **mutually exclusive** if for all pairs,

$$E_i \cap E_j = \varnothing$$

For a collection of mutually exclusive events,

$$P(E_1 \cup E_2 \cup \ldots \cup E_k) = P(E_1) + P(E_2) + \ldots P(E_k) \qquad (2\text{-}8)$$



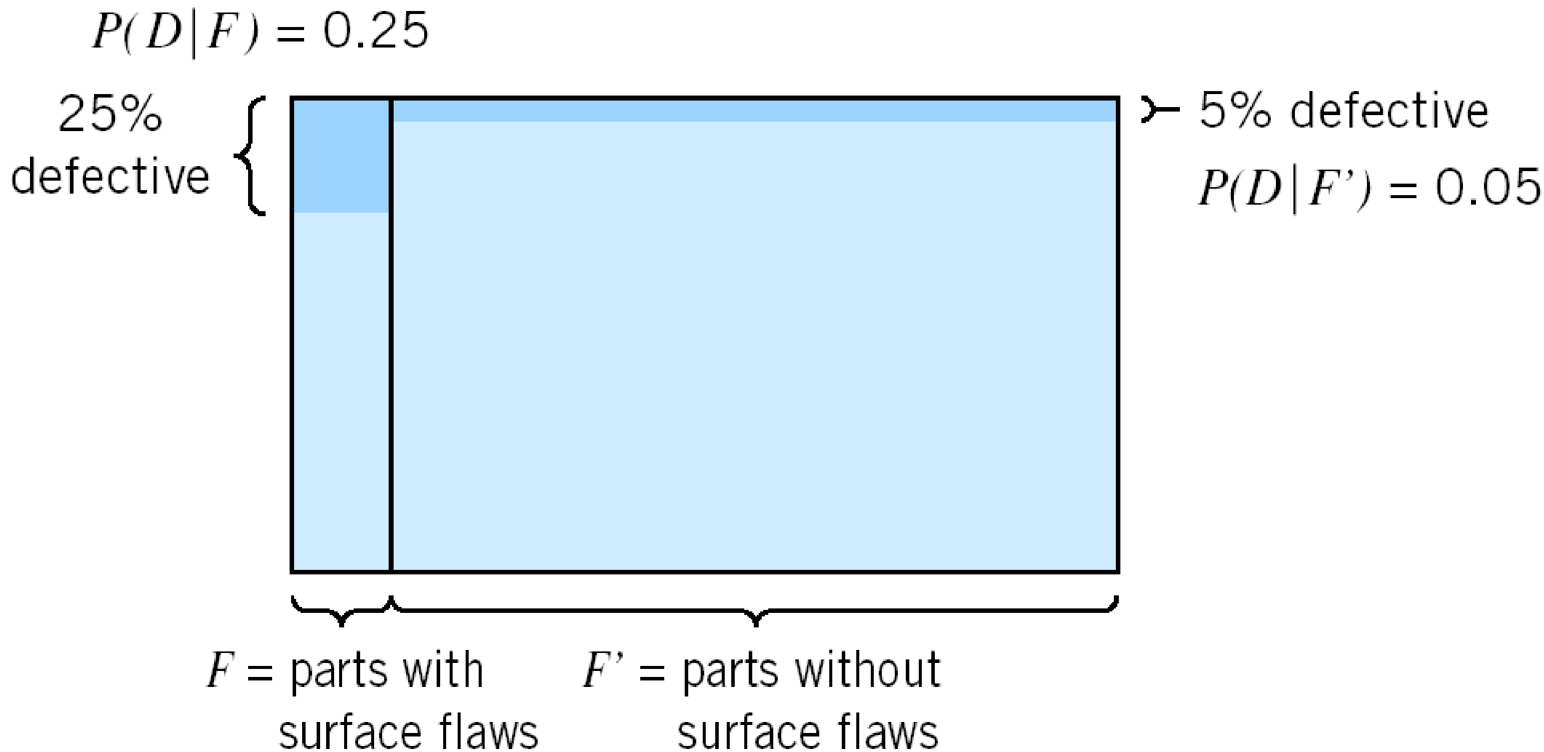
2.4 Conditional Probability

- $P(B|A)$ ---This notation is read as the **conditional probability** of B given A, and it is interpreted as the probability that a part is defective, given that the part has a surface flaw.

Table 2-3 Parts Classified

		Surface Flaws		
		Yes (event F)	No	Total
Defective	Yes (event D)	10	18	28
	No	30	342	372
	Total	40	360	400

- Let D denote the event that a part is defective and let F denote the event that a part has a surface flaw.
  - ☞ $P(D|F)$=?



➢Definition

The **conditional probability** of an event B given an event A, denoted as $P(B|A)$, is

$$P(B|A) = P(A \cap B)/P(A) \qquad (2\text{-}9)$$

for $P(A) > 0$.

- If there are n total outcomes,

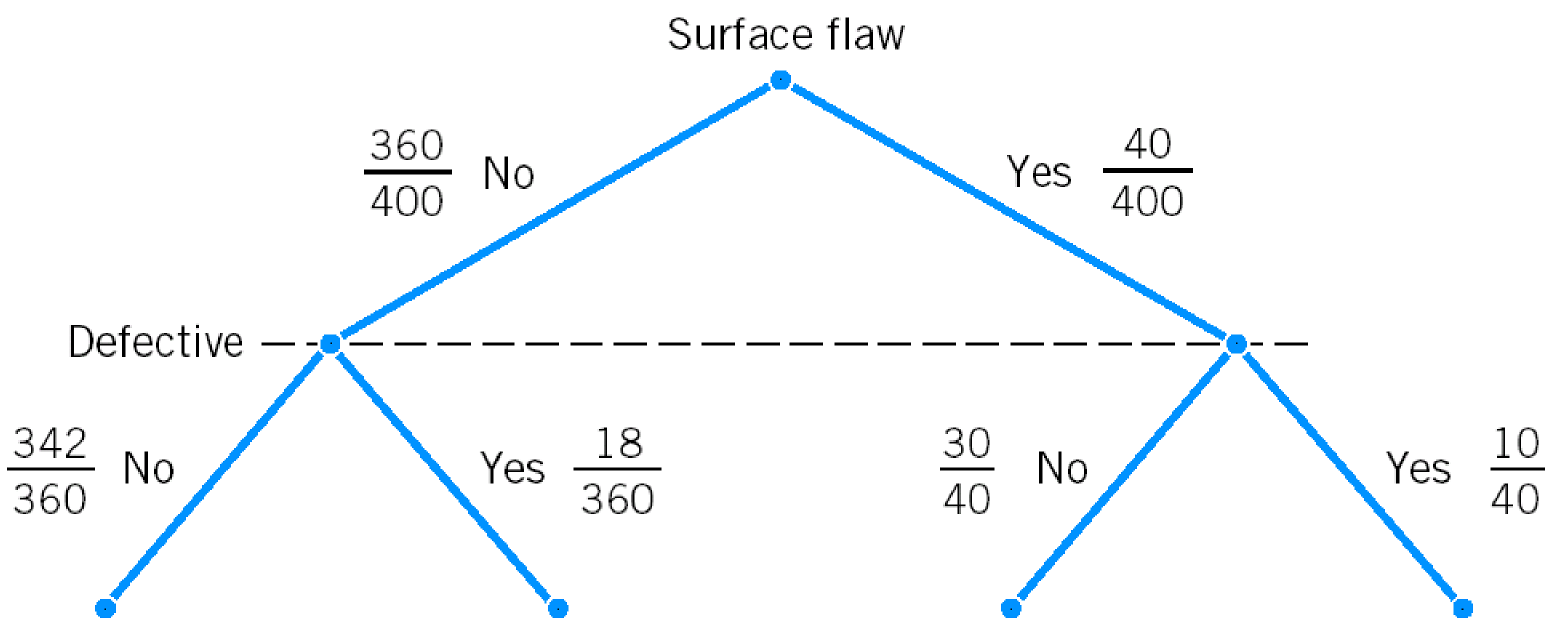
$$P(A) = (\text{number of outcomes in } A)/n$$

$$P(A \cap B) = (\text{number of outcomes in } A \cap B)/n$$

$$P(A \cap B)/P(A) = \frac{\text{number of outcomes in } A \cap B}{\text{number of outcomes in } A}$$

- $P(F)=0.1, P(F')=0.9, P(D|F)=0.25, P(D|F')=0.05, P(D'|F)=0.75, P(D'|F')=0.95$

- Tree diagram for parts classified

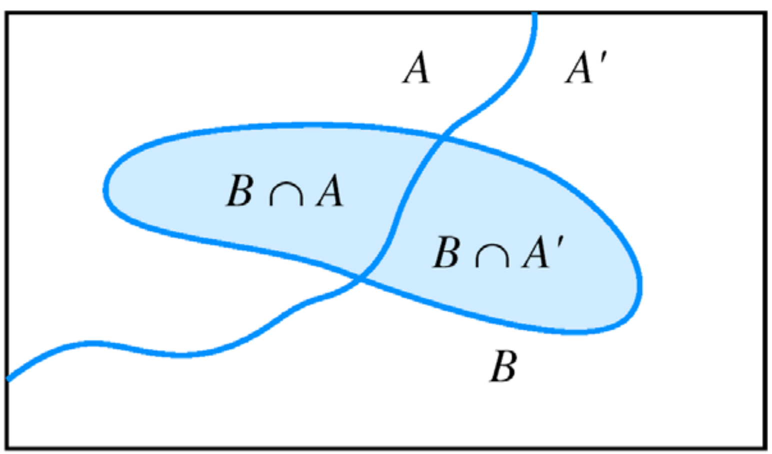


- Random Samples and Conditional Probability
  - ➢ Sampling with and without replacement were defined and illustrated for the simple case of a batch with three items {*a*, *b*, *c*}.
    - ☞ If two items are selected randomly from this batch without replacement, each of the six outcomes in the ordered sample space $S_{\text{without}} = \{ab, ac, ba, bc, ca, cb\}$ has probability 1/6.
    - ☞ If the unordered sample space is used, each of the three outcomes in {{*a*, *b*}, {*a*, *c*}, {*b*, *c*}} has probability 1/3.
  - ➢ Random Samples---To select randomly implies that at each step of the sample, the items that remain in the batch are equally likely to be selected.

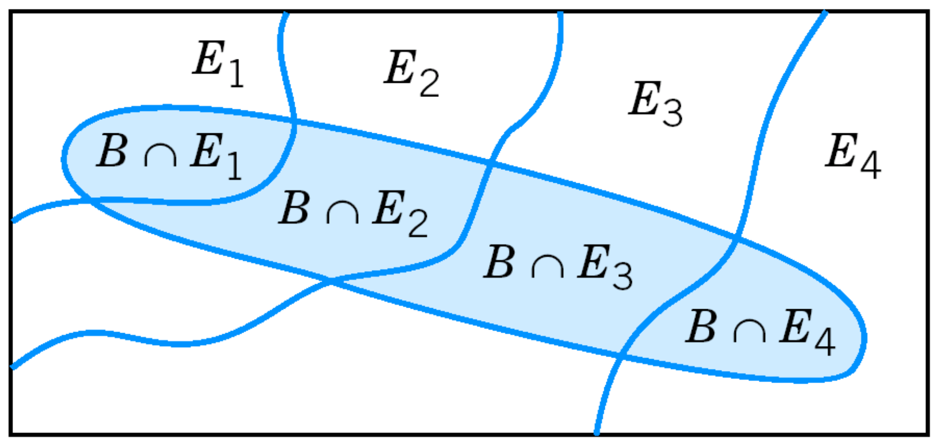
2.5 Multiplication and total probability rules

- Multiplication rule
  - $P(A \cap B) = P(B|A)P(A) = P(A|B)P(B)$
- Total probability rule (two events)
  - For any events *A* and *B*

$$P(B) = P(B \cap A) + P(B \cap A') = P(B|A)P(A) + P(B|A')P(A') \quad (2\text{-}11)$$



B=(A∩B)∪(A′∩B)



$B = (B \cap E_1) \cup (B \cap E_2) \cup (B \cap E_3) \cup (B \cap E_4)$

- Total probability rule (multiple events)
  - Assume $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive sets. Then

$$P(B)=P(B\cap E_1)+P(B\cap E_2)+\ldots+P(B\cap E_k)$$

$$=P(B|E_1)P(E_1)+P(B|E_2)P(E_2)+\ldots+P(B|E_k)P(E_k) \qquad (2\text{-}12)$$

  - Example 2-28

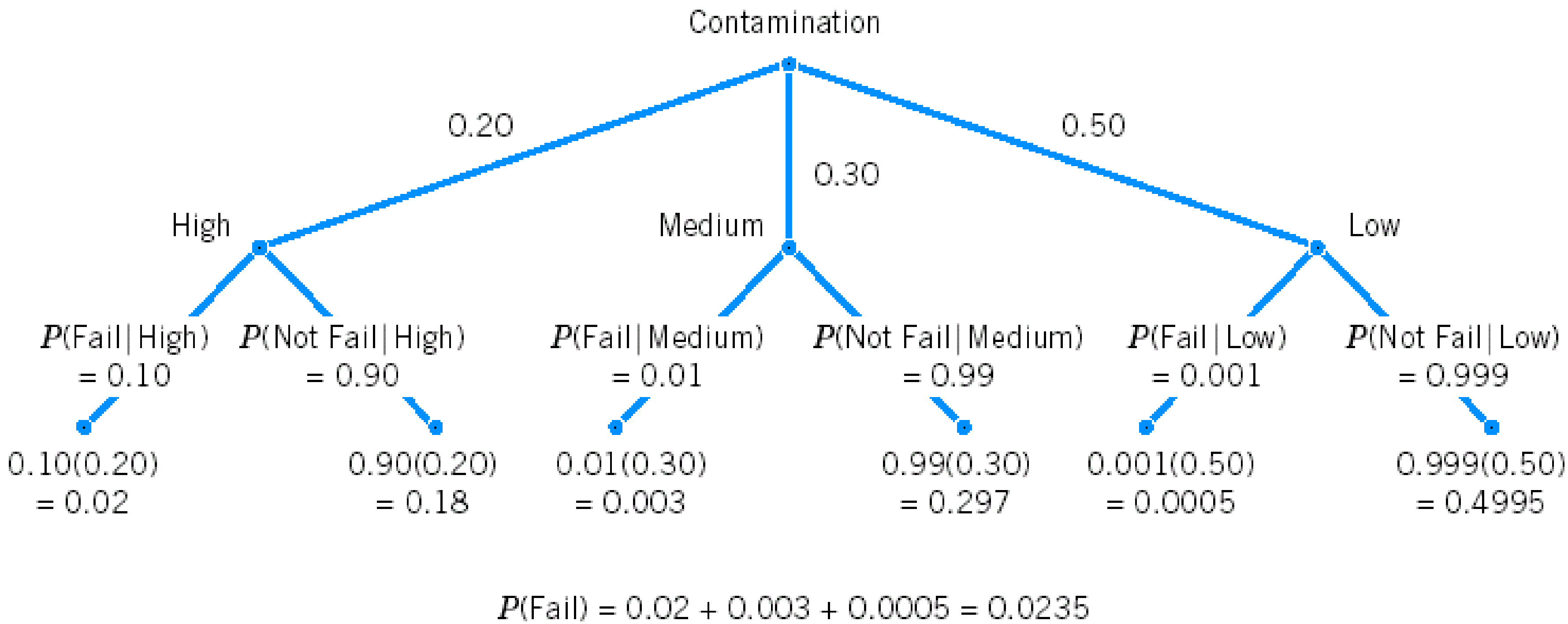
In a particular production run, 20% of the chips are subjected to high levels of contamination, 30% to medium levels of contamination, and 50% to low levels of contamination. What is the probability that a product using one of these chips fails?

Probability of Failure	Level of Contamination
0.10	High
0.01	Medium
0.001	Low

H denote the event that a chip is exposed to high levels of contamination

M denote the event that a chip is exposed to medium levels of contamination

L denote the event that a chip is exposed to low levels of contamination



➢ $P(F)=P(F|H)P(H)+P(F|M)P(M)+P(F|L)P(L)$

$=0.1\bullet0.2+0.01\bullet0.3+0.001\bullet0.5=0.0235$

- Independence
  - Example 2-30

Table 2-4 Parts Classified

		Surface Flaws		
		Yes (event F)	No	Total
Defective	Yes (event D)	2	18	20
	No	38	342	380
	Total	40	360	400

- $P(D|F)$=2/40=0.05 and $P(D)$=20/400=0.05

  The probability that the part is defective does not depend on whether it has surface flaws.

- $P(F|D)$=2/20=0.1 and $P(F)$=40/400=0.1

  The probability of a surface flaw does not depend on whether the part is defective.

- In the special case that $P(B|A)=P(B)$, we obtain

$$P(A\cap B)=P(B|A)P(A)=P(B)P(A)$$

and

$$P(A|B)=P(A\cap B)/P(B)=P(B)P(A)/P(B)=P(A)$$

Two events are **independent** if any one of the following equivalent statements is true:

(1) $P(A|B) = P(A)$

(2) $P(B|A) = P(B)$

(3) $P(A \cap B) = P(A)P(B)$ (2-13)

$$P(A\cap B')=? \quad P(A'\cap B')=?$$

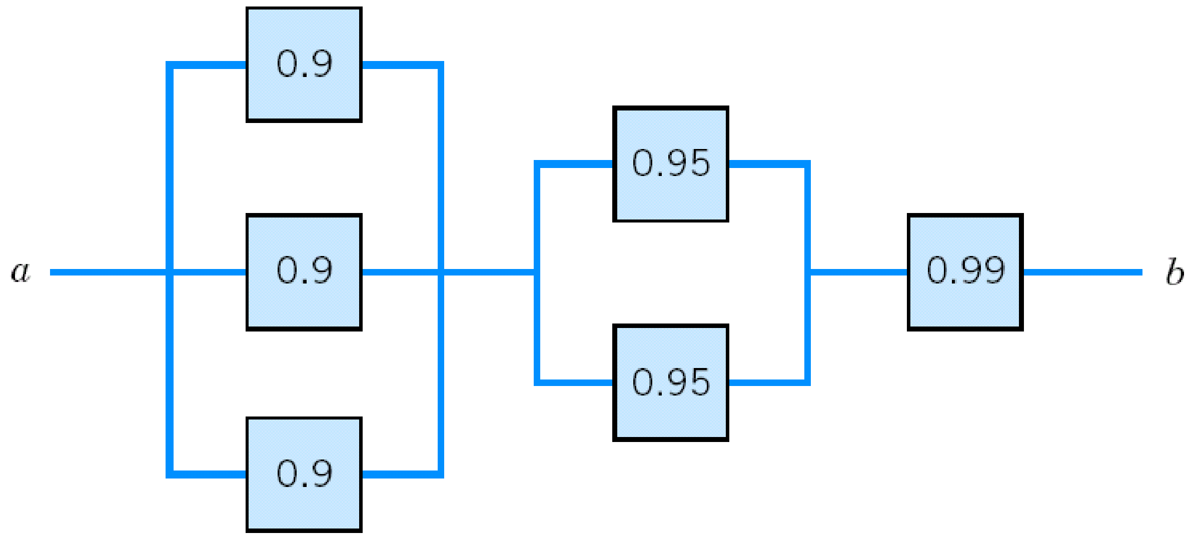
- Definition

The events $E_1, E_2, \ldots, E_n$ are independent if and only if for any subset of these events $E_{i_1}, E_{i_2}, \ldots, E_{i_k}$,

$$P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) = P(E_{i_1}) \times P(E_{i_2}) \times \cdots \times P(E_{i_k}) \quad (2\text{-}14)$$

➢Example 2-35

The following circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates?



$$(1-0.1^3)(1-0.05^2)0.99=0.987$$

- Q2-134. A credit card contains 16 digits. It also contains a month and year of expiration. Suppose there are one million credit card holders with unique card numbers. A hacker ran-domly selects a 16-digit credit card number. (a)What is the probability that it belongs to a user? (b)Suppose a hacker has a 25% chance of correctly guessing the year your card expires and randomly selects one ofthe12 months. What is the probability that the hacker correctly selects the month and year of expiration?

2.7 Bayes' theorem

From the definition of conditional probability,

- $P(A \cap B) = P(A|B)P(B) = P(B \cap A) = P(B|A)P(A)$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \quad \text{for} \quad P(B) > 0 \qquad (2\text{-}15)$$

This is a useful result that enables us to solve $P(A|B)$ for in terms of $P(B|A)$.

- Bayes' theorem

If $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive events and B is any event,

$$P(E_1|B) = \frac{P(B|E_1)P(E_1)}{P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \cdots + P(B|E_k)P(E_k)} \qquad (2\text{-}16)$$

for $P(B) > 0$

2.8 Random variables

- Definition---random variable
  - A **random variable** is a function that assigns a real number to each outcome in the sample space of a random experiment.
  - A random variable is denoted by an uppercase letter such as X. After an experiment is conducted, the measured value of the random variable is denoted by a lowercase letter such as x=70 milliamperes.

- Definition
  - A **discrete** random variable is a random variable with a finite (or countably infinite) range.
  - A **continuous** random variable is a random variable with an interval (either finite orinfinite) of real numbers for its range.

- Examples of random variables
  - Examples of **continuous** random variables:

    electrical current, length, pressure, temperature, time, voltage, weight.

  - Examples of **discrete** random variables:

    number of scratches on a surface, proportion of defective parts among 1000 tested, number of transmitted bits received in error.