4-2 P.D.F.

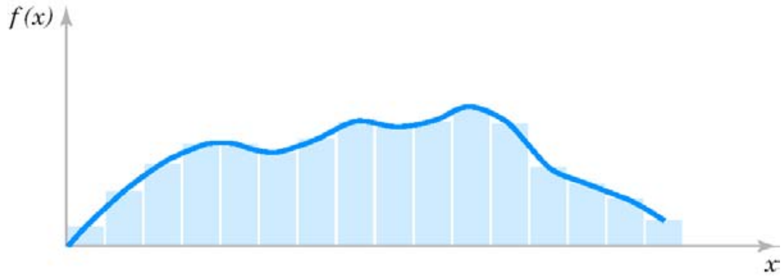
- A **probability density function** $f(x)$ can be used to describe the probability distribution of a **continuous random variable** X.
- Def. For a continuous random variable X, a **probability density function** is a function such that

$$(1)\quad f(x) \geq 0$$

$$(2)\quad \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$(3)\quad P(a \leq X \leq b) = \int_{a}^{b} f(x)\,dx = \text{ area under } f(x) \text{ from } a \text{ to } b \text{ for any } a \text{ and } b$$

➢ A **histogram** is an approximation to a probability density function.



- $P(X=x)=0$, Ex. 14.47: $14.465 \le x \le 14.475$.
- One need not distinguish between inequalities such as < or ≤ for continuous random variables.
- If X is a **continuous random variable,** for any x_1 and x_2,

$$P(x_1 \le X \le x_2) = P(x_1 < X \le x_2) = P(x_1 \le X < x_2) = P(x_1 < X < x_2) \quad (4\text{-}2)$$

4-3 C.D.F.

- Def.

The **cumulative distribution function** of a continuous random variable X is

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\, du \quad \text{for } -\infty < x < \infty.$$

Note: $P(X>x)=1-P(X\le x)=1-F(x)$

- $\frac{d}{dx}\int_{-\infty}^{x} f(u)du = f(x) \qquad f(x) = \frac{dF(x)}{dx}$

- $P(5<X<10)=F(10)-F(5)$

4-4 mean and variance

- Def. Suppose X is a continuous random variable with probability density function $f(x)$. The **mean** or **expected value** of X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)\,dx$$

The **variance** of X, denoted as $V(X)$ or σ^2 is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

The **standard deviation** of X is $\sigma=\sqrt{\sigma^2}$.

- If X is a continuous random variable with probability density function $f(x)$,

$$E[h(X)] = \int_{-\infty}^{\infty} h(x) f(x)\, dx$$

4-5 CONTINUOUS UNIFORM DISTRIBUTION

- Def. A continuous random variable X with probability density function $f(x)=1/(b-a)$, $a \le x \le b$ is a continuous uniform random variable.
- If X is a continuous uniform random variable over $a \le x \le b$,
  - $\mu = E[X] = (a+b)/2$
  - $\sigma^2 = (b-a)^2/12$

$$E(X) = \int_a^b \frac{x}{b-a}\,dx = \frac{0.5x^2}{b-a}\Bigg|_a^b = \frac{(a+b)}{2}$$

$$V(X) = \int_a^b \frac{\left(x - \left(\frac{a+b}{2}\right)\right)^2}{b-a}\,dx = \frac{\left(x - \frac{a+b}{2}\right)^3}{3(b-a)}\Bigg|_a^b = \frac{(b-a)^2}{12}$$

- The cumulative distribution function of a continuous uniform random variable

$$F(x) = \int_a^x 1/(b-a)\,du = x/(b-a) - a/(b-a)$$

$$F(x) = \begin{cases} 0 & x < a \\ (x-a)/(b-a) & a \le x < b \\ 1 & b \le x \end{cases}$$

- Ex. X is uniformly distributed in [-a a]. Find a, so that the following is satisfied (1)$P(X>1)=1/3$ (2)$P(X<0.5)=0.7$

$$f(x)=\begin{cases}\frac{1}{2a}, & -a\le x\le a\\ 0, & \text{otherwise}\end{cases} \qquad F(x)=\int_{-a}^{x}\frac{1}{2a}du=\frac{x}{2a}+\frac{1}{2}$$

$$P(x>1)=1-P(x\le 1)=1-F(1)=1-\frac{1}{2a}-\frac{1}{2}=\frac{1}{2}-\frac{1}{2a}=\frac{1}{3}$$

$$a=3$$

4-6 Normal (Gaussian) distribution

A random variable X with probability density function

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad -\infty<x<\infty \tag{4-8}$$

is a **normal random variable** with parameters μ, where $-\infty<\mu<\infty$, and $\sigma>0$.

$$E(X) = \mu \quad \text{and} \quad V(X) = \sigma^2 \qquad (4\text{-}9)$$

and the notation $N(\mu, \sigma^2)$ is used to denote the distribution. The mean and variance of X are shown to equal μ and σ^2, respectively, at the end of this Section 5-6.

- Some useful results concerning a normal distribution are summarized below

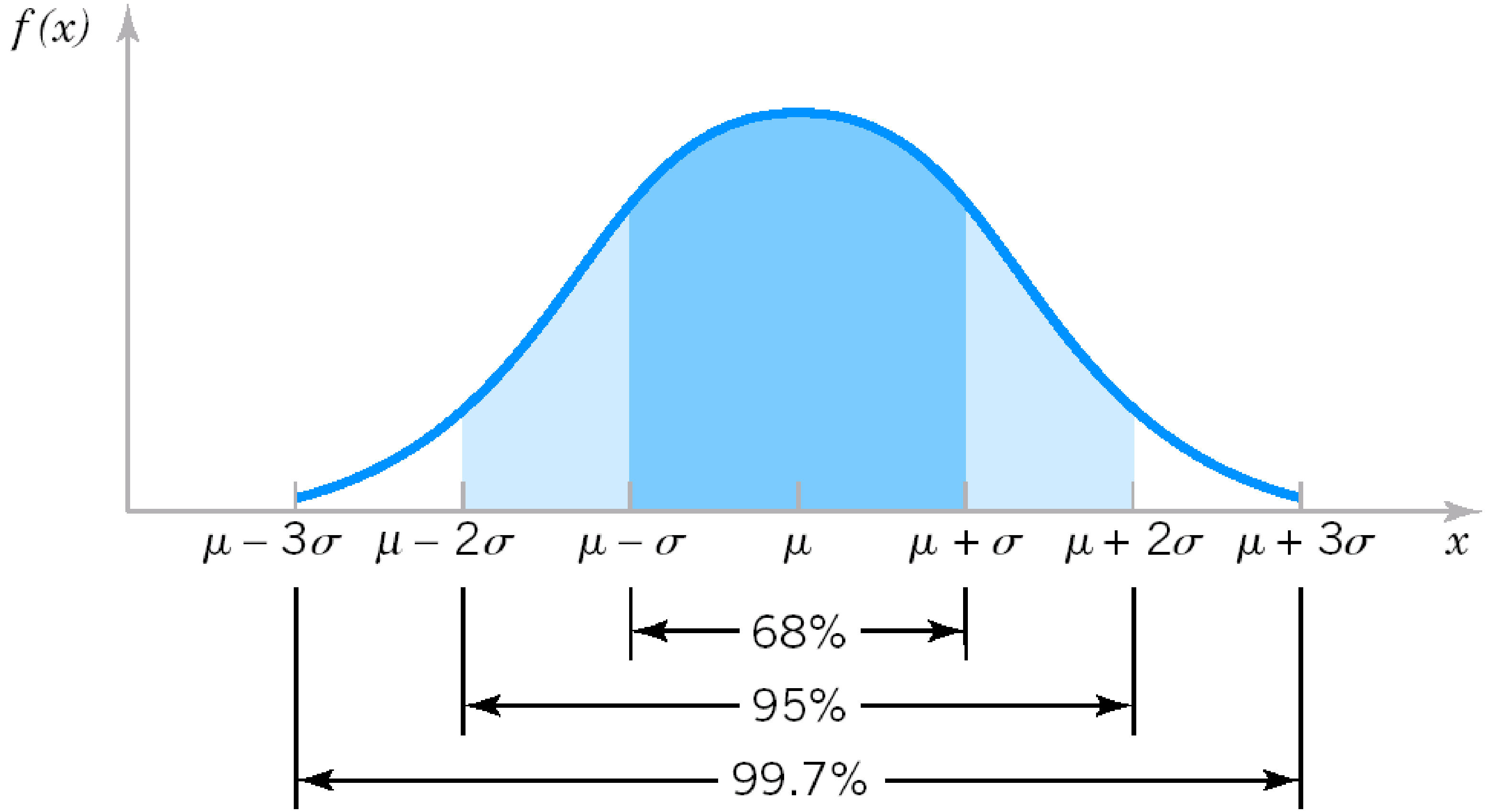
1. $P(\mu-\sigma<X<\mu+\sigma)=0.6827$, $P(\mu-2\sigma<X<\mu+2\sigma)=0.9545$

   $P(\mu-3\sigma<X<\mu+3\sigma)=0.9973$

2. $P(X>\mu)=P(X<\mu)=0.5$

3. 6σ is often referred to as the **width** of a normal distribution.

$P(\mu<X<\mu+\sigma)= 0.34135$, $P(X<\mu-\sigma)= 0.15865$



- **Def.--** A normal random variable with $\mu=0$ and $\sigma^2=1$ is called a **standard normal random variable** and is denoted as *Z*. The cumulative distribution function of a standard normal random variable is denoted as

$$\Phi(z) = P(Z \leq z)$$

If X is a normal random variable with $E(X) = \mu$ and $V(X) = \sigma^2$, the random variable

$$Z = \frac{X - \mu}{\sigma} \tag{4-10}$$

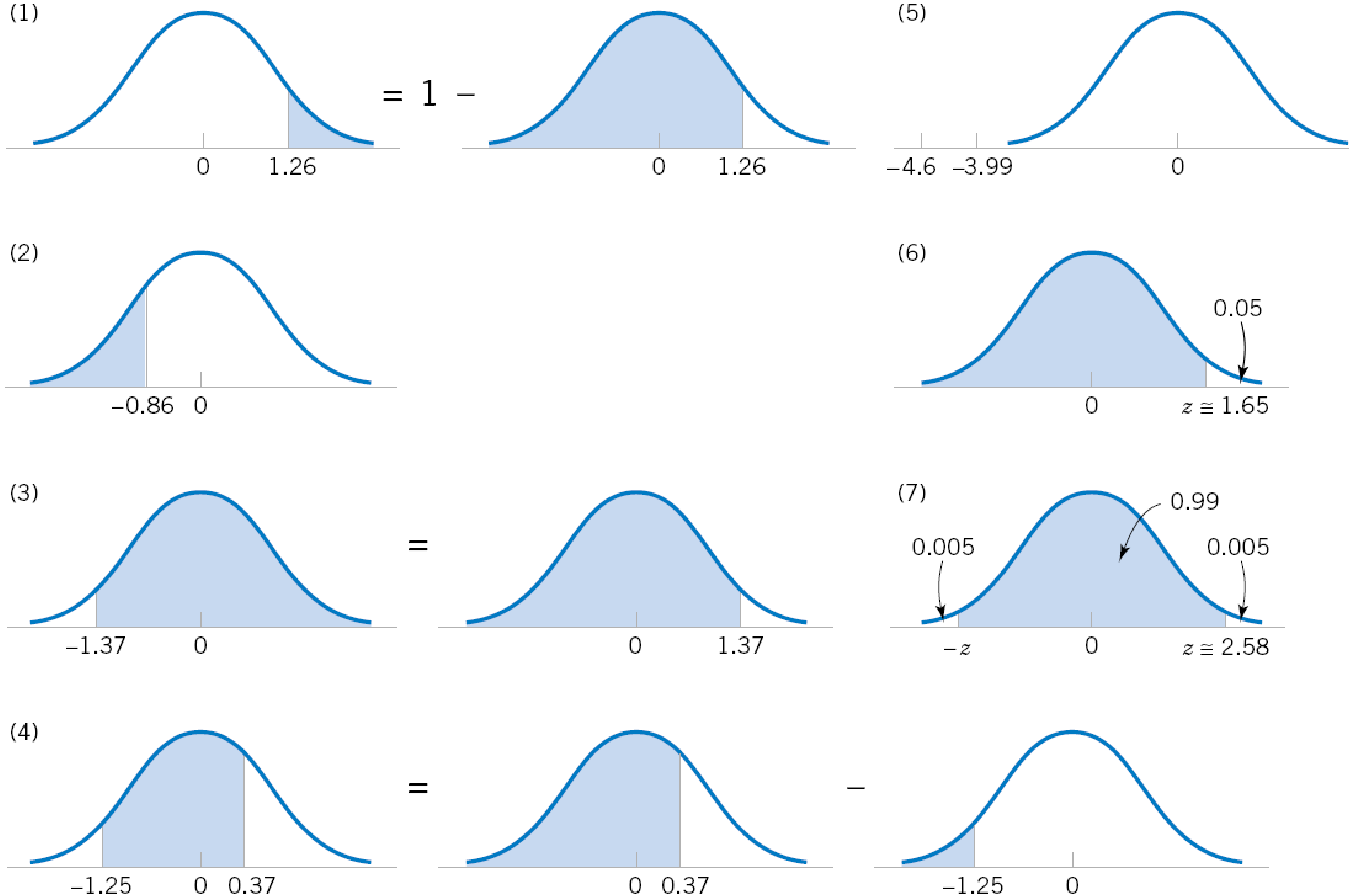
is a normal random variable with $E(Z) = 0$ and $V(Z) = 1$. That is, Z is a standard normal random variable.

- It is the key step to calculate a probability for an arbitrary normal random variable. (p.A-6, Table III)
- Ex. 4-12 (1) $P(Z>1.26)$, (2) $P(Z<-0.86)$, (3) $P(Z>-1.37)$

(4) $P(-1.25<Z<0.37)$, (5) $P(Z\leq-4.6)$, (6) $P(Z>z)=0.05$, z=?

(7) $P(-z<Z<z)=0.99$, z=?

- Ex. 4-13 Normal distribution, mean=10, variance=4, $P(X>13)$=?
  - (13-10)/2=1.5, $P(Z>1.5)=1-P(Z<1.5)=1-0.93319=0.06681$ (from Table III)



(1)
0
1.26
= 1 −
0
1.26
(5)
−4.6
−3.99
0
(2)
−0.86
0
(6)
0.05
0
z ≅ 1.65
(3)
−1.37
0
=
0
1.37
(7)
0.99
0.005
0.005
−z
0
z ≅ 2.58
(4)
−1.25
0
0.37
=
0
0.37
−
−1.25
0

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.532922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273
1.8	0.964070	0.964852	0.965621	0.966375	0.967116	0.967843	0.968557	0.969258	0.969946	0.970621
1.9	0.971283	0.971933	0.972571	0.973197	0.973810	0.974412	0.975002	0.975581	0.976148	0.976705
2.0	0.977250	0.977784	0.978308	0.978822	0.979325	0.979818	0.980301	0.980774	0.981237	0.981691
2.1	0.982136	0.982571	0.982997	0.983414	0.983823	0.984222	0.984614	0.984997	0.985371	0.985738
2.2	0.986097	0.986447	0.986791	0.987126	0.987455	0.987776	0.988089	0.988396	0.988696	0.988989
2.3	0.989276	0.989556	0.989830	0.990097	0.990358	0.990613	0.990863	0.991106	0.991344	0.991576
2.4	0.991802	0.992024	0.992240	0.992451	0.992656	0.992857	0.993053	0.993244	0.993431	0.993613
2.5	0.993790	0.993963	0.994132	0.994297	0.994457	0.994614	0.994766	0.994915	0.995060	0.995201
2.6	0.995339	0.995473	0.995604	0.995731	0.995855	0.995975	0.996093	0.996207	0.996319	0.996427
2.7	0.996533	0.996636	0.996736	0.996833	0.996928	0.997020	0.997110	0.997197	0.997282	0.997365
2.8	0.997445	0.997523	0.997599	0.997673	0.997744	0.997814	0.997882	0.997948	0.998012	0.998074
2.9	0.998134	0.998193	0.998250	0.998305	0.998359	0.998411	0.998462	0.998511	0.998559	0.998605
3.0	0.998650	0.998694	0.998736	0.998777	0.998817	0.998856	0.998893	0.998930	0.998965	0.998999
3.1	0.999032	0.999065	0.999096	0.999126	0.999155	0.999184	0.999211	0.999238	0.999264	0.999289
3.2	0.999313	0.999336	0.999359	0.999381	0.999402	0.999423	0.999443	0.999462	0.999481	0.999499
3.3	0.999517	0.999533	0.999550	0.999566	0.999581	0.999596	0.999610	0.999624	0.999638	0.999650
3.4	0.999663	0.999675	0.999687	0.999698	0.999709	0.999720	0.999730	0.999740	0.999749	0.999758
3.5	0.999767	0.999776	0.999784	0.999792	0.999800	0.999807	0.999815	0.999821	0.999828	0.999835
3.6	0.999841	0.999847	0.999853	0.999858	0.999864	0.999869	0.999874	0.999879	0.999883	0.999888
3.7	0.999892	0.999896	0.999900	0.999904	0.999908	0.999912	0.999915	0.999918	0.999922	0.999925
3.8	0.999928	0.999931	0.999933	0.999936	0.999938	0.999941	0.999943	0.999946	0.999948	0.999950
3.9	0.999952	0.999954	0.999956	0.999958	0.999959	0.999961	0.999963	0.999964	0.999966	0.999967

- Suppose X is a normal random variable with mean μ and variance σ^2. Then,

$$P(X \leq x) = P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) = P(Z \leq z)$$

where Z is a **standard normal random variable,** and $z=(x- \mu)/\sigma$ is the ***z*-value** obtained by **standardizing** X.

- ➢The probability is obtained by entering Appendix Table III with $z=(x- \mu)/\sigma$

4-7 NORMAL APPROXIMATION TO THE BINOMIAL AND POISSON DISTRIBUTIONS

- ➢The normal distribution can be used to approximate binomial probabilities for cases in which n is large.

- Ex. 4-17 A bit is received in error is 10^{-5}, If 16,000,000 bits are transmitted, what is the probability that 150 or fewer errors occur?

$$P(X \le 150) = \sum_{x=0}^{150} \binom{16,000,000}{x} (10^{-5})^x (1-10^{-5})^{16,000,000-x}$$

  - ➢The probability of Ex. 4-17 is difficult to compute.

- Normal approximation to the Binomial distribution
  - ➢If X is a binomial random variable, with parameters n and p,

$$Z = \frac{X - np}{\sqrt{np(1-p)}}$$

is approximately a standard normal random variable.

- To approximate a binomial probability with a normal distribution a continuity correction is applied as follows

$$P(X \le x) = P(X \le x + 0.5) \cong P(Z \le \frac{x + 0.5 - np}{\sqrt{np(1-p)}})$$

$$\text{and } P(x \le X) = P(x - 0.5 \le X) \cong P(\frac{x - 0.5 - np}{\sqrt{np(1-p)}} \le Z)$$

The approximation is good for $np > 5$ and $n(1 - p) > 5$

- Ex 4-18 $P(X \le 150) = P(X \le 150.5) \cong P(\frac{X - 160}{\sqrt{160(1-10^{-5})}} \le \frac{150.5 - 160}{\sqrt{160(1-10^{-5})}})$

$$= P(Z \le -0.75) = 0.227$$

  - Because $np = 16 \times 10^6 10^{-5} = 160$ and $n(1-p)$ is much larger, the approximation is expected to work well in this case.

  - Fig. 4-20

- hypergometric distribution $\approx$ binomial distribution $\approx$ normal distribution

hypergometric distribution	$\approx$	binomial distribution	$\approx$	normal distribution
	$\frac{n}{N} < 0.1$		$np > 5$ $n(1-p) > 5$	

- Normal approximation to the Poisson distribution
  - If X is a Poisson random variable with $E(X)=\lambda$ and $V(X)=\lambda$

$$Z = \frac{X - \lambda}{\sqrt{\lambda}}$$

is approximately a standard normal random variable. The approximation is good for $\lambda>5$.

  - Ex. 4-20 Assume that the number of asbestos particles in a squared meter of dust on a surface follows a Poisson distribution with a mean of 1000. If a squared meter of dust is analyzed, what is the probability that less than 950 particles are found?

- $P(X \leq 950) = \sum_{x=0}^{950} \frac{e^{-1000} 1000^{x}}{x!}$

$$P(X \leq x) = P(Z \leq \frac{950 - 1000)}{\sqrt{1000}}) = P(Z \leq -1.58) = 0.057$$

-

- $Z = \dfrac{X-\mu}{\sigma} = \dfrac{50-45}{4.975} = 1.005$

  $P(X \geq 50) = P(Z > 1.005) = 1 - P(Z \leq 1.005) \approx 1 - 0.841 = 0.159$

  $P(X \geq 50) \cong P(X \geq \dfrac{50-0.5-45}{4.975}) = P(X \geq 0.9045) \approx 0.171$

4-8 EXPONENTIAL DISTRIBUTION

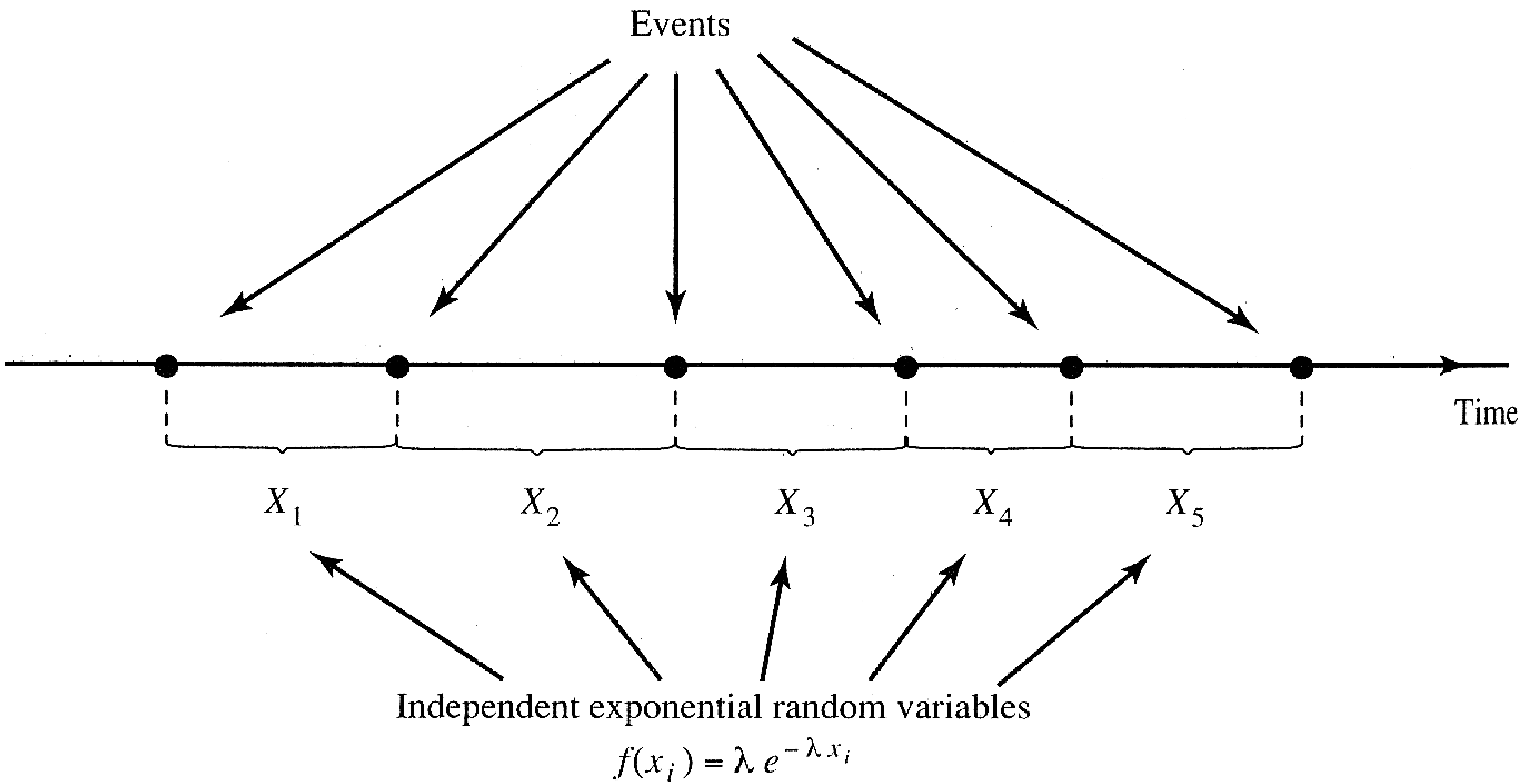
- Def. The random variable X that equals the distance between successive events of a Poisson process with mean number of events $\lambda > 0$ per unit interval is an **exponential random variable** with parameter λ. The probability density function of X is

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } 0 \le x < \infty$$

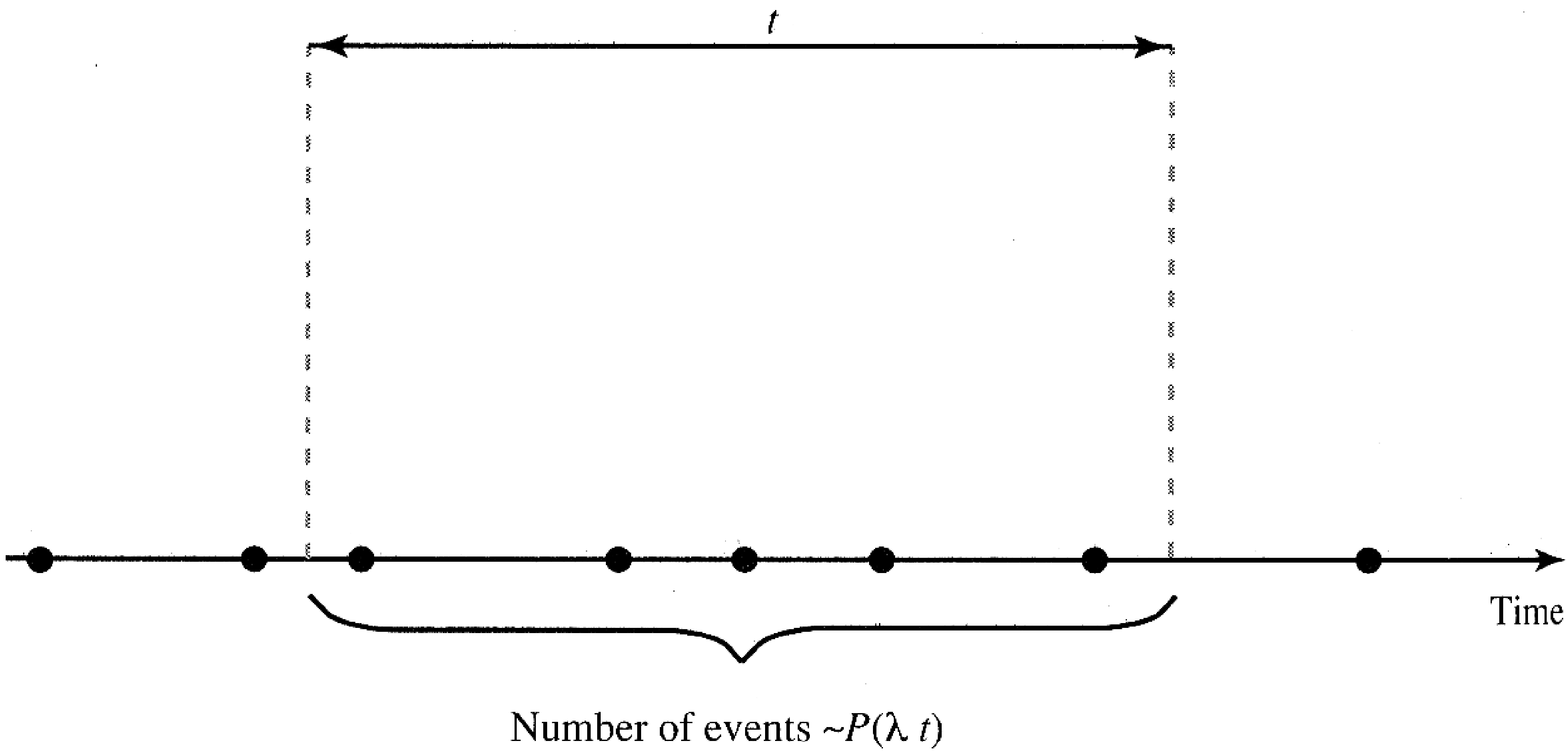
  - If the random variable X has an exponential distribution with parameter λ ,

$$\mu = E[X] = \frac{1}{\lambda}, \ \sigma^2 = V[X] = \frac{1}{\lambda^2}$$

- A Poisson process. The time intervals between events have independent exponential distributions with parameter λ



- A Poisson process. The number of events occurring in a time interval of length t has a Poison distribution with mean λt



- Ex. The discussion of the Poisson distribution defined a random variable to be the number of flaws along a length of copper wire.
  - ☞ Let the random variable *N* denote the number of flaws in *x* millimeters of wire.
  - ☞ If the mean number of flaws is λ per millimeter, *N* has a Poisson distribution with mean λ*x*.
  - ☞ We assume that the wire is longer than the value of *x*.

$$P(X > x) = P(N = 0) = \frac{e^{-\lambda x}(\lambda x)^0}{0!} = e^{-\lambda x}$$

Therefore, $F(x) = P(X \le x) = 1 - e^{-\lambda x},\ x \ge 0$

☞ P.D.F --- $f(x) = \lambda e^{-\lambda x}$

- Ex. At a checkout counter, customers arrive at an average of 2 per hour following a Poisson process. Here λ =2 customers per hour. Time between arrivals are exponentially distributed with parameter λ, implying mean of ½ hours =30 minutes between arrivals.

What is the probability that no customers arrive in first 3 hours?

☞ $P(X>3)$: Time until the first arrival, is greater than 3 hours.

☞ $P(X > 3) = 1 - P(X \le 3) = 1 - F(3) = 1 - \int_0^3 2e^{-2x} dx = e^{-6}$

By using the Poisson process:

☞ $\lambda_3 = 3\lambda = 3*2 = 6$ customers per 3 hours.

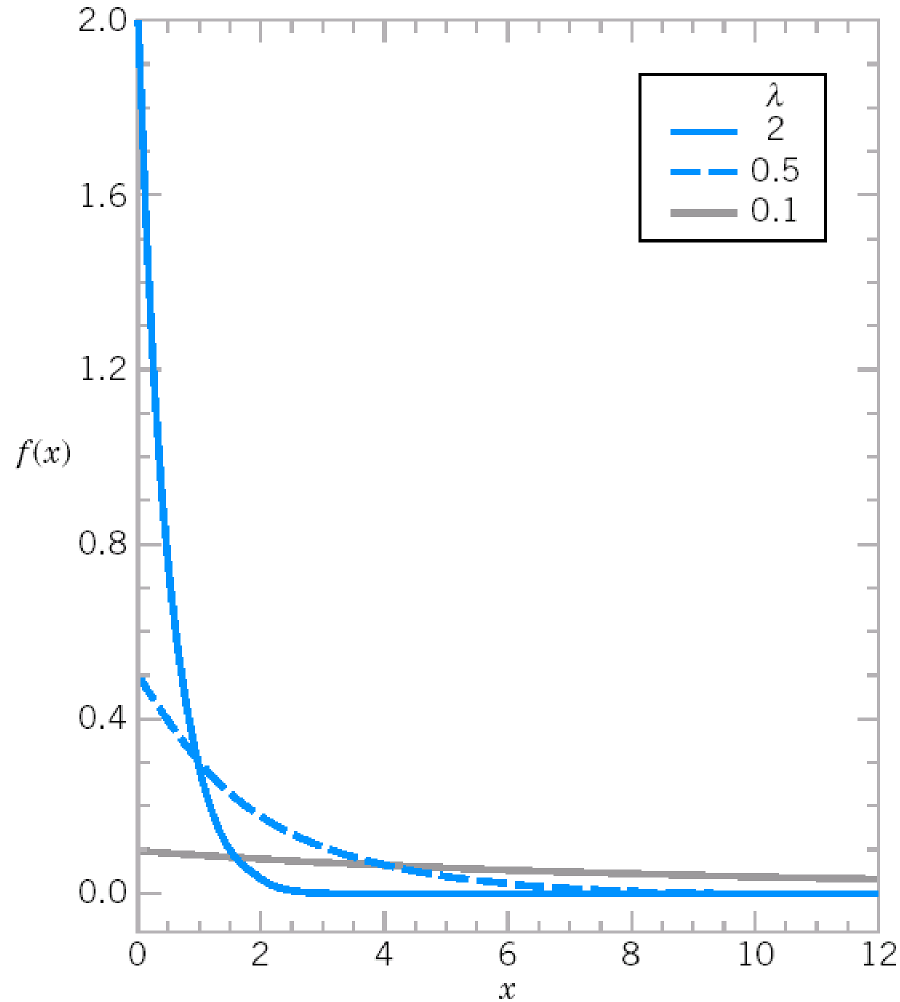
☞ No arrival occurs in first 3 hours.

$$P(Y_3=0) = (e^{-6} * 6^0) / 0! = e^{-6}$$

- Another form

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

where $\beta > 0$



- Lack of Memory Property
  - ☞ For an exponential random variable X,

$$P(X<t_1+t_2 \mid X>t_1) = P(X<t_2)$$

- ■ $P(X<t_1+t_2 \mid X>t_1)= P((X<t_1+t_2) \cap (X>t_1))/P(X>t_1)$

Ex. The lifetime of a semiconductor chip might be modeled as an exponential random variable with a mean of 40,000 hours.

☞ That is, regardless of how long the device has been operating, the probability of a failure in the next 1000 hours is the same as the probability of a failure in the first 1000 hours of operation.