Natural Logarithms (Sect. 7.2)

- Definition as an integral.
- The derivative and properties.
- The graph of the natural logarithm.
- Integrals involving logarithms.
- Logarithmic differentiation.

Definition as an integral

Recall:

(a) The derivative of $y = x^n$ is $y' = n\, x^{(n-1)}$, for n integer.

(b) The integral of $y = x^n$ is $\displaystyle\int x^n\, dx = \frac{x^{(n+1)}}{(n+1)}$, for $n \neq -1$.

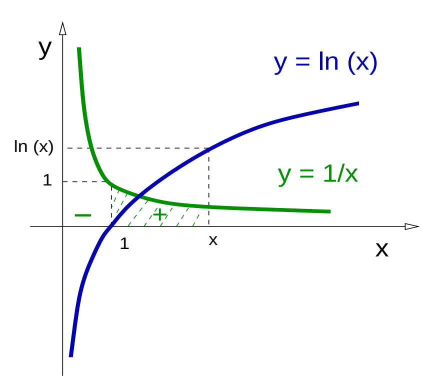
(c) Case $n = -1$: $\displaystyle\int \frac{dx}{x}$ is neither rational nor trigonometric function. This is a new function.

Definition

The *natural logarithm* is the function

$$\ln(x) = \int_1^x \frac{dt}{t}, \qquad x \in (0, \infty).$$

In particular: $\ln(1) = 0$.



Definition as an integral

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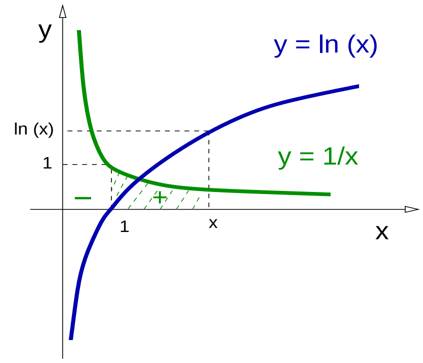
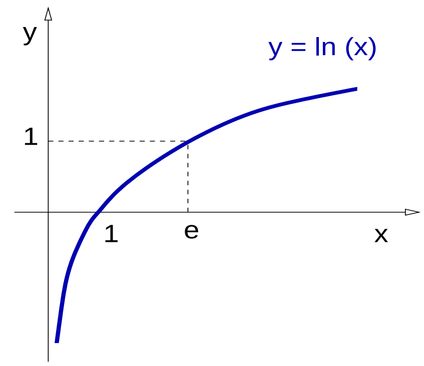


Definition
The *number e* is the number satisfying $\ln(e) = 1$, that is,

$$\int_1^e \frac{dt}{t} = 1.$$

$(e = 2.718281...)$.



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The derivative and properties

Theorem (Derivative of ln)

The Fundamental Theorem of Calculus implies $\ln'(x) = \dfrac{1}{x}$.

Proof:

$$\ln(x) = \int_1^x \frac{dt}{t} \quad \Rightarrow \quad \ln'(x) = \frac{1}{x}.$$

□

Theorem (Chain rule)

For every differentiable function u holds $\big[\ln(u)\big]' = \dfrac{u'}{u}$.

Proof:

$$\frac{d\ln(u)}{dx} = \frac{d\ln}{du}(u)\,\frac{du}{dx} = \frac{1}{u}\,u' \quad \Rightarrow \quad \frac{d\ln(u)}{dx}(x) = \frac{u'(x)}{u(x)}.$$

□

The derivative and properties

Example

Find the derivative of $y(x) = \ln(3x)$, and $z(x) = \ln\big(2x^2 + \cos(x)\big)$.

Solution: We use the chain rule.

$$y'(x) = \frac{1}{(3x)}\,(3) = \frac{1}{x} \quad \Rightarrow \quad y'(x) = \frac{1}{x}.$$

We also use chain rule,

$$z'(x) = \frac{1}{\big(2x^2 + \cos(x)\big)}\,\big(4x - \sin(x)\big)$$

$$z'(x) = \frac{4x - \sin(x)}{2x^2 + \cos(x)}.$$

◁

Remark: $y(x) = \ln(3x)$, satisfies $y'(x) = \ln'(x)$.

The derivative and properties

Theorem (Algebraic properties)

For every positive real numbers a and b holds,

(a) $\ln(ab) = \ln(a) + \ln(b)$, (product rule);

(b) $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$, (quotient rule);

(c) $\ln\left(\frac{1}{a}\right) = -\ln(a)$, (reciprocal rule);

(d) $\ln(a^b) = b\ln(a)$, (power rule).

Proof of (a): (only)

The function $y(x) = \ln(ax)$ satisfies $y'(x) = \frac{1}{ax}\,a = \frac{1}{x} = \ln'(x)$

Therefore $\ln(ax) = \ln(x) + c$. Evaluating at $x = 1$ we obtain c.

$$\ln(a) = \ln(1) + c \;\Rightarrow\; c = \ln(a) \;\Rightarrow\; \ln(ax) = \ln(x) + \ln(a).$$

The derivative and properties

Example

Compute the derivative of $y(x) = \ln\left[\frac{(x+1)^2}{3(x+2)}\right]$.

Solution: Before computing the derivative of y, we simplify it,

$$y = \ln\left[(x+1)^2\right] - \ln\left[3(x+2)\right],$$

$$y = 2\,\ln(x+1) - \left[\ln(3) + \ln(x+2)\right].$$

The derivative of function y is: $y' = 2\,\frac{1}{(x+1)} - \frac{1}{(x+2)}$.

$$y' = \frac{2(x+2) - (x+1)}{(x+1)(x+2)} \quad\Rightarrow\quad y' = \frac{(x+3)}{(x+1)(x+2)}. \qquad \triangleleft$$

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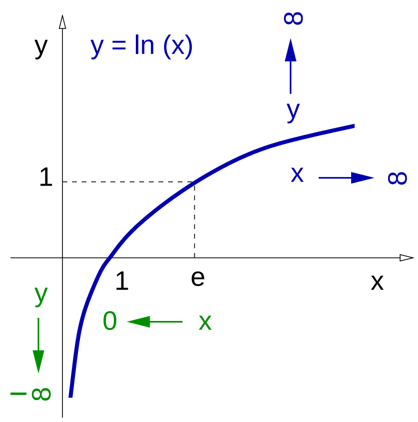
The graph of the natural logarithm

Remarks:

The graph of ln function has:

(a) A vertical asymptote at $x = 0$.

(b) No horizontal asymptote.



Proof: Recall $e = 2.718281... > 1$ and $\ln(e) = 1$.

(a): If $x = e^n$, then $\ln(e^n) = n\ln(e) = n$. Hence

$$\lim_{x\to\infty} \ln(x) = \infty.$$

(b): If $x = \dfrac{1}{e^n}$, then $\ln\Big(\dfrac{1}{e^n}\Big) = -\ln(e^n) - n\ln(e) = -n$. Hence

$$\lim_{x\to 0^+} \ln(x) = -\infty.$$

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Integrals involving logarithms.

Remark: It holds $\displaystyle\int \frac{dx}{x} = \ln(|x|) + c$ for $x \neq 0$ and $c \in \mathbb{R}$.

Indeed, for $x > 0$ this is the definition of logarithm.
And for $x < 0$, we have that $-x > 0$, then,

$$\int \frac{dx}{x} = \int \frac{(-dx)}{(-x)} = \ln(-x) + c, \qquad -x > 0.$$

We conclude,

$$\int \frac{dx}{x} = \begin{cases} \ln(-x) + c & \text{if } x < 0, \\ \ln(x) + c & \text{if } x > 0. \end{cases}$$

Remark: It also holds $\displaystyle\int \frac{f'(x)}{f(x)}\, dx = \ln(|f(x)|) + c$, for $f(x) \neq 0$.

Integrals involving logarithms.

Remarks:

(a) $\int \tan(x)\,dx = -\ln(|\cos(x)|) + c$. Indeed,

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx \quad u = \cos(x), \quad du = -\sin(x)\,dx.$$

$$\int \tan(x)\,dx = -\int \frac{du}{u} = -\ln(|u|) + c = -\ln(|\cos(x)|) + c.$$

(b) $\int \cot(x)\,dx = \ln(|\sin(x)|) + c$. Indeed,

$$\int \cot(x)\,dx = \int \frac{\cos(x)}{\sin(x)}\,dx \quad u = \sin(x), \quad du = \cos(x)\,dx.$$

$$\int \cot(x)\,dx = \int \frac{du}{u} = \ln(|u|) + c = \ln(|\sin(x)|) + c.$$

Integrals involving logarithms.

Example

Find $y(t) = \int \frac{3\sin(t)}{(2+\cos(t))}\,dt$.

Solution:

$$y(t) = \int \frac{3\sin(t)}{(2+\cos(t))}\,dt, \quad u = 2+\cos(t), \quad du = -\sin(t)\,dt.$$

$$y(t) = \int \frac{3(-du)}{u} = -3\int \frac{du}{u} = -3\,\ln(|u|) + c$$

We conclude that $y(t) = -3\,\ln\big(|2+\cos(t)|\big) + c$. $\triangleleft$

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Logarithmic differentiation

Remark: Logarithms can be used to simplify the derivative of complicated functions.

Example

Find the derivative of $y(x) = \dfrac{x^3(x+2)^2}{\cos^3(x)}$.

Solution: First compute $\ln[y(x)] = \ln\Big[\dfrac{x^3(x+2)^2}{\cos^3(x)}\Big]$,

$$\ln[y(x)] = \ln\big[x^3(x+2)^2\big] - \ln\big[\cos^3(x)\big],$$

$$\ln[y(x)] = \ln\big[x^3\big] + \ln\big[(x+2)^2\big] - \ln\big[\cos^3(x)\big],$$

$$\ln[y(x)] = 3\ln(x) + 2\ln(x+2) - 3\ln\big[\cos(x)\big].$$

Logarithmic differentiation

Example

Find the derivative of $y(x) = \dfrac{x^3(x+2)^2}{\cos^3(x)}$.

Solution: Recall: $\ln[y(x)] = 3\ln(x) + 2\ln(x+2) - 3\ln\big[\cos(x)\big]$.

$$\frac{y'(x)}{y(x)} = \frac{3}{x} + \frac{2}{(x+2)} + \frac{3\sin(x)}{\cos(x)}.$$

$$y'(x) = \left[\frac{3}{x} + \frac{2}{(x+2)} + \frac{3\sin(x)}{\cos(x)}\right] y(x).$$

We conclude that

$$y'(x) = \left[\frac{3}{x} + \frac{2}{(x+2)} + \frac{3\sin(x)}{\cos(x)}\right] \frac{\cos^3(x)}{x^3(x+2)^2}.$$

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