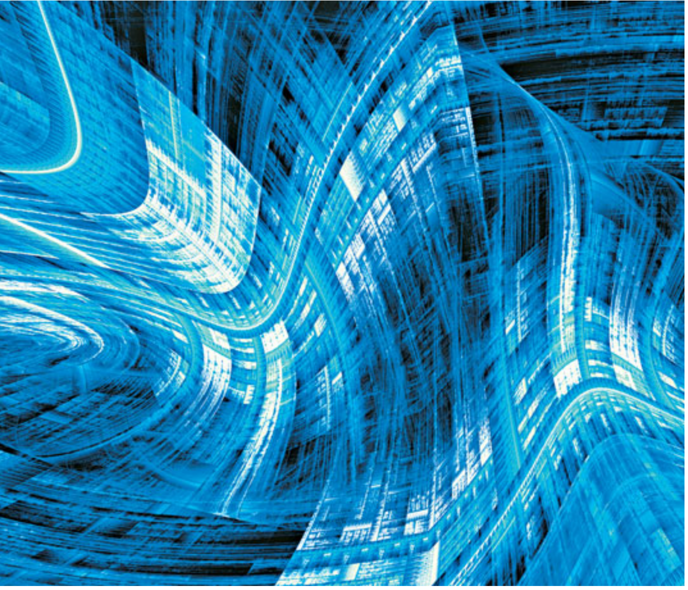

8

Techniques of Integration

OVERVIEW The Fundamental Theorem tells us how to evaluate a definite integral once we have an antiderivative for the integrand function. However, finding antiderivatives (or indefinite integrals) is not as straightforward as finding derivatives. We need to develop some techniques to help us. Nevertheless, we note that it is not always possible to find an antiderivative expressed in terms of elementary functions.

In this chapter we study a number of important techniques which apply to finding integrals for specialized classes of functions such as trigonometric functions, products of certain functions, and rational functions. Since we cannot always find an antiderivative, we also develop some numerical methods for calculating definite integrals. Finally, we extend the idea of the definite integral to *improper integrals*, and we apply them to finding probabilities.

8.1 Using Basic Integration Formulas

Table 8.1 summarizes the forms of indefinite integrals for many of the functions we have studied so far, and the substitution method helps us use the table to evaluate more complicated functions involving these basic ones. In this section we combine the Substitution Rules (studied in Chapter 5) with algebraic methods and trigonometric identities to help us use Table 8.1. A more extensive Table of Integrals is given at the back of the book, and we discuss its use in Section 8.6.

Sometimes we have to rewrite an integral to match it to a standard form in Table 8.1. We have used this procedure before, but here is another example.

EXAMPLE 1 Evaluate the integral

$$\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx.$$

Solution We rewrite the integral and apply the Substitution Rule for Definite Integrals presented in Section 5.6, to find

$$\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx = \int_1^{11} \frac{du}{\sqrt{u}} \qquad \begin{aligned}&u = x^2-3x+1,\ du=(2x-3)\,dx;\\ &u=1 \text{ when } x=3,\ u=11 \text{ when } x=5\end{aligned}$$

$$= \int_1^{11} u^{-1/2}\,du$$

$$= 2\sqrt{u}\,\Big]_1^{11} = 2\left(\sqrt{11}-1\right) \approx 4.63. \qquad \text{Table 8.1, Formula 2}$$

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TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C \quad$ (any number k)
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$
21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \quad (a > 0)$
22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \quad (x > a > 0)$

EXAMPLE 2 Complete the square to evaluate

$$\int \frac{dx}{\sqrt{8x - x^2}}.$$

Solution We complete the square to simplify the denominator:

$$\begin{aligned} 8x - x^2 &= -(x^2 - 8x) = -(x^2 - 8x + 16 - 16) \\ &= -(x^2 - 8x + 16) + 16 = 16 - (x - 4)^2. \end{aligned}$$

Then

$$\begin{aligned} \int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{dx}{\sqrt{16 - (x - 4)^2}} \\ &= \int \frac{du}{\sqrt{a^2 - u^2}} && a = 4,\ u = (x - 4),\ du = dx \\ &= \sin^{-1}\left(\frac{u}{a}\right) + C && \text{Table 8.1, Formula 18} \\ &= \sin^{-1}\left(\frac{x - 4}{4}\right) + C. \end{aligned}$$

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EXAMPLE 3 Evaluate the integral

$$\int (\cos x \sin 2x + \sin x \cos 2x)\, dx.$$

Solution Here we can replace the integrand with an equivalent trigonometric expression using the Sine Addition Formula to obtain a simple substitution:

$$\begin{aligned}\int (\cos x \sin 2x + \sin x \cos 2x)\, dx &= \int (\sin(x + 2x))\, dx \\ &= \int \sin 3x\, dx \\ &= \int \frac{1}{3} \sin u\, du && u = 3x,\ du = 3\, dx \\ &= -\frac{1}{3} \cos 3x + C. && \text{Table 8.1, Formula 6}\end{aligned}$$

■

In Section 5.5 we found the indefinite integral of the secant function by multiplying it by a fractional form identically equal to one, and then integrating the equivalent result. We can use that same procedure in other instances as well, which we illustrate next.

EXAMPLE 4 Find $\displaystyle\int_0^{\pi/4} \frac{dx}{1 - \sin x}$.

Solution We multiply the numerator and denominator of the integrand by $1 + \sin x$, which is simply a multiplication by a form of the number one. This procedure transforms the integral into one we can evaluate:

$$\begin{aligned}\int_0^{\pi/4} \frac{dx}{1 - \sin x} &= \int_0^{\pi/4} \frac{1}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x}\, dx \\ &= \int_0^{\pi/4} \frac{1 + \sin x}{1 - \sin^2 x}\, dx \\ &= \int_0^{\pi/4} \frac{1 + \sin x}{\cos^2 x}\, dx \\ &= \int_0^{\pi/4} (\sec^2 x + \sec x \tan x)\, dx && \text{Use Table 8.1, Formulas 8 and 10} \\ &= \Big[\tan x + \sec x\Big]_0^{\pi/4} = \left(1 + \sqrt{2} - (0 + 1)\right) = \sqrt{2}.\end{aligned}$$

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EXAMPLE 5 Evaluate

$$\int \frac{3x^2 - 7x}{3x + 2}\, dx.$$

$$\begin{array}{r} x - 3 \\ 3x + 2\overline{)3x^2 - 7x} \\ \underline{3x^2 + 2x} \\ -9x \\ \underline{-9x - 6} \\ +6 \end{array}$$

Solution The integrand is an improper fraction since the degree of the numerator is greater than the degree of the denominator. To integrate it, we perform long division to obtain a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

Therefore,

$$\int \frac{3x^2 - 7x}{3x + 2}\,dx = \int \left(x - 3 + \frac{6}{3x + 2}\right) dx = \frac{x^2}{2} - 3x + 2\ln|3x + 2| + C.$$

Reducing an improper fraction by long division (Example 5) does not always lead to an expression we can integrate directly. We see what to do about that in Section 8.5.

EXAMPLE 6 Evaluate

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\,dx.$$

Solution We first separate the integrand to get

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\,dx = 3\int \frac{x\,dx}{\sqrt{1 - x^2}} + 2\int \frac{dx}{\sqrt{1 - x^2}}.$$

In the first of these new integrals, we substitute

$$u = 1 - x^2, \qquad du = -2x\,dx, \qquad \text{so} \qquad x\,dx = -\frac{1}{2}\,du.$$

Then we obtain

$$\begin{aligned} 3\int \frac{x\,dx}{\sqrt{1 - x^2}} &= 3\int \frac{(-1/2)\,du}{\sqrt{u}} = -\frac{3}{2}\int u^{-1/2}\,du \\ &= -\frac{3}{2} \cdot \frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{1 - x^2} + C_1. \end{aligned}$$

The second of the new integrals is a standard form,

$$2\int \frac{dx}{\sqrt{1 - x^2}} = 2\sin^{-1} x + C_2. \qquad \text{Table 8.1, Formula 18}$$

Combining these results and renaming $C_1 + C_2$ as C gives

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\,dx = -3\sqrt{1 - x^2} + 2\sin^{-1} x + C.$$

The question of what to substitute for in an integrand is not always quite so clear. Sometimes we simply proceed by trial-and-error, and if nothing works out, we then try another method altogether. The next several sections of the text present some of these new methods, but substitution works in the next example.

EXAMPLE 7 Evaluate

$$\int \frac{dx}{(1 + \sqrt{x})^3}.$$

Solution We might try substituting for the term $\sqrt{x}$, but we quickly realize the derivative factor $1/\sqrt{x}$ is missing from the integrand, so this substitution will not help. The other possibility is to substitute for $(1 + \sqrt{x})$, and it turns out this works:

$$\begin{aligned} \int \frac{dx}{(1 + \sqrt{x})^3} &= \int \frac{2(u - 1)\,du}{u^3} \qquad u = 1 + \sqrt{x},\ du = \frac{1}{2\sqrt{x}}\,dx;\ dx = 2\sqrt{x}\,du = 2(u - 1)\,du \\ &= \int \left(\frac{2}{u^2} - \frac{2}{u^3}\right) du \end{aligned}$$

$$= -\frac{2}{u} + \frac{1}{u^2} + C$$
$$= \frac{1 - 2u}{u^2} + C$$
$$= \frac{1 - 2(1 + \sqrt{x})}{(1 + \sqrt{x})^2} + C$$
$$= C - \frac{1 + 2\sqrt{x}}{(1 + \sqrt{x})^2}.$$

When evaluating definite integrals, a property of the integrand may help us in calculating the result.

EXAMPLE 8 Evaluate $\int_{-\pi/2}^{\pi/2} x^3 \cos x \, dx$.

Solution No substitution or algebraic manipulation is clearly helpful here. But we observe that the interval of integration is the symmetric interval $[-\pi/2, \pi/2]$. Moreover, the factor x^3 is an odd function, and $\cos x$ is an even function, so their product is odd. Therefore,

$$\int_{-\pi/2}^{\pi/2} x^3 \cos x \, dx = 0. \qquad \text{Theorem 8, Section 5.6}$$

Exercises 8.1

Assorted Integrations

The integrals in Exercises 1–40 are in no particular order. Evaluate each integral using any algebraic method or trigonometric identity you think is appropriate, and then use a substitution to reduce it to a standard form.

1. $\int_0^1 \frac{16x}{8x^2 + 2} dx$
2. $\int \frac{x^2}{x^2 + 1} dx$
3. $\int (\sec x - \tan x)^2 dx$
4. $\int_{\pi/4}^{\pi/3} \frac{dx}{\cos^2 x \tan x}$
5. $\int \frac{1 - x}{\sqrt{1 - x^2}} dx$
6. $\int \frac{dx}{x - \sqrt{x}}$
7. $\int \frac{e^{-\cot z}}{\sin^2 z} dz$
8. $\int \frac{2^{\ln z^3}}{16z} dz$
9. $\int \frac{dz}{e^z + e^{-z}}$
10. $\int_1^2 \frac{8\, dx}{x^2 - 2x + 2}$
11. $\int_{-1}^0 \frac{4\, dx}{1 + (2x + 1)^2}$
12. $\int_{-1}^3 \frac{4x^2 - 7}{2x + 3} dx$
13. $\int \frac{dt}{1 - \sec t}$
14. $\int \csc t \sin 3t \, dt$
15. $\int_0^{\pi/4} \frac{1 + \sin\theta}{\cos^2\theta} d\theta$
16. $\int \frac{d\theta}{\sqrt{2\theta - \theta^2}}$
17. $\int \frac{\ln y}{y + 4y \ln^2 y} dy$
18. $\int \frac{2^{\sqrt{y}}\, dy}{2\sqrt{y}}$
19. $\int \frac{d\theta}{\sec\theta + \tan\theta}$
20. $\int \frac{dt}{t\sqrt{3 + t^2}}$
21. $\int \frac{4t^3 - t^2 + 16t}{t^2 + 4} dt$
22. $\int \frac{x + 2\sqrt{x - 1}}{2x\sqrt{x - 1}} dx$
23. $\int_0^{\pi/2} \sqrt{1 - \cos\theta}\, d\theta$
24. $\int (\sec t + \cot t)^2 dt$
25. $\int \frac{dy}{\sqrt{e^{2y} - 1}}$
26. $\int \frac{6\, dy}{\sqrt{y}\,(1 + y)}$
27. $\int \frac{2\, dx}{x\sqrt{1 - 4\ln^2 x}}$
28. $\int \frac{dx}{(x - 2)\sqrt{x^2 - 4x + 3}}$
29. $\int (\csc x - \sec x)(\sin x + \cos x)\, dx$
30. $\int 3 \sinh\left(\frac{x}{2} + \ln 5\right) dx$
31. $\int_{\sqrt{2}}^3 \frac{2x^3}{x^2 - 1} dx$
32. $\int_{-1}^1 \sqrt{1 + x^2} \sin x \, dx$
33. $\int_{-1}^0 \sqrt{\frac{1 + y}{1 - y}} dy$
34. $\int e^{z + e^z} dz$

35. $\displaystyle\int \frac{7\,dx}{(x-1)\sqrt{x^2-2x-48}}$

36. $\displaystyle\int \frac{dx}{(2x+1)\sqrt{4x+4x^2}}$

37. $\displaystyle\int \frac{2\theta^3-7\theta^2+7\theta}{2\theta-5}\,d\theta$

38. $\displaystyle\int \frac{d\theta}{\cos\theta-1}$

39. $\displaystyle\int \frac{dx}{1+e^x}$

Hint: Use long division.

40. $\displaystyle\int \frac{\sqrt{x}}{1+x^3}\,dx$

Hint: Let $u = x^{3/2}$.

Theory and Examples

41. Area Find the area of the region bounded above by $y = 2\cos x$ and below by $y = \sec x$, $-\pi/4 \le x \le \pi/4$.

42. Volume Find the volume of the solid generated by revolving the region in Exercise 41 about the x-axis.

43. Arc length Find the length of the curve $y = \ln(\cos x)$, $0 \le x \le \pi/3$.

44. Arc length Find the length of the curve $y = \ln(\sec x)$, $0 \le x \le \pi/4$.

45. Centroid Find the centroid of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

46. Centroid Find the centroid of the region bounded by the x-axis, the curve $y = \csc x$, and the lines $x = \pi/6$, $x = 5\pi/6$.

47. The functions $y = e^{x^3}$ and $y = x^3e^{x^3}$ do not have elementary antiderivatives, but $y = (1+3x^3)e^{x^3}$ does. Evaluate

$$\int (1+3x^3)e^{x^3}\,dx.$$

48. Use the substitution $u = \tan x$ to evaluate the integral

$$\int \frac{dx}{1+\sin^2 x}.$$

49. Use the substitution $u = x^4 + 1$ to evaluate the integral

$$\int x^7\sqrt{x^4+1}\,dx.$$

50. Using different substitutions Show that the integral

$$\int ((x^2-1)(x+1))^{-2/3}dx$$

can be evaluated with any of the following substitutions.

a. $u = 1/(x+1)$

b. $u = ((x-1)/(x+1))^k$ for $k = 1, 1/2, 1/3, -1/3, -2/3,$ and -1

c. $u = \tan^{-1} x$

d. $u = \tan^{-1}\sqrt{x}$

e. $u = \tan^{-1}((x-1)/2)$

f. $u = \cos^{-1} x$

g. $u = \cosh^{-1} x$

What is the value of the integral?

8.2 Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\,dx.$$

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integrals

$$\int x\cos x\,dx \qquad \text{and} \qquad \int x^2e^x\,dx$$

are such integrals because $f(x) = x$ or $f(x) = x^2$ can be differentiated repeatedly to become zero, and $g(x) = \cos x$ or $g(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int \ln x\,dx \qquad \text{and} \qquad \int e^x\cos x\,dx.$$

In the first case, $f(x) = \ln x$ is easy to differentiate and $g(x) = 1$ easily integrates to x. In the second case, each part of the integrand appears again after repeated differentiation or integration.

Product Rule in Integral Form

If f and g are differentiable functions of x, the Product Rule says that

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$