Trigonometric Integrals

In this section we use trigonometric identities to integrate certain combinations of trigonometric functions. We start with powers of sine and cosine.

EXAMPLE 1 Evaluate $\int \cos^3 x \, dx$.

SOLUTION Simply substituting $u = \cos x$ isn't helpful, since then $du = -\sin x \, dx$. In order to integrate powers of cosine, we would need an extra $\sin x$ factor. Similarly, a power of sine would require an extra $\cos x$ factor. Thus, here we can separate one cosine factor and convert the remaining $\cos^2 x$ factor to an expression involving sine using the identity $\sin^2 x + \cos^2 x = 1$:

$$\cos^3 x = \cos^2 x \cdot \cos x = (1 - \sin^2 x) \cos x$$

We can then evaluate the integral by substituting $u = \sin x$, so $du = \cos x \, dx$ and

$$\int \cos^3 x \, dx = \int \cos^2 x \cdot \cos x \, dx = \int (1 - \sin^2 x) \cos x \, dx$$

$$= \int (1 - u^2) \, du = u - \tfrac{1}{3} u^3 + C$$

$$= \sin x - \tfrac{1}{3} \sin^3 x + C \qquad \blacksquare\blacksquare$$

In general, we try to write an integrand involving powers of sine and cosine in a form where we have only one sine factor (and the remainder of the expression in terms of cosine) or only one cosine factor (and the remainder of the expression in terms of sine). The identity $\sin^2 x + \cos^2 x = 1$ enables us to convert back and forth between even powers of sine and cosine.

EXAMPLE 2 Find $\int \sin^5 x \cos^2 x \, dx$

SOLUTION We could convert $\cos^2 x$ to $1 - \sin^2 x$, but we would be left with an expression in terms of $\sin x$ with no extra $\cos x$ factor. Instead, we separate a single sine factor and rewrite the remaining $\sin^4 x$ factor in terms of $\cos x$:

$$\sin^5 x \cos^2 x = (\sin^2 x)^2 \cos^2 x \sin x = (1 - \cos^2 x)^2 \cos^2 x \sin x$$

Substituting $u = \cos x$, we have $du = -\sin x \, dx$ and so

$$\int \sin^5 x \cos^2 x \, dx = \int (\sin^2 x)^2 \cos^2 x \sin x \, dx$$

$$= \int (1 - \cos^2 x)^2 \cos^2 x \sin x \, dx$$

$$= \int (1 - u^2)^2 u^2 (-du) = -\int (u^2 - 2u^4 + u^6) \, du$$

$$= -\left(\frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7} \right) + C$$

$$= -\tfrac{1}{3} \cos^3 x + \tfrac{2}{5} \cos^5 x - \tfrac{1}{7} \cos^7 x + C \qquad \blacksquare\blacksquare$$

■■ Figure 1 shows the graphs of the integrand $\sin^5 x \cos^2 x$ in Example 2 and its indefinite integral (with $C = 0$). Which is which?
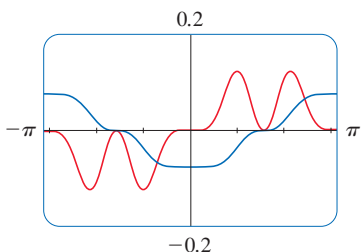

FIGURE 1

In the preceding examples, an odd power of sine or cosine enabled us to separate a single factor and convert the remaining even power. If the integrand contains even powers of both sine and cosine, this strategy fails. In this case, we can take advantage of the following half-angle identities (see Equations 17b and 17a in Appendix C):

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \text{and} \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

EXAMPLE 3 Evaluate $\int_0^\pi \sin^2 x \, dx$.

SOLUTION If we write $\sin^2 x = 1 - \cos^2 x$, the integral is no simpler to evaluate. Using the half-angle formula for $\sin^2 x$, however, we have

$$\int_0^\pi \sin^2 x \, dx = \tfrac{1}{2} \int_0^\pi (1 - \cos 2x) \, dx = \left[\tfrac{1}{2}\left(x - \tfrac{1}{2}\sin 2x\right) \right]_0^\pi$$

$$= \tfrac{1}{2}\left(\pi - \tfrac{1}{2}\sin 2\pi\right) - \tfrac{1}{2}\left(0 - \tfrac{1}{2}\sin 0\right) = \tfrac{1}{2}\pi$$

Notice that we mentally made the substitution $u = 2x$ when integrating $\cos 2x$. Another method for evaluating this integral was given in Exercise 33 in Section 5.6.

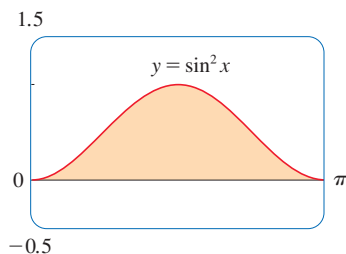■ ■ Example 3 shows that the area of the region shown in Figure 2 is $\pi/2$.

FIGURE 2

EXAMPLE 4 Find $\int \sin^4 x \, dx$.

SOLUTION We could evaluate this integral using the reduction formula for $\int \sin^n x \, dx$ (Equation 5.6.7) together with Example 3 (as in Exercise 33 in Section 5.6), but a better method is to write $\sin^4 x = (\sin^2 x)^2$ and use a half-angle formula:

$$\int \sin^4 x \, dx = \int (\sin^2 x)^2 \, dx$$

$$= \int \left(\frac{1 - \cos 2x}{2} \right)^2 dx$$

$$= \tfrac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) \, dx$$

Since $\cos^2 2x$ occurs, we must use another half-angle formula

$$\cos^2 2x = \tfrac{1}{2}(1 + \cos 4x)$$

This gives

$$\int \sin^4 x \, dx = \tfrac{1}{4} \int \left[1 - 2\cos 2x + \tfrac{1}{2}(1 + \cos 4x) \right] dx$$

$$= \tfrac{1}{4} \int \left(\tfrac{3}{2} - 2\cos 2x + \tfrac{1}{2}\cos 4x \right) dx$$

$$= \tfrac{1}{4}\left(\tfrac{3}{2}x - \sin 2x + \tfrac{1}{8}\sin 4x \right) + C$$

To summarize, we list guidelines to follow when evaluating integrals of the form $\int \sin^m x \cos^n x \, dx$, where $m \geqslant 0$ and $n \geqslant 0$ are integers.

Strategy for Evaluating $\int \sin^m x \, \cos^n x \, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \, \cos^{2k+1} x \, dx = \int \sin^m x \, (\cos^2 x)^k \cos x \, dx$$

$$= \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \, \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \, \sin x \, dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \, \sin x \, dx$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2} \sin 2x$$

We can use a similar strategy to evaluate integrals of the form $\int \tan^m x \, \sec^n x \, dx$. Since $(d/dx) \tan x = \sec^2 x$, we can separate a $\sec^2 x$ factor and convert the remaining (even) power of secant to an expression involving tangent using the identity $\sec^2 x = 1 + \tan^2 x$. Or, since $(d/dx) \sec x = \sec x \tan x$, we can separate a $\sec x \tan x$ factor and convert the remaining (even) power of tangent to secant.

EXAMPLE 5 Evaluate $\int \tan^6 x \, \sec^4 x \, dx$.

SOLUTION If we separate one $\sec^2 x$ factor, we can express the remaining $\sec^2 x$ factor in terms of tangent using the identity $\sec^2 x = 1 + \tan^2 x$. We can then evaluate the integral by substituting $u = \tan x$ with $du = \sec^2 x \, dx$:

$$\int \tan^6 x \, \sec^4 x \, dx = \int \tan^6 x \, \sec^2 x \, \sec^2 x \, dx$$

$$= \int \tan^6 x \, (1 + \tan^2 x) \, \sec^2 x \, dx$$

$$= \int u^6 (1 + u^2) \, du = \int (u^6 + u^8) \, du$$

$$= \frac{u^7}{7} + \frac{u^9}{9} + C$$

$$= \tfrac{1}{7} \tan^7 x + \tfrac{1}{9} \tan^9 x + C$$

EXAMPLE 6 Find $\int \tan^5\theta \, \sec^7\theta \, d\theta$.

SOLUTION If we separate a $\sec^2\theta$ factor, as in the preceding example, we are left with a $\sec^5\theta$ factor, which isn't easily converted to tangent. However, if we separate a $\sec \theta \, \tan \theta$ factor, we can convert the remaining power of tangent to an expression involving only secant using the identity $\tan^2\theta = \sec^2\theta - 1$. We can then evaluate the integral by substituting $u = \sec \theta$, so $du = \sec \theta \, \tan \theta \, d\theta$:

$$\int \tan^5\theta \, \sec^7\theta \, d\theta = \int \tan^4\theta \, \sec^6\theta \, \sec \theta \, \tan \theta \, d\theta$$

$$= \int (\sec^2\theta - 1)^2 \sec^6\theta \, \sec \theta \, \tan \theta \, d\theta$$

$$= \int (u^2 - 1)^2 u^6 \, du = \int (u^{10} - 2u^8 + u^6) \, du$$

$$= \frac{u^{11}}{11} - 2\frac{u^9}{9} + \frac{u^7}{7} + C$$

$$= \tfrac{1}{11} \sec^{11}\theta - \tfrac{2}{9} \sec^9\theta + \tfrac{1}{7} \sec^7\theta + C \qquad ■ ■$$

The preceding examples demonstrate strategies for evaluating integrals of the form $\int \tan^m x \, \sec^n x \, dx$ for two cases, which we summarize here.

Strategy for Evaluating $\int \tan^m x \, \sec^n x \, dx$

(a) If the power of secant is even ($n = 2k$, $k \geqslant 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \, \sec^{2k} x \, dx = \int \tan^m x \, (\sec^2 x)^{k-1} \sec^2 x \, dx$$

$$= \int \tan^m x \, (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \, \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \, \sec x \, \tan x \, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \, \sec x \, \tan x \, dx$$

Then substitute $u = \sec x$.

For other cases, the guidelines are not as clear-cut. We may need to use identities, integration by parts, and occasionally a little ingenuity. We will sometimes need to be able to integrate $\tan x$ by using the formula established in Example 5 in Section 5.5:

$$\int \tan x \, dx = \ln |\sec x| + C$$

We will also need the indefinite integral of secant:

1

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

We could verify Formula 1 by differentiating the right side, or as follows. First we multiply numerator and denominator by $\sec x + \tan x$:

$$\int \sec x \, dx = \int \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

If we substitute $u = \sec x + \tan x$, then $du = (\sec x \tan x + \sec^2 x) \, dx$, so the integral becomes $\int (1/u) \, du = \ln |u| + C$. Thus, we have

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

EXAMPLE 7 Find $\int \tan^3 x \, dx$.

SOLUTION Here only $\tan x$ occurs, so we use $\tan^2 x = \sec^2 x - 1$ to rewrite a $\tan^2 x$ factor in terms of $\sec^2 x$:

$$\int \tan^3 x \, dx = \int \tan x \tan^2 x \, dx$$

$$= \int \tan x \, (\sec^2 x - 1) \, dx$$

$$= \int \tan x \sec^2 x \, dx - \int \tan x \, dx$$

$$= \frac{\tan^2 x}{2} - \ln |\sec x| + C$$

In the first integral we mentally substituted $u = \tan x$ so that $du = \sec^2 x \, dx$. ■ ■

If an even power of tangent appears with an odd power of secant, it is helpful to express the integrand completely in terms of $\sec x$. Powers of $\sec x$ may require integration by parts, as shown in the following example.

EXAMPLE 8 Find $\int \sec^3 x \, dx$.

SOLUTION Here we integrate by parts with

$$u = \sec x \qquad\qquad dv = \sec^2 x \, dx$$
$$du = \sec x \tan x \, dx \qquad v = \tan x$$

Then

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$$= \sec x \tan x - \int \sec x \, (\sec^2 x - 1) \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

Using Formula 1 and solving for the required integral, we get

$$\int \sec^3 x \, dx = \tfrac{1}{2}\left(\sec x \tan x + \ln |\sec x + \tan x|\right) + C$$

■ ■

Integrals such as the one in the preceding example may seem very special but they occur frequently in applications of integration, as we will see in Chapter 6. Integrals of the form $\int \cot^m x \csc^n x \, dx$ can be found by similar methods because of the identity $1 + \cot^2 x = \csc^2 x$.

Finally, we can make use of another set of trigonometric identities:

■ ■ These product identities are discussed in Appendix C.

> **2** To evaluate the integrals (a) $\int \sin mx \cos nx \, dx$, (b) $\int \sin mx \sin nx \, dx$, or (c) $\int \cos mx \cos nx \, dx$, use the corresponding identity:
>
> (a) $\sin A \cos B = \tfrac{1}{2}[\sin(A - B) + \sin(A + B)]$
>
> (b) $\sin A \sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)]$
>
> (c) $\cos A \cos B = \tfrac{1}{2}[\cos(A - B) + \cos(A + B)]$

EXAMPLE 9 Evaluate $\int \sin 4x \cos 5x \, dx$.

SOLUTION This integral could be evaluated using integration by parts, but it's easier to use the identity in Equation 2(a) as follows:

$$\int \sin 4x \cos 5x \, dx = \int \tfrac{1}{2}[\sin(-x) + \sin 9x] \, dx$$

$$= \tfrac{1}{2}\int (-\sin x + \sin 9x) \, dx$$

$$= \tfrac{1}{2}\left(\cos x - \tfrac{1}{9}\cos 9x\right) + C$$

■ ■

Exercises

1–47 ■ Evaluate the integral.

1. $\int \sin^3 x \cos^2 x \, dx$

2. $\int \sin^6 x \cos^3 x \, dx$

3. $\int_{\pi/2}^{3\pi/4} \sin^5 x \cos^3 x \, dx$

4. $\int_0^{\pi/2} \cos^5 x \, dx$

5. $\int \cos^5 x \sin^4 x \, dx$

6. $\int \sin^3(mx) \, dx$

7. $\int_0^{\pi/2} \cos^2 \theta \, d\theta$

8. $\int_0^{\pi/2} \sin^2(2\theta) \, d\theta$

9. $\int_0^{\pi} \sin^4(3t) \, dt$

10. $\int_0^{\pi} \cos^6 \theta \, d\theta$

11. $\int (1 + \cos \theta)^2 \, d\theta$

12. $\int x \cos^2 x \, dx$

13. $\int_0^{\pi/4} \sin^4 x \cos^2 x \, dx$

14. $\int_0^{\pi/2} \sin^2 x \cos^2 x \, dx$

15. $\int \sin^3 x \sqrt{\cos x} \, dx$

16. $\int \cos \theta \cos^5(\sin \theta) \, d\theta$

17. $\int \cos^2 x \tan^3 x \, dx$

18. $\int \cot^5 \theta \sin^4 \theta \, d\theta$

19. $\int \dfrac{1 - \sin x}{\cos x} \, dx$

20. $\int \cos^2 x \sin 2x \, dx$

21. $\int \sec^2 x \tan x \, dx$

22. $\int_0^{\pi/2} \sec^4(t/2) \, dt$

23. $\int \tan^2 x \, dx$

24. $\int \tan^4 x \, dx$

25. $\int \sec^6 t \, dt$

26. $\int_0^{\pi/4} \sec^4 \theta \tan^4 \theta \, d\theta$

27. $\int_0^{\pi/3} \tan^5 x \sec^4 x \, dx$

28. $\int \tan^3(2x) \sec^5(2x) \, dx$

29. $\int \tan^3 x \sec x \, dx$

30. $\int_0^{\pi/3} \tan^5 x \sec^6 x \, dx$

31. $\int \tan^5 x \, dx$

32. $\int \tan^6(ay) \, dy$

33. $\displaystyle\int \frac{\tan^3\theta}{\cos^4\theta}\, d\theta$

34. $\displaystyle\int \tan^2 x \sec x \, dx$

35. $\displaystyle\int_{\pi/6}^{\pi/2} \cot^2 x \, dx$

36. $\displaystyle\int_{\pi/4}^{\pi/2} \cot^3 x \, dx$

37. $\displaystyle\int \cot^3\alpha \csc^3\alpha \, d\alpha$

38. $\displaystyle\int \csc^4 x \cot^6 x \, dx$

39. $\displaystyle\int \csc x \, dx$

40. $\displaystyle\int_{\pi/6}^{\pi/3} \csc^3 x \, dx$

41. $\displaystyle\int \sin 5x \sin 2x \, dx$

42. $\displaystyle\int \sin 3x \cos x \, dx$

43. $\displaystyle\int \cos 7\theta \cos 5\theta \, d\theta$

44. $\displaystyle\int \frac{\cos x + \sin x}{\sin 2x}\, dx$

45. $\displaystyle\int \frac{1 - \tan^2 x}{\sec^2 x}\, dx$

46. $\displaystyle\int \frac{dx}{\cos x - 1}$

47. $\displaystyle\int t \sec^2(t^2) \tan^4(t^2) \, dt$

48. If $\int_0^{\pi/4} \tan^6 x \sec x \, dx = I$, express the value of $\int_0^{\pi/4} \tan^8 x \sec x \, dx$ in terms of I.

49–52 ■ Evaluate the indefinite integral. Illustrate, and check that your answer is reasonable, by graphing both the integrand and its antiderivative (taking $C = 0$).

49. $\displaystyle\int \sin^5 x \, dx$

50. $\displaystyle\int \sin^4 x \cos^4 x \, dx$

51. $\displaystyle\int \sin 3x \sin 6x \, dx$

52. $\displaystyle\int \sec^4 \frac{x}{2}\, dx$

53. Find the average value of the function $f(x) = \sin^2 x \cos^3 x$ on the interval $[-\pi, \pi]$.

54. Evaluate $\int \sin x \cos x \, dx$ by four methods: (a) the substitution $u = \cos x$, (b) the substitution $u = \sin x$, (c) the identity $\sin 2x = 2 \sin x \cos x$, and (d) integration by parts. Explain the different appearances of the answers.

55–56 ■ Find the area of the region bounded by the given curves.

55. $y = \sin x, \quad y = \sin^3 x, \quad x = 0, \quad x = \pi/2$

56. $y = \sin x, \quad y = 2 \sin^2 x, \quad x = 0, \quad x = \pi/2$

57–58 ■ Use a graph of the integrand to guess the value of the integral. Then use the methods of this section to prove that your guess is correct.

57. $\displaystyle\int_0^{2\pi} \cos^3 x \, dx$

58. $\displaystyle\int_0^2 \sin 2\pi x \cos 5\pi x \, dx$

59–62 ■ Find the volume obtained by rotating the region bounded by the given curves about the specified axis.

59. $y = \sin x, \ x = \pi/2, \ x = \pi, \ y = 0;$ about the x-axis

60. $y = \tan^2 x, \ y = 0, \ x = 0, \ x = \pi/4;$ about the x-axis

61. $y = \cos x, \ y = 0, \ x = 0, \ x = \pi/2;$ about $y = -1$

62. $y = \cos x, \ y = 0, \ x = 0, \ x = \pi/2;$ about $y = 1$

63. A particle moves on a straight line with velocity function $v(t) = \sin \omega t \cos^2 \omega t$. Find its position function $s = f(t)$ if $f(0) = 0$.

64. Household electricity is supplied in the form of alternating current that varies from 155 V to -155 V with a frequency of 60 cycles per second (Hz). The voltage is thus given by the equation

$$E(t) = 155 \sin(120\pi t)$$

where t is the time in seconds. Voltmeters read the RMS (root-mean-square) voltage, which is the square root of the average value of $[E(t)]^2$ over one cycle.
(a) Calculate the RMS voltage of household current.
(b) Many electric stoves require an RMS voltage of 220 V. Find the corresponding amplitude A needed for the voltage $E(t) = A \sin(120\pi t)$.

65–67 ■ Prove the formula, where m and n are positive integers.

65. $\displaystyle\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$

66. $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$

67. $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$

68. A *finite Fourier series* is given by the sum

$$f(x) = \sum_{n=1}^{N} a_n \sin nx$$
$$= a_1 \sin x + a_2 \sin 2x + \cdots + a_N \sin Nx$$

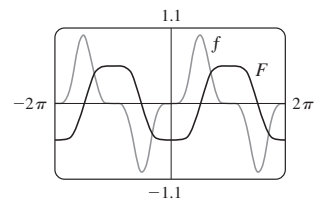Show that the mth coefficient a_m is given by the formula

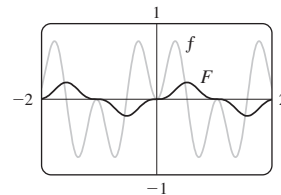$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx$$

Answers

1. $\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$ **3.** $-\frac{11}{384}$

5. $\frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$ **7.** $\pi/4$ **9.** $3\pi/8$

11. $\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta + C$ **13.** $(3\pi - 4)/192$

15. $\left(\frac{2}{7}\cos^3 x - \frac{2}{3}\cos x\right)\sqrt{\cos x} + C$

17. $\frac{1}{2}\cos^2 x - \ln|\cos x| + C$ **19.** $\ln(1 + \sin x) + C$

21. $\frac{1}{2}\tan^2 x + C$ **23.** $\tan x - x + C$

25. $\frac{1}{5}\tan^5 t + \frac{2}{3}\tan^3 t + \tan t + C$ **27.** $\frac{117}{8}$

29. $\frac{1}{3}\sec^3 x - \sec x + C$

31. $\frac{1}{4}\sec^4 x - \tan^2 x + \ln|\sec x| + C$

33. $\frac{1}{6}\tan^6\theta + \frac{1}{4}\tan^4\theta + C$ **35.** $\sqrt{3} - (\pi/3)$

37. $\frac{1}{3}\csc^3\alpha - \frac{1}{5}\csc^5\alpha + C$ **39.** $\ln|\csc x - \cot x| + C$

41. $\frac{1}{6}\sin 3x - \frac{1}{14}\sin 7x + C$ **43.** $\frac{1}{4}\sin 2\theta + \frac{1}{24}\sin 12\theta + C$

45. $\frac{1}{2}\sin 2x + C$ **47.** $\frac{1}{10}\tan^5(t^2) + C$

49. $-\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + C$

51. $\frac{1}{6}\sin 3x - \frac{1}{18}\sin 9x + C$

53. 0 **55.** $\frac{1}{3}$ **57.** 0 **59.** $\pi^2/4$ **61.** $2\pi + (\pi^2/4)$

63. $s = (1 - \cos^3\omega t)/(3\omega)$

Solutions: Trigonometric Integrals

The symbols $\overset{s}{=}$ and $\overset{c}{=}$ indicate the use of the substitutions $\{u = \sin x, du = \cos x\, dx\}$ and $\{u = \cos x, du = -\sin x\, dx\}$, respectively.

1. $\int \sin^3 x \, \cos^2 x \, dx = \int \sin^2 x \, \cos^2 x \, \sin x \, dx = \int \left(1 - \cos^2 x\right)\cos^2 x \, \sin x \, dx \overset{c}{=} \int \left(1 - u^2\right)u^2(-du)$

$\qquad = \int \left(u^2 - 1\right)u^2 \, du = \int \left(u^4 - u^2\right) du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$

3. $\int_{\pi/2}^{3\pi/4} \sin^5 x \, \cos^3 x \, dx = \int_{\pi/2}^{3\pi/4} \sin^5 x \, \cos^2 x \, \cos x \, dx = \int_{\pi/2}^{3\pi/4} \sin^5 x \left(1 - \sin^2 x\right) \cos x \, dx$

$\qquad \overset{s}{=} \int_1^{\sqrt{2}/2} u^5 \left(1 - u^2\right) du = \int_1^{\sqrt{2}/2} \left(u^5 - u^7\right) du = \left[\frac{1}{6}u^6 - \frac{1}{8}u^8\right]_1^{\sqrt{2}/2}$

$\qquad = \left(\frac{1/8}{6} - \frac{1/16}{8}\right) - \left(\frac{1}{6} - \frac{1}{8}\right) = -\frac{11}{384}$

5. $\int \cos^5 x \, \sin^4 x \, dx = \int \cos^4 x \, \sin^4 x \, \cos x \, dx = \int \left(1 - \sin^2 x\right)^2 \sin^4 x \, \cos x \, dx \overset{s}{=} \int \left(1 - u^2\right)^2 u^4 \, du$

$\qquad = \int \left(1 - 2u^2 + u^4\right) u^4 \, du = \int \left(u^4 - 2u^6 + u^8\right) du = \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C$

$\qquad = \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$

7. $\int_0^{\pi/2} \cos^2 \theta \, d\theta = \int_0^{\pi/2} \frac{1}{2}(1 + \cos 2\theta) \, d\theta$ [half-angle identity]

$\qquad = \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{1}{2}\left[\left(\frac{\pi}{2} + 0\right) - (0 + 0)\right] = \frac{\pi}{4}$

9. $\int_0^{\pi} \sin^4(3t) \, dt = \int_0^{\pi} \left[\sin^2(3t)\right]^2 dt = \int_0^{\pi} \left[\frac{1}{2}(1 - \cos 6t)\right]^2 dt = \frac{1}{4}\int_0^{\pi}\left(1 - 2\cos 6t + \cos^2 6t\right) dt$

$\qquad = \frac{1}{4}\int_0^{\pi} \left[1 - 2\cos 6t + \frac{1}{2}(1 + \cos 12t)\right] dt = \frac{1}{4}\int_0^{\pi}\left(\frac{3}{2} - 2\cos 6t + \frac{1}{2}\cos 12t\right) dt$

$\qquad = \frac{1}{4}\left[\frac{3}{2}t - \frac{1}{3}\sin 6t + \frac{1}{24}\sin 12t\right]_0^{\pi} = \frac{1}{4}\left[\left(\frac{3\pi}{2} - 0 + 0\right) - (0 - 0 + 0)\right] = \frac{3\pi}{8}$

11. $\int (1 + \cos\theta)^2 \, d\theta = \int (1 + 2\cos\theta + \cos^2\theta) \, d\theta = \theta + 2\sin\theta + \frac{1}{2}\int (1 + \cos 2\theta) \, d\theta$

$\qquad = \theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C = \frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta + C$

13. $\int_0^{\pi/4} \sin^4 x \, \cos^2 x \, dx = \int_0^{\pi/4} \sin^2 x \, (\sin x \cos x)^2 \, dx = \int_0^{\pi/4} \frac{1}{2}(1 - \cos 2x)\left(\frac{1}{2}\sin 2x\right)^2 dx$

$\qquad = \frac{1}{8}\int_0^{\pi/4}(1 - \cos 2x) \sin^2 2x \, dx = \frac{1}{8}\int_0^{\pi/4} \sin^2 2x \, dx - \frac{1}{8}\int_0^{\pi/4} \sin^2 2x \cos 2x \, dx$

$\qquad = \frac{1}{16}\int_0^{\pi/4}(1 - \cos 4x) \, dx - \frac{1}{16}\left[\frac{1}{3}\sin^3 2x\right]_0^{\pi/4} = \frac{1}{16}\left[x - \frac{1}{4}\sin 4x - \frac{1}{3}\sin^3 2x\right]_0^{\pi/4}$

$\qquad = \frac{1}{16}\left(\frac{\pi}{4} - 0 - \frac{1}{3}\right) = \frac{1}{192}(3\pi - 4)$

15. $\int \sin^3 x \, \sqrt{\cos x} \, dx = \int \left(1 - \cos^2 x\right) \sqrt{\cos x} \, \sin x \, dx \overset{c}{=} \int \left(1 - u^2\right) u^{1/2} \, (-du) = \int \left(u^{5/2} - u^{1/2}\right) du$

$\qquad = \frac{2}{7}u^{7/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{7}(\cos x)^{7/2} - \frac{2}{3}(\cos x)^{3/2} + C$

$\qquad = \left(\frac{2}{7}\cos^3 x - \frac{2}{3}\cos x\right)\sqrt{\cos x} + C$

17. $\int \cos^2 x \tan^3 x \, dx = \int \frac{\sin^3 x}{\cos x} \, dx \overset{c}{=} \int \frac{\left(1 - u^2\right)(-du)}{u} = \int \left[\frac{-1}{u} + u\right] du$

$\qquad = -\ln|u| + \frac{1}{2}u^2 + C = \frac{1}{2}\cos^2 x - \ln|\cos x| + C$

19. $\int \dfrac{1 - \sin x}{\cos x}\, dx = \int (\sec x - \tan x)\, dx = \ln|\sec x + \tan x| - \ln|\sec x| + C$ $\begin{bmatrix} \text{by (1) and the boxed} \\ \text{formula above it} \end{bmatrix}$

$\qquad\qquad = \ln|(\sec x + \tan x)\cos x| + C = \ln|1 + \sin x| + C$

$\qquad\qquad = \ln(1 + \sin x) + C \quad \text{since } 1 + \sin x \geq 0$

$\quad$ *Or:* $\int \dfrac{1 - \sin x}{\cos x}\, dx = \int \dfrac{1 - \sin x}{\cos x} \cdot \dfrac{1 + \sin x}{1 + \sin x}\, dx = \int \dfrac{(1 - \sin^2 x)\, dx}{\cos x\,(1 + \sin x)} = \int \dfrac{\cos x\, dx}{1 + \sin x}$

$\qquad\qquad = \int \dfrac{dw}{w} \qquad [\text{where } w = 1 + \sin x,\ dw = \cos x\, dx]$

$\qquad\qquad = \ln|w| + C = \ln|1 + \sin x| + C = \ln(1 + \sin x) + C$

21. Let $u = \tan x$, $du = \sec^2 x\, dx$. Then $\int \sec^2 x \tan x\, dx = \int u\, du = \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2 x + C$.

$\quad$ *Or:* Let $v = \sec x$, $dv = \sec x \tan x\, dx$. Then $\int \sec^2 x \tan x\, dx = \int v\, dv = \frac{1}{2}v^2 + C = \frac{1}{2}\sec^2 x + C$.

23. $\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \tan x - x + C$

25. $\int \sec^6 t\, dt = \int \sec^4 t \cdot \sec^2 t\, dt = \int (\tan^2 t + 1)^2 \sec^2 t\, dt = \int (u^2 + 1)^2\, du \qquad [u = \tan t,\ du = \sec^2 t\, dt]$

$\qquad\qquad = \int (u^4 + 2u^2 + 1)\, du = \frac{1}{5}u^5 + \frac{2}{3}u^3 + u + C = \frac{1}{5}\tan^5 t + \frac{2}{3}\tan^3 t + \tan t + C$

27. $\int_0^{\pi/3} \tan^5 x \sec^4 x\, dx = \int_0^{\pi/3} \tan^5 x\,(\tan^2 x + 1)\sec^2 x\, dx$

$\qquad\qquad = \int_0^{\sqrt{3}} u^5 (u^2 + 1)\, du \qquad [u = \tan x,\ du = \sec^2 x\, dx]$

$\qquad\qquad = \int_0^{\sqrt{3}} (u^7 + u^5)\, du = \left[\frac{1}{8}u^8 + \frac{1}{6}u^6\right]_0^{\sqrt{3}} = \frac{81}{8} + \frac{27}{6} = \frac{81}{8} + \frac{9}{2} = \frac{81}{8} + \frac{36}{8} = \frac{117}{8}$

$\quad$ *Alternate solution:*

$\quad \int_0^{\pi/3} \tan^5 x \sec^4 x\, dx = \int_0^{\pi/3} \tan^4 x \sec^3 x \sec x \tan x\, dx = \int_0^{\pi/3} (\sec^2 x - 1)^2 \sec^3 x \sec x \tan x\, dx$

$\qquad\qquad = \int_1^2 (u^2 - 1)^2 u^3\, du \qquad [u = \sec x,\ du = \sec x \tan x\, dx]$

$\qquad\qquad = \int_1^2 (u^4 - 2u^2 + 1)u^3\, du = \int_1^2 (u^7 - 2u^5 + u^3)\, du$

$\qquad\qquad = \left[\frac{1}{8}u^8 - \frac{1}{3}u^6 + \frac{1}{4}u^4\right]_1^2 = \left(32 - \frac{64}{3} + 4\right) - \left(\frac{1}{8} - \frac{1}{3} + \frac{1}{4}\right) = \frac{117}{8}$

29. $\int \tan^3 x \sec x\, dx = \int \tan^2 x \sec x \tan x\, dx = \int (\sec^2 x - 1)\sec x \tan x\, dx$

$\qquad\qquad = \int (u^2 - 1)\, du \qquad [u = \sec x,\ du = \sec x \tan x\, dx]$

$\qquad\qquad = \frac{1}{3}u^3 - u + C = \frac{1}{3}\sec^3 x - \sec x + C$

31. $\int \tan^5 x\, dx = \int (\sec^2 x - 1)^2 \tan x\, dx = \int \sec^4 x \tan x\, dx - 2\int \sec^2 x \tan x\, dx + \int \tan x\, dx$

$\qquad\qquad = \int \sec^3 x \sec x \tan x\, dx - 2\int \tan x \sec^2 x\, dx + \int \tan x\, dx$

$\qquad\qquad = \frac{1}{4}\sec^4 x - \tan^2 x + \ln|\sec x| + C \quad [\text{or } \frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C\,]$

33. $\int \dfrac{\tan^3 \theta}{\cos^4 \theta}\, d\theta = \int \tan^3 \theta \sec^4 \theta\, d\theta = \int \tan^3 \theta \cdot (\tan^2 \theta + 1) \cdot \sec^2 \theta\, d\theta$

$\qquad\qquad = \int u^3 (u^2 + 1)\, du \qquad [u = \tan \theta,\ du = \sec^2 \theta\, d\theta]$

$\qquad\qquad = \int (u^5 + u^3)\, du = \frac{1}{6}u^6 + \frac{1}{4}u^4 + C = \frac{1}{6}\tan^6 \theta + \frac{1}{4}\tan^4 \theta + C$

35. $\int_{\pi/6}^{\pi/2} \cot^2 x\, dx = \int_{\pi/6}^{\pi/2} (\csc^2 x - 1)\, dx = [-\cot x - x]_{\pi/6}^{\pi/2} = \left(0 - \frac{\pi}{2}\right) - \left(-\sqrt{3} - \frac{\pi}{6}\right) = \sqrt{3} - \frac{\pi}{3}$

37. $\int \cot^3 \alpha \csc^3 \alpha\, d\alpha = \int \cot^2 \alpha \csc^2 \alpha \cdot \csc \alpha \cot \alpha\, d\alpha = \int (\csc^2 \alpha - 1)\csc^2 \alpha \cdot \csc \alpha \cot \alpha\, d\alpha$

$\qquad\qquad = \int (u^2 - 1)u^2 \cdot (-du) \qquad [u = \csc \alpha,\ du = -\csc \alpha \cot \alpha\, d\alpha]$

$\qquad\qquad = \int (u^2 - u^4)\, du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C = \frac{1}{3}\csc^3 \alpha - \frac{1}{5}\csc^5 \alpha + C$

39. $I = \int \csc x \, dx = \int \dfrac{\csc x \, (\csc x - \cot x)}{\csc x - \cot x} \, dx = \int \dfrac{-\csc x \cot x + \csc^2 x}{\csc x - \cot x} \, dx$. Let $u = \csc x - \cot x \Rightarrow$

$du = (-\csc x \cot x + \csc^2 x) \, dx$. Then $I = \int du/u = \ln |u| = \ln |\csc x - \cot x| + C$.

41. Use Equation 2(b):

$\int \sin 5x \sin 2x \, dx = \int \frac{1}{2}[\cos(5x - 2x) - \cos(5x + 2x)] \, dx = \frac{1}{2} \int (\cos 3x - \cos 7x) \, dx$

$\qquad = \frac{1}{6} \sin 3x - \frac{1}{14} \sin 7x + C$

43. Use Equation 2(c):

$\int \cos 7\theta \cos 5\theta \, d\theta = \int \frac{1}{2}[\cos(7\theta - 5\theta) + \cos(7\theta + 5\theta)] \, d\theta = \frac{1}{2} \int (\cos 2\theta + \cos 12\theta) \, d\theta$

$\qquad = \frac{1}{2}\left(\frac{1}{2} \sin 2\theta + \frac{1}{12} \sin 12\theta\right) + C = \frac{1}{4} \sin 2\theta + \frac{1}{24} \sin 12\theta + C$

45. $\displaystyle\int \frac{1 - \tan^2 x}{\sec^2 x} \, dx = \int (\cos^2 x - \sin^2 x) \, dx = \int \cos 2x \, dx = \frac{1}{2} \sin 2x + C$
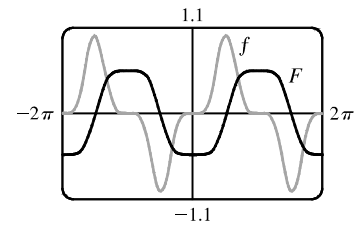
47. Let $u = \tan(t^2) \Rightarrow du = 2t \sec^2(t^2) \, dt$. Then

$\int t \sec^2(t^2) \tan^4(t^2) \, dt = \int u^4 \left(\frac{1}{2} du\right) = \frac{1}{10} u^5 + C = \frac{1}{10} \tan^5(t^2) + C.$

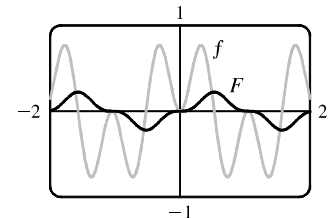**49.** Let $u = \cos x \Rightarrow du = -\sin x \, dx$. Then

$\int \sin^5 x \, dx = \int (1 - \cos^2 x)^2 \sin x \, dx = \int (1 - u^2)^2 (-du)$

$\qquad = \int (-1 + 2u^2 - u^4) \, du = -\frac{1}{5} u^5 + \frac{2}{3} u^3 - u + C$

$\qquad = -\frac{1}{5} \cos^5 x + \frac{2}{3} \cos^3 x - \cos x + C$

Notice that F is increasing when $f(x) > 0$, so the graphs serve as a

check on our work.

51. $\int \sin 3x \sin 6x \, dx = \int \frac{1}{2}[\cos(3x - 6x) - \cos(3x + 6x)] \, dx$

$\qquad = \frac{1}{2} \int (\cos 3x - \cos 9x) \, dx$

$\qquad = \frac{1}{6} \sin 3x - \frac{1}{18} \sin 9x + C$
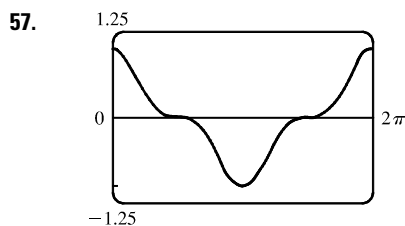
Notice that $f(x) = 0$ whenever F has a horizontal tangent.

53. $f_{\text{ave}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin^2 x \cos^3 x \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin^2 x \left(1 - \sin^2 x\right) \cos x \, dx$

$\qquad = \frac{1}{2\pi} \int_0^0 u^2 \left(1 - u^2\right) du \qquad$ [where $u = \sin x$]

$\qquad = 0$

55. For $0 < x < \frac{\pi}{2}$, we have $0 < \sin x < 1$, so $\sin^3 x < \sin x$. Hence the area is

$\int_0^{\pi/2} (\sin x - \sin^3 x) \, dx = \int_0^{\pi/2} \sin x \left(1 - \sin^2 x\right) dx = \int_0^{\pi/2} \cos^2 x \sin x \, dx$. Now let $u = \cos x \Rightarrow$

$du = -\sin x \, dx$. Then area $= \int_1^0 u^2 (-du) = \int_0^1 u^2 \, du = \left[\frac{1}{3} u^3\right]_0^1 = \frac{1}{3}.$

57. It seems from the graph that $\int_0^{2\pi} \cos^3 x \, dx = 0$, since the area below the

x-axis and above the graph looks about equal to the area above the axis

and below the graph. By Example 1, the integral is

$\left[\sin x - \frac{1}{3} \sin^3 x\right]_0^{2\pi} = 0$. Note that due to symmetry, the integral of

any odd power of $\sin x$ or $\cos x$ between limits which differ by $2n\pi$

(n any integer) is 0.

59. $V = \int_{\pi/2}^{\pi} \pi \sin^2 x \, dx = \pi \int_{\pi/2}^{\pi} \frac{1}{2}(1 - \cos 2x) \, dx = \pi \left[\frac{1}{2}x - \frac{1}{4}\sin 2x \right]_{\pi/2}^{\pi} = \pi \left(\frac{\pi}{2} - 0 - \frac{\pi}{4} + 0 \right) = \frac{\pi^2}{4}$

61. Volume $= \pi \int_0^{\pi/2} \left[(1 + \cos x)^2 - 1^2 \right] dx = \pi \int_0^{\pi/2} \left(2\cos x + \cos^2 x \right) dx$

$$= \pi \left[2\sin x + \frac{1}{2}x + \frac{1}{4}\sin 2x \right]_0^{\pi/2} = \pi \left(2 + \frac{\pi}{4} \right) = 2\pi + \frac{\pi^2}{4}$$

63. $s = f(t) = \int_0^t \sin \omega u \cos^2 \omega u \, du$. Let $y = \cos \omega u \quad \Rightarrow \quad dy = -\omega \sin \omega u \, du$. Then

$s = -\frac{1}{\omega} \int_1^{\cos \omega t} y^2 dy = -\frac{1}{\omega} \left[\frac{1}{3}y^3 \right]_1^{\cos \omega t} = \frac{1}{3\omega} \left(1 - \cos^3 \omega t \right)$.

65. Just note that the integrand is odd $[f(-x) = -f(x)]$.

Or: If $m \neq n$, calculate

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = \int_{-\pi}^{\pi} \frac{1}{2}[\sin(m - n)x + \sin(m + n)x] \, dx$$

$$= \frac{1}{2} \left[-\frac{\cos(m - n)x}{m - n} - \frac{\cos(m + n)x}{m + n} \right]_{-\pi}^{\pi} = 0$$

If $m = n$, then the first term in each set of brackets is zero.

67. $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \int_{-\pi}^{\pi} \frac{1}{2}[\cos(m - n)x + \cos(m + n)x] \, dx$. If $m \neq n$,

this is equal to $\frac{1}{2} \left[\frac{\sin(m - n)x}{m - n} + \frac{\sin(m + n)x}{m + n} \right]_{-\pi}^{\pi} = 0$. If $m = n$, we get

$\int_{-\pi}^{\pi} \frac{1}{2}[1 + \cos(m + n)x] \, dx = \left[\frac{1}{2}x \right]_{-\pi}^{\pi} + \left[\frac{\sin(m + n)x}{2(m + n)} \right]_{-\pi}^{\pi} = \pi + 0 = \pi$.