7.4 Integration by Partial Fractions

The method of partial fractions is used to integrate *rational functions.* That is, we want to compute

$$\int \frac{P(x)}{Q(x)}\,\mathrm{d}x \quad \text{where } P, Q \text{ are polynomials.}$$

First reduce[1] the integrand to the form $S(x) + \frac{R(x)}{Q(x)}$ where $°R < °Q$.

Example Here we write the integrand as a polynomial plus a rational function $\frac{7}{x+2}$ whose denominator has higher degreee than its numerator. Thankfully, this expression can be easily integrated using logarithms.

$$\frac{x^2+3}{x+2} = \frac{x(x+2)-2x+3}{x+2} = x + \frac{-2(x+2)+4+3}{x+2} = x-2+\frac{7}{x+2}$$

$$\implies \int \frac{x^2+3}{x+2}\,\mathrm{d}x = \int x-2+\frac{7}{x+2}\,\mathrm{d}x = \frac{1}{2}x^2 - 2x + 7\ln|x+2| + c$$

What if $°Q \geq 2$?

If the denominator $Q(x)$ is quadratic or has higher degree, we need another trick:

Theorem. *Suppose that $°R < °Q$. Then the rational function $\frac{R(x)}{Q(x)}$ can be written as a sum of fractions of the form*

$$\frac{A}{(ax+b)^m} \qquad \frac{Ax+B}{(ax^2+bx+c)^n}$$

where A, B, a, b, c are constants and m, n are positive integers.

Expressions such as the above can all be integrated using either logarithms or trigonometric substitutions.

Example With a little experimenting, you should be convinced that

$$\frac{3x^2+2x+3}{x^3+x} = \frac{3}{x} + \frac{2}{1+x^2}$$

It follows that

$$\int \frac{3x^2+2x+3}{x^3+x}\,\mathrm{d}x = 3\ln|x| + 2\tan^{-1}x + c$$

The burning question is *how* to find the expressions in the Therorem. The approach depends on the form of the denominator $Q(x)$.

[1] By Long Division or some other Torture. . .

Case 1: Distinct Linear Factors

Suppose that our denominator can be factorized completely into *distinct* linear factors. That is

$$Q(x) = (x - a_1)(x - a_2) \cdots (x - a_n)$$

where the values $a_1, \ldots, a_n$ are all different.[2]

Theorem. *For such a Q, there exist constants $A_1, \ldots, A_n$ such that*

$$\frac{R(x)}{Q(x)} = \sum_{i=1}^{n} \frac{A_i}{x - a_i} = \frac{A_1}{x - a_1} + \cdots + \frac{A_n}{x - a_n} \qquad (*)$$

whence the integral can be easily computed term-by-term:

$$\int \frac{R(x)}{Q(x)}\,\mathrm{d}x = \sum_{i=1}^{n} \int \frac{A_i}{x - a_i}\,\mathrm{d}x = \sum_{i=1}^{n} A_i \ln|x - a_i| + c$$

We find the constants A_i by putting the right hand side of $(*)$ over the common denominator $Q(x)$

$$\frac{R(x)}{Q(x)} = \frac{R(x)}{(x - a_1)\cdots(x - a_n)} = \frac{A_1}{x - a_1} + \cdots + \frac{A_n}{x - a_n}$$

and comparing numerators.

Examples

1. According to the Theorem, there exist constants A, B such that

   $$\frac{x+8}{x^2 + x - 2} = \frac{x+8}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

   Summing the right hand side, we obtain

   $$\frac{x+8}{(x-1)(x+2)} = \frac{A(x+2) + B(x-1)}{(x-1)(x+2)}$$

   Since the denominators are equal, it follows that the numerators are equal:

   $$x + 8 = A(x+2) + B(x-1)$$

   This is a relationship between A, B which holds *for all*[3] x: every value of x gives a valid relationship between A and B. Evaluating at $x = 1$ and $x = -2$ gives two very simple expressions:

   $$\begin{aligned} x = 1: &\qquad 9 = 3A \implies A = 3 \\ x = -2: &\qquad 6 = -3B \implies B = -2 \end{aligned}$$

   Putting it all together, we have

   $$\begin{aligned} \int \frac{x+8}{x^2+x-2}\,\mathrm{d}x &= \int \frac{3}{x-1} - \frac{2}{x+2}\,\mathrm{d}x = 3\ln|x-1| - 2\ln|x+2| + c \\ &= \ln\frac{|x-1|^3}{|x+2|^2} + c \end{aligned}$$

[2] We assume for clarity that the leading term of $Q(x)$ is x^n (coefficient 1). If not, absorb it into the numerator!

[3] You might worry that it doesn't when $x = 1$ or $x = -2$ because of the denominator. The fact fact that polynomials are *continuous* combined with $x + 8 = A(x+2) + B(x-1)$ everywhere else guarantees that we have equality everywhere.

2. We know that there exist constants A, B, C such that

$$\frac{x^2+2}{x^3-x} = \frac{x^2+2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

Combining the right hand side yields

$$x^2+2 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

Now evaluate at $x = 0, \pm 1$:

$$\begin{aligned} x = 0: &\qquad 2 = -A \implies A = -1 \\ x = 1: &\qquad 3 = 2B \implies B = \frac{3}{2} \\ x = -1: &\qquad 3 = 2C \implies C = \frac{3}{2} \end{aligned}$$

It follows that

$$\begin{aligned} \int \frac{x^2+2}{x^3-x}\,\mathrm{d}x &= \int \frac{-2}{x} + \frac{3}{2(x-1)} + \frac{3}{2(x+1)}\,\mathrm{d}x \\ &= -2\ln|x| + \frac{3}{2}(\ln|x-1| + \ln|x+1|) + c \\ &= \ln\frac{|x^2-1|^{\frac{3}{2}}}{x^2} + c \end{aligned}$$

Case 2: Repeated Linear Factors

Suppose that when we factorize $Q(x)$ we obtain a repeated linear factor. That is, some term of the form $(x-a)^m$ where $m \geq 2$. In a partial fractions decomposition, such a factor produces m seperate contributions:

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_m}{(x-a)^m}$$

each of which can be integrated normally. One way to remember this is to count the constants: $(x-a)^m$ has degree m and must therefore correspond to m distinct terms.

Examples

1. $\frac{x-2}{x^2(x-1)}$ has a repeated factor of x in the denominator. The single factor of $x-1$ behaves exactly as in Case 1. We therefore have constants A, B, C such that

$$\frac{x-2}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

Combining the right hand side and cancelling the denominators yields[4]

$$x-2 = Ax(x-1) + B(x-1) + Cx^2 \qquad (\dagger)$$

[4] Be careful: think about what each term is missing compared to the common denominator.

There are only two nice places at which to evaluate this expression:

$$\begin{aligned} x = 0: &\quad -2 = -B \implies B = 2 \\ x = 1: &\quad -1 = C \end{aligned}$$

To obtain A we have choices. Either evaluate (†) at another value of x, or compare coefficients. For example, it is easy to see that the coefficient of x^2 on the right side of (†) is $A + C$. This is clearly zero, since ther is no x^2 term on the left. We might write this as

$$\text{coeff}(x^2): \quad 0 = A + C \implies A = -C = 1$$

Putting it all together, we have

$$\int \frac{x-2}{x^2(x-1)}\,dx = \int \frac{1}{x} + \frac{2}{x^2} - \frac{1}{x-1}\,dx = \ln\frac{|x|}{|x-1|} - \frac{2}{x} + c$$

2. Suppose we want to integrate $\frac{x^3+3x+1}{(x+1)^2(x-2)^2}$. We have two repeated factors, whence there exist constants A, B, C, D such that

$$\frac{x^3+3x+1}{(x+1)^2(x-2)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$$

Combining the right hand side and cancelling the denominators yields

$$x^3 + 3x + 1 = A(x+1)(x-2)^2 + B(x-2)^2 + C(x+1)^2(x-2) + D(x+1)^2$$

We evaluate at the two nice places then compare some coefficients and evaluate at $x = 0$:

$$\begin{aligned} x = 2: &\quad 15 = 9D \implies D = \frac{5}{3} \\ x = -1: &\quad -3 = 9B \implies B = -\frac{1}{3} \\ \text{coeff}(x^3): &\quad 1 = A + C \\ x = 0: &\quad 1 = 4A + 4B - 2C + D \implies 2A - C = \frac{1}{3} \end{aligned}$$

The last two equations can be solved to obtain $A = \frac{4}{9}$ and $C = \frac{5}{9}$. The final integral is then

$$\begin{aligned} \int \frac{x^3+3x+1}{(x+1)^2(x-2)^2}\,dx &= \int \frac{4}{9(x+1)} - \frac{1}{3(x+1)^2} + \frac{5}{9(x-2)} + \frac{5}{3(x-2)^2}\,dx \\ &= \frac{4}{9}\ln|x+1| + \frac{1}{3(x+1)} + \frac{5}{9}\ln|x-2| - \frac{5}{3(x-2)} + c \\ &= \frac{1}{9}\ln|x+1|^4\,|x-2|^5 + \frac{1}{3(x+1)} - \frac{5}{3(x-2)} + c \end{aligned}$$

Case 3: Quadratic Factors

Suppose that the denominator $Q(x)$ contains an *irreducible quadratic* term: a term of the form[5]

$$ax^2 + bx + c \quad \text{where} \quad b^2 - 4ac < 0$$

Each such factor generates a partial fraction of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

which can be integrated using logarithms and/or tangent substitutions.[6]

Example The rational function $\dfrac{x^2 - x + 2}{x^3 + 4x} = \dfrac{x^2 - x + 2}{x(x^2 + 4)}$ contains the irreduciuble quadratic $x^2 + 4$ in its denominator. We therefore know that there exist constants A, B, C such that

$$\frac{x^2 - x + 2}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Combining the right hand side and equating numerators yields

$$x^2 - x + 2 = A(x^2 + 4) + (Bx + C)x$$

which can be solved (try it!) to obtain

$$A = \frac{1}{2}, \quad B = \frac{1}{2}, \quad C = -1$$

It follows that

$$\begin{aligned}\int \frac{x^2 - x + 2}{x^3 + 4x}\,\mathrm{d}x &= \int \frac{1}{2x} + \frac{x - 2}{2(x^2 + 4)}\,\mathrm{d}x = \frac{1}{2}\ln|x| + \int \frac{x}{2(x^2 + 4)} - \frac{1}{x^2 + 4}\,\mathrm{d}x \\ &= \frac{1}{2}\ln|x| + \frac{1}{4}\ln(x^2 + 4) - \frac{1}{2}\tan^{-1}\frac{x}{2} + c\end{aligned}$$

We had to be a little creative with the quadratic term in order to find an anti-derivative.

Case 4: Repeated Quadratic Factors (very hard!)

If $Q(x)$ contains a repeated factor $(ax^2 + bx + c)^m$ where $ax^2 + bx + c$ is irreducible and $m \geq 2$, then each such expression yields the m terms

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}$$

Each term may be integrated similarly to Case 3: part by inspection, part by completing the square.

[5]Thus $ax^2 + bx + c$ cannot be factored (over $\mathbb{R}$) into linear terms.
[6]Warning: These examples are often very involved. Master Cases 1 and 2 first!

(Partial) Example To integrate $\frac{x^3+2x^2+4}{(x^2+2x+5)^2(x-3)^4(x-2)^2}$ we first seek a partial fraction decomposition:

$$\begin{aligned}\frac{x^3+2x^2+4}{(x^2+2x+5)^2(x-3)^4(x-2)^2} = \frac{Ax+B}{x^2+2x+5} + \frac{Cx+D}{(x^2+2x+5)^2} \\ + \frac{E}{x-3} + \frac{F}{(x-3)^2} + \frac{G}{(x-3)^3} + \frac{H}{(x-3)^4} \\ + \frac{I}{x-2} + \frac{J}{(x-2)^2}\end{aligned}$$

This is long and messy. The first two terms may be integrated by completing the square and substituting $u = x+1$

$$x^2+2x+5 = (x+1)^2+4 = u^2+1$$

The integral of these terms will then be a combination of expressions such as

$$\tan^{-1}\frac{u}{2}, \quad \ln(u^2+1), \quad (u^2+1)^{-1}$$

If you're interested in the solution, ask a computer to help: the mathematician in you should be comfortable believing that it could be done!

Rationalizing

A clever substitution can sometimes convert an irrational expression into a rational one, to which the partial fractions method may be applied.
For example, the substitution $u^3 = x-7$ $(\mathrm{d}x = 3u^2\,\mathrm{d}u)$ gives

$$\begin{aligned}\int \frac{\sqrt[3]{x-7}}{x+1}\,\mathrm{d}x &= \int \frac{3u^3}{u^3+8}\,\mathrm{d}u = \int 3 - \frac{24}{(u+2)(u^2-2u+4)}\,\mathrm{d}u \\ &= 3u + \ln\frac{u^2-2u+4}{(u+2)^2} - 2\sqrt{3}\tan^{-1}\frac{u-1}{\sqrt{3}} + c \qquad \text{(partial fractions in here)} \\ &= 3(x-7)^{1/3} + \ln\frac{(x-7)^{2/3}-2(x-7)^{1/3}+4}{((x-7)^{1/3}+2)^2} - 2\sqrt{3}\tan^{-1}\frac{(x-7)^{1/3}-1}{\sqrt{3}} + c\end{aligned}$$

A similar approach (substituting $u = \sqrt{x-2}$) rationalizes the integral

$$\int \frac{1}{(x-2)(x-2+\sqrt{x-2})}\,\mathrm{d}x = \int \frac{2\,\mathrm{d}u}{u^2(u+1)}$$

Suggested problems

1. Evaluate the integrals:

   (a) $\int \frac{8}{(x-2)(x+6)}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{x}{(x-6)(x+2)^2}\,\mathrm{d}x$

2. Evaluate the integrals:

   (a) $\displaystyle\int_1^2 \frac{8-x^2}{x(x^2+5x+8)}\,\mathrm{d}x$

   (b) $\displaystyle\int \frac{1}{y^4+3y^2+1}\,\mathrm{d}y$

3. Evaluate $\int \frac{\mathrm{d}x}{x^2-1}$ in two ways: using partial fractions and using a trigonometric substitution.[7] Reconcile your two answers.

[7] Look up the integral of $\csc\theta$ if you need to. . .