lowest such line. None of the points (n, a_n) lies above $y = L$, but some do lie above any lower line $y = L - \epsilon$, if ϵ is a positive number. The sequence converges to L because

a. $a_n \le L$ for *all* values of n, and

b. given any $\epsilon > 0$, there exists at least one integer N for which $a_N > L - \epsilon$.

The fact that $\{a_n\}$ is nondecreasing tells us further that

$$a_n \ge a_N > L - \epsilon \qquad \text{for all } n \ge N.$$

Thus, *all* the numbers a_n beyond the Nth number lie within ϵ of L. This is precisely the condition for L to be the limit of the sequence $\{a_n\}$.

The proof for nonincreasing sequences bounded from below is similar. ■

It is important to realize that Theorem 6 does not say that convergent sequences are monotonic. The sequence $\{(-1)^{n+1}/n\}$ converges and is bounded, but it is not monotonic since it alternates between positive and negative values as it tends toward zero. What the theorem does say is that a nondecreasing sequence converges when it is bounded from above, but it diverges to infinity otherwise.

Exercises 10.1

Finding Terms of a Sequence

Each of Exercises 1–6 gives a formula for the nth term a_n of a sequence $\{a_n\}$. Find the values of a_1, a_2, a_3, and a_4.

1. $a_n = \dfrac{1-n}{n^2}$

2. $a_n = \dfrac{1}{n!}$

3. $a_n = \dfrac{(-1)^{n+1}}{2n-1}$

4. $a_n = 2 + (-1)^n$

5. $a_n = \dfrac{2^n}{2^{n+1}}$

6. $a_n = \dfrac{2^n - 1}{2^n}$

Each of Exercises 7–12 gives the first term or two of a sequence along with a recursion formula for the remaining terms. Write out the first ten terms of the sequence.

7. $a_1 = 1, \quad a_{n+1} = a_n + (1/2^n)$

8. $a_1 = 1, \quad a_{n+1} = a_n/(n+1)$

9. $a_1 = 2, \quad a_{n+1} = (-1)^{n+1}a_n/2$

10. $a_1 = -2, \quad a_{n+1} = na_n/(n+1)$

11. $a_1 = a_2 = 1, \quad a_{n+2} = a_{n+1} + a_n$

12. $a_1 = 2, \quad a_2 = -1, \quad a_{n+2} = a_{n+1}/a_n$

Finding a Sequence's Formula

In Exercises 13–26, find a formula for the nth term of the sequence.

13. The sequence $1, -1, 1, -1, 1, \ldots$ — 1's with alternating signs

14. The sequence $-1, 1, -1, 1, -1, \ldots$ — 1's with alternating signs

15. The sequence $1, -4, 9, -16, 25, \ldots$ — Squares of the positive integers, with alternating signs

16. The sequence $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$ — Reciprocals of squares of the positive integers, with alternating signs

17. $\frac{1}{9}, \frac{2}{12}, \frac{2^2}{15}, \frac{2^3}{18}, \frac{2^4}{21}, \ldots$ — Powers of 2 divided by multiples of 3

18. $-\frac{3}{2}, -\frac{1}{6}, \frac{1}{12}, \frac{3}{20}, \frac{5}{30}, \ldots$ — Integers differing by 2 divided by products of consecutive integers

19. The sequence $0, 3, 8, 15, 24, \ldots$ — Squares of the positive integers diminished by 1

20. The sequence $-3, -2, -1, 0, 1, \ldots$ — Integers, beginning with -3

21. The sequence $1, 5, 9, 13, 17, \ldots$ — Every other odd positive integer

22. The sequence $2, 6, 10, 14, 18, \ldots$ — Every other even positive integer

23. $\frac{5}{1}, \frac{8}{2}, \frac{11}{6}, \frac{14}{24}, \frac{17}{120}, \ldots$ — Integers differing by 3 divided by factorials

24. $\frac{1}{25}, \frac{8}{125}, \frac{27}{625}, \frac{64}{3125}, \frac{125}{15{,}625}, \ldots$ — Cubes of positive integers divided by powers of 5

25. The sequence $1, 0, 1, 0, 1, \ldots$ — Alternating 1's and 0's

26. The sequence $0, 1, 1, 2, 2, 3, 3, 4, \ldots$ — Each positive integer repeated

Convergence and Divergence

Which of the sequences $\{a_n\}$ in Exercises 27–90 converge, and which diverge? Find the limit of each convergent sequence.

27. $a_n = 2 + (0.1)^n$

28. $a_n = \dfrac{n + (-1)^n}{n}$

29. $a_n = \dfrac{1-2n}{1+2n}$

30. $a_n = \dfrac{2n+1}{1 - 3\sqrt{n}}$

31. $a_n = \dfrac{1-5n^4}{n^4 + 8n^3}$

32. $a_n = \dfrac{n+3}{n^2 + 5n + 6}$

33. $a_n = \dfrac{n^2 - 2n + 1}{n - 1}$

34. $a_n = \dfrac{1 - n^3}{70 - 4n^2}$

35. $a_n = 1 + (-1)^n$

36. $a_n = (-1)^n\left(1 - \frac{1}{n}\right)$

37. $a_n = \left(\frac{n+1}{2n}\right)\left(1 - \frac{1}{n}\right)$

38. $a_n = \left(2 - \frac{1}{2^n}\right)\left(3 + \frac{1}{2^n}\right)$

39. $a_n = \frac{(-1)^{n+1}}{2n - 1}$

40. $a_n = \left(-\frac{1}{2}\right)^n$

41. $a_n = \sqrt{\frac{2n}{n+1}}$

42. $a_n = \frac{1}{(0.9)^n}$

43. $a_n = \sin\left(\frac{\pi}{2} + \frac{1}{n}\right)$

44. $a_n = n\pi\cos(n\pi)$

45. $a_n = \frac{\sin n}{n}$

46. $a_n = \frac{\sin^2 n}{2^n}$

47. $a_n = \frac{n}{2^n}$

48. $a_n = \frac{3^n}{n^3}$

49. $a_n = \frac{\ln(n+1)}{\sqrt{n}}$

50. $a_n = \frac{\ln n}{\ln 2n}$

51. $a_n = 8^{1/n}$

52. $a_n = (0.03)^{1/n}$

53. $a_n = \left(1 + \frac{7}{n}\right)^n$

54. $a_n = \left(1 - \frac{1}{n}\right)^n$

55. $a_n = \sqrt[n]{10n}$

56. $a_n = \sqrt[n]{n^2}$

57. $a_n = \left(\frac{3}{n}\right)^{1/n}$

58. $a_n = (n+4)^{1/(n+4)}$

59. $a_n = \frac{\ln n}{n^{1/n}}$

60. $a_n = \ln n - \ln(n+1)$

61. $a_n = \sqrt[n]{4^n n}$

62. $a_n = \sqrt[n]{3^{2n+1}}$

63. $a_n = \frac{n!}{n^n}$ (*Hint:* Compare with $1/n$.)

64. $a_n = \frac{(-4)^n}{n!}$

65. $a_n = \frac{n!}{10^{6n}}$

66. $a_n = \frac{n!}{2^n \cdot 3^n}$

67. $a_n = \left(\frac{1}{n}\right)^{1/(\ln n)}$

68. $a_n = \ln\left(1 + \frac{1}{n}\right)^n$

69. $a_n = \left(\frac{3n+1}{3n-1}\right)^n$

70. $a_n = \left(\frac{n}{n+1}\right)^n$

71. $a_n = \left(\frac{x^n}{2n+1}\right)^{1/n}, \quad x > 0$

72. $a_n = \left(1 - \frac{1}{n^2}\right)^n$

73. $a_n = \frac{3^n \cdot 6^n}{2^{-n} \cdot n!}$

74. $a_n = \frac{(10/11)^n}{(9/10)^n + (11/12)^n}$

75. $a_n = \tanh n$

76. $a_n = \sinh(\ln n)$

77. $a_n = \frac{n^2}{2n-1}\sin\frac{1}{n}$

78. $a_n = n\left(1 - \cos\frac{1}{n}\right)$

79. $a_n = \sqrt{n}\sin\frac{1}{\sqrt{n}}$

80. $a_n = (3^n + 5^n)^{1/n}$

81. $a_n = \tan^{-1} n$

82. $a_n = \frac{1}{\sqrt{n}}\tan^{-1} n$

83. $a_n = \left(\frac{1}{3}\right)^n + \frac{1}{\sqrt{2^n}}$

84. $a_n = \sqrt[n]{n^2 + n}$

85. $a_n = \frac{(\ln n)^{200}}{n}$

86. $a_n = \frac{(\ln n)^5}{\sqrt{n}}$

87. $a_n = n - \sqrt{n^2 - n}$

88. $a_n = \frac{1}{\sqrt{n^2-1} - \sqrt{n^2+n}}$

89. $a_n = \frac{1}{n}\int_1^n \frac{1}{x}\,dx$

90. $a_n = \int_1^n \frac{1}{x^p}\,dx, \quad p > 1$

Recursively Defined Sequences

In Exercises 91–98, assume that each sequence converges and find its limit.

91. $a_1 = 2, \quad a_{n+1} = \frac{72}{1 + a_n}$

92. $a_1 = -1, \quad a_{n+1} = \frac{a_n + 6}{a_n + 2}$

93. $a_1 = -4, \quad a_{n+1} = \sqrt{8 + 2a_n}$

94. $a_1 = 0, \quad a_{n+1} = \sqrt{8 + 2a_n}$

95. $a_1 = 5, \quad a_{n+1} = \sqrt{5a_n}$

96. $a_1 = 3, \quad a_{n+1} = 12 - \sqrt{a_n}$

97. $2, 2 + \frac{1}{2}, 2 + \frac{1}{2 + \frac{1}{2}}, 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}, \ldots$

98. $\sqrt{1}, \sqrt{1 + \sqrt{1}}, \sqrt{1 + \sqrt{1 + \sqrt{1}}}, \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}, \ldots$

Theory and Examples

99. The first term of a sequence is $x_1 = 1$. Each succeeding term is the sum of all those that come before it:

$$x_{n+1} = x_1 + x_2 + \cdots + x_n.$$

Write out enough early terms of the sequence to deduce a general formula for x_n that holds for $n \geq 2$.

100. A sequence of rational numbers is described as follows:

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots, \frac{a}{b}, \frac{a+2b}{a+b}, \ldots.$$

Here the numerators form one sequence, the denominators form a second sequence, and their ratios form a third sequence. Let x_n and y_n be, respectively, the numerator and the denominator of the nth fraction $r_n = x_n/y_n$.

a. Verify that $x_1^2 - 2y_1^2 = -1$, $x_2^2 - 2y_2^2 = +1$ and, more generally, that if $a^2 - 2b^2 = -1$ or $+1$, then

$$(a + 2b)^2 - 2(a + b)^2 = +1 \quad \text{or} \quad -1,$$

respectively.

b. The fractions $r_n = x_n/y_n$ approach a limit as n increases. What is that limit? (*Hint:* Use part (a) to show that $r_n^2 - 2 = \pm(1/y_n)^2$ and that y_n is not less than n.)

101. **Newton's method** The following sequences come from the recursion formula for Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$