53. ***Income taxes*** The 2007 tax brackets for a single person claiming one personal exemption are given in the following table.

Taxable Income *I*	Tax Due *T*
\$0–\$7825	10% *I*
\$7826–\$31,850	\$782.50 + 15%(*I* − 7825)
\$31,851–\$77,100	\$4386.25 + 25%(*I* − 31,850)
\$77,101–\$160,850	\$15,698.75 + 28%(*I* − 77,100)
\$160,851–\$349,700	\$39,148.75 + 33%(*I* − 160,850)
Over \$349,700	\$101,469.25 + 35%(*I* − 349,700)

Source: Internal Revenue Service

(a) Write the last three taxable income ranges as inequalities.
(b) If an individual has a taxable income of \$31,850, calculate the tax due. Repeat this calculation for a taxable income of \$77,100.
(c) Write an interval that represents the amount of tax due for a taxable income between \$31,850 and \$77,100.

0.3 Integral Exponents

If \$1000 is placed in a 5-year savings certificate that pays an interest rate of 10% per year, compounded annually, then the amount returned after 5 years is given by

$$1000(1.1)^5$$

The 5 in this expression is an **exponent.** Exponents provide an easier way to denote certain multiplications. For example,

$$(1.1)^5 = (1.1)(1.1)(1.1)(1.1)(1.1)$$

An understanding of the properties of exponents is fundamental to the algebra needed to study functions and solve equations. Furthermore, the definition of exponential and logarithmic functions and many of the techniques in calculus also require an understanding of the properties of exponents.

For any real number a,

$$a^2 = a \cdot a, \quad a^3 = a \cdot a \cdot a, \quad \text{and} \quad a^n = a \cdot a \cdot a \cdot \cdots \cdot a \quad (n \text{ factors})$$

for any positive integer n. The positive integer n is called the **exponent,** the number a is called the **base,** and a^n is read "a to the nth power."

Note that $4a^n$ means $4(a^n)$, which is different from $(4a)^n$. The 4 is the coefficient of a^n in $4a^n$. Note also that $-x^n$ is not equivalent to $(-x)^n$ when n is even. For example, $-3^4 = -81$, but $(-3)^4 = 81$.

Some of the rules of exponents follow.

Positive Integer Exponents

For any real numbers a and b and positive integers m and n,

1. $a^m \cdot a^n = a^{m+n}$
2. For $a \neq 0$, $\dfrac{a^m}{a^n} = \begin{cases} a^{m-n} & \text{if } m > n \\ 1 & \text{if } m = n \\ 1/a^{n-m} & \text{if } m < n \end{cases}$
3. $(ab)^m = a^m b^m$
4. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m} \quad (b \neq 0)$
5. $(a^m)^n = a^{mn}$

● EXAMPLE 1 *Positive Integer Exponents*

Use properties of positive integer exponents to rewrite each of the following. Assume all denominators are nonzero.

(a) $\frac{5^6}{5^4}$ (b) $\frac{x^2}{x^5}$ (c) $\left(\frac{x}{y}\right)^4$ (d) $(3x^2y^3)^4$ (e) $3^3 \cdot 3^2$

Solution

(a) $\frac{5^6}{5^4} = 5^{6-4} = 5^2$ (b) $\frac{x^2}{x^5} = \frac{1}{x^{5-2}} = \frac{1}{x^3}$ (c) $\left(\frac{x}{y}\right)^4 = \frac{x^4}{y^4}$

(d) $(3x^2y^3)^4 = 3^4(x^2)^4(y^3)^4 = 81x^8y^{12}$ (e) $3^3 \cdot 3^2 = 3^{3+2} = 3^5$

For certain calculus operations, use of negative exponents is necessary in order to write problems in the proper form. We can extend the rules for positive integer exponents to all integers by defining a^0 and a^{-n}. Clearly $a^m \cdot a^0$ should equal $a^{m+0} = a^m$, and it will if $a^0 = 1$.

Zero Exponent

For any nonzero real number a, we define $a^0 = 1$. We leave 0^0 undefined.

In Section 0.2, we defined a^{-1} as $1/a$ for $a \neq 0$, so we define a^{-n} as $(a^{-1})^n$.

Negative Exponents

$$a^{-n} = (a^{-1})^n = \left(\frac{1}{a}\right)^n = \frac{1}{a^n} \quad (a \neq 0)$$

$$\left(\frac{a}{b}\right)^{-n} = \left[\left(\frac{a}{b}\right)^{-1}\right]^n = \left(\frac{b}{a}\right)^n \quad (a \neq 0, b \neq 0)$$

● EXAMPLE 2 *Negative and Zero Exponents*

Write each of the following without exponents.

(a) $6 \cdot 3^0$ (b) 6^{-2} (c) $\left(\frac{1}{3}\right)^{-1}$ (d) $-\left(\frac{2}{3}\right)^{-4}$ (e) $(-4)^{-2}$

Solution

(a) $6 \cdot 3^0 = 6 \cdot 1 = 6$ (b) $6^{-2} = \frac{1}{6^2} = \frac{1}{36}$

(c) $\left(\frac{1}{3}\right)^{-1} = \frac{3}{1} = 3$ (d) $-\left(\frac{2}{3}\right)^{-4} = -\left(\frac{3}{2}\right)^4 = \frac{-81}{16}$

(e) $(-4)^{-2} = \frac{1}{(-4)^2} = \frac{1}{16}$

As we'll see in the chapter on the mathematics of finance (Chapter 6), negative exponents arise in financial calculations when we have a future goal for an investment and want to know how much to invest now. For example, if money can be invested at 9%, compounded annually, then the amount we must invest now (which is called the present value) in order to have \$10,000 in the account after 7 years is given by $\$10,000(1.09)^{-7}$. Calculations such as this are often done directly with a calculator.

Using the definitions of zero and negative exponents enables us to extend the rules of exponents to all integers and to express them more simply.

Rules of Exponents

For real numbers a and b and *integers* m and n,

1. $a^m \cdot a^n = a^{m+n}$
2. $a^m/a^n = a^{m-n} \quad (a \neq 0)$
3. $(ab)^m = a^m b^m$
4. $(a^m)^n = a^{mn}$
5. $(a/b)^m = a^m/b^m \quad (b \neq 0)$
6. $a^0 = 1 \quad (a \neq 0)$
7. $a^{-n} = 1/a^n \quad (a \neq 0)$
8. $(a/b)^{-n} = (b/a)^n \quad (a, b \neq 0)$

Throughout the remainder of the text, we will assume all expressions are defined.

● EXAMPLE 3 *Operations with Exponents*

Use the rules of exponents and the definitions of a^0 and a^{-n} to simplify each of the following with positive exponents.

(a) $2(x^2)^{-2}$ (b) $x^{-2} \cdot x^{-5}$ (c) $\dfrac{x^{-8}}{x^{-4}}$ (d) $\left(\dfrac{2x^3}{3x^{-5}}\right)^{-2}$

Solution

(a) $2(x^2)^{-2} = 2x^{-4} = 2\left(\dfrac{1}{x^4}\right) = \dfrac{2}{x^4}$ (b) $x^{-2} \cdot x^{-5} = x^{-2-5} = x^{-7} = \dfrac{1}{x^7}$

(c) $\dfrac{x^{-8}}{x^{-4}} = x^{-8-(-4)} = x^{-4} = \dfrac{1}{x^4}$ (d) $\left(\dfrac{2x^3}{3x^{-5}}\right)^{-2} = \left(\dfrac{2x^8}{3}\right)^{-2} = \left(\dfrac{3}{2x^8}\right)^2 = \dfrac{9}{4x^{16}}$

● *Checkpoint*

1. Complete the following.

   (a) $x^3 \cdot x^8 = x^?$ (b) $x \cdot x^4 \cdot x^{-3} = x^?$ (c) $\dfrac{1}{x^4} = x^?$

   (d) $x^{24} \div x^{-3} = x^?$ (e) $(x^4)^2 = x^?$ (f) $(2x^4y)^3 = ?$

2. True or false:

   (a) $3x^{-2} = \dfrac{1}{9x^2}$ (b) $-x^{-4} = \dfrac{-1}{x^4}$ (c) $x^{-3} = -x^3$

3. Evaluate the following, if possible. For any that are meaningless, so state. Assume $x > 0$.

   (a) 0^4 (b) 0^0 (c) x^0 (d) 0^x (e) 0^{-4} (f) -5^{-2}

● EXAMPLE 4 *Rewriting a Quotient*

Write $(x^2y)/(9wz^3)$ with all factors in the numerator.

Solution

$$\frac{x^2y}{9wz^3} = x^2y\left(\frac{1}{9wz^3}\right) = x^2y\left(\frac{1}{9}\right)\left(\frac{1}{w}\right)\left(\frac{1}{z^3}\right) = x^2y \cdot 9^{-1}w^{-1}z^{-3}$$
$$= 9^{-1}x^2yw^{-1}z^{-3}$$

● EXAMPLE 5 *Rewriting with Positive Exponents*

Simplify the following so all exponents are positive.

(a) $(2^3x^{-4}y^5)^{-2}$ (b) $\dfrac{2x^4(x^2y)^0}{(4x^{-2}y)^2}$

Solution

(a) $(2^3x^{-4}y^5)^{-2} = 2^{-6}x^8y^{-10} = \frac{1}{2^6} \cdot x^8 \cdot \frac{1}{y^{10}} = \frac{x^8}{64y^{10}}$

(b) $\frac{2x^4(x^2y)^0}{(4x^{-2}y)^2} = \frac{2x^4 \cdot 1}{4^2x^{-4}y^2} = \frac{2}{4^2} \cdot \frac{x^4}{x^{-4}} \cdot \frac{1}{y^2} = \frac{2}{16} \cdot \frac{x^8}{1} \cdot \frac{1}{y^2} = \frac{x^8}{8y^2}$

Checkpoint Solutions

1. (a) $x^3 \cdot x^8 = x^{3+8} = x^{11}$ (b) $x \cdot x^4 \cdot x^{-3} = x^{1+4+(-3)} = x^2$
   (c) $\frac{1}{x^4} = x^{-4}$ (d) $x^{24} \div x^{-3} = x^{24-(-3)} = x^{27}$
   (e) $(x^4)^2 = x^{(4)(2)} = x^8$ (f) $(2x^4y)^3 = 2^3(x^4)^3y^3 = 8x^{12}y^3$
2. (a) False. $3x^{-2} = 3\left(\frac{1}{x^2}\right) = \frac{3}{x^2}$
   (b) True. $-x^{-4} = (-1)\left(\frac{1}{x^4}\right) = \frac{-1}{x^4}$
   (c) False. $x^{-3} = \frac{1}{x^3}$
3. (a) $0^4 = 0$ (b) Meaningless. 0^0 is undefined.
   (c) $x^0 = 1$ since $x \neq 0$ (d) $0^x = 0$ because $x > 0$
   (e) Meaningless. 0^{-4} would be $\frac{1}{0^4}$, which is not defined.
   (f) $-5^{-2} = (-1)\left(\frac{1}{5^2}\right) = \frac{-1}{25}$

0.3 Exercises

Evaluate in Problems 1–8. Write all answers without using exponents.

1. $(-4)^4$ 2. -5^3 3. -2^6 4. $(-2)^5$
5. 3^{-2} 6. 6^{-1} 7. $-\left(\frac{3}{2}\right)^2$ 8. $\left(\frac{2}{3}\right)^3$

In Problems 9–16, use rules of exponents to simplify the expressions. Express answers with positive exponents.

9. $6^5 \cdot 6^3$ 10. $\frac{7^8}{7^3}$ 11. $\frac{10^8}{10^9}$
12. $\frac{5^4}{(5^{-2} \cdot 5^3)}$ 13. $(3^3)^3$ 14. $(2^{-3})^{-2}$
15. $\left(\frac{2}{3}\right)^{-2}$ 16. $\left(\frac{-2}{5}\right)^{-4}$

In Problems 17–20, simplify by expressing answers with positive exponents ($x, y, z \neq 0$).

17. $(x^2)^{-3}$ 18. x^{-4} 19. $xy^{-2}z^0$ 20. $(xy^{-2})^0$

In Problems 21–34, use the rules of exponents to simplify so that only positive exponents remain.

21. $x^3 \cdot x^4$ 22. $a^5 \cdot a$ 23. $x^{-5} \cdot x^3$
24. $y^{-5} \cdot y^{-2}$ 25. $\frac{x^8}{x^4}$ 26. $\frac{a^5}{a^{-1}}$
27. $\frac{y^5}{y^{-7}}$ 28. $\frac{y^{-3}}{y^{-4}}$ 29. $(x^4)^3$ 30. $(y^3)^{-2}$
31. $(xy)^2$ 32. $(2m)^3$ 33. $\left(\frac{2}{x^5}\right)^4$ 34. $\left(\frac{8}{a^3}\right)^3$

In Problems 35–46, compute and simplify so that only positive exponents remain.

35. $(2x^{-2}y)^{-4}$ 36. $(-32x^5)^{-3}$
37. $(-8a^{-3}b^2)(2a^5b^{-4})$ 38. $(-3m^2y^{-1})(2m^{-3}y^{-1})$
39. $(2x^{-2}) \div (x^{-1}y^2)$ 40. $(-8a^{-3}b^2c) \div (2a^5b^4)$
41. $\left(\frac{x^3}{y^{-2}}\right)^{-3}$ 42. $\left(\frac{x^{-2}}{y}\right)^{-3}$
43. $\left(\frac{a^{-2}b^{-1}c^{-4}}{a^4b^{-3}c^0}\right)^{-3}$ 44. $\left(\frac{4x^{-1}y^{-40}}{2^{-2}x^4y^{-10}}\right)^{-2}$
45. (a) $\frac{2x^{-2}}{(2x)^2}$ (b) $\frac{(2x)^{-2}}{(2x)^2}$
    (c) $\frac{2x^{-2}}{2x^2}$ (d) $\frac{2x^{-2}}{(2x)^{-2}}$
46. (a) $\frac{2^{-1}x^{-2}}{(2x)^2}$ (b) $\frac{2^{-1}x^{-2}}{2x^2}$
    (c) $\frac{(2x^{-2})^{-1}}{(2x)^{-2}}$ (d) $\frac{(2x^{-2})^{-1}}{2x^2}$

In many applications it is often necessary to write expressions in the form cx^n, where c is a constant and n is an integer. In Problems 47–54, write the expressions in this form.

47. $\frac{1}{x}$ 48. $\frac{1}{x^2}$ 49. $(2x)^3$ 50. $(3x)^2$

51. $\frac{1}{4x^2}$ 52. $\frac{3}{2x^4}$ 53. $\left(\frac{-x}{2}\right)^3$ 54. $\left(\frac{-x}{3}\right)^2$

In Problems 55–58, use a calculator to evaluate the indicated powers.

55. 1.2^4 56. $(-3.7)^3$ 57. $(1.5)^{-5}$ 58. $(-0.8)^{-9}$

APPLICATIONS

Compound interest If $\$P$ is invested for n years at rate i (as a decimal), compounded annually, the future value that accrues is given by $S = P(1 + i)^n$, and the interest earned is $I = S - P$. In Problems 59–62, find S and I for the given P, n, and i.

59. \$1200 for 5 years at 12%
60. \$1800 for 7 years at 10%
61. \$5000 for 6 years at 11.5%
62. \$800 for 20 years at 10.5%

Present value If an investment has a goal (future value) of $\$S$ after n years, invested at interest rate i (as a decimal), compounded annually, then the present value P that must be invested is given by $P = S(1 + i)^{-n}$. In Problems 63 and 64, find P for the given S, n, and i.

63. \$15,000 after 6 years at 11.5%
64. \$80,000 after 20 years at 10.5%

65. ***Personal income*** For selected years from 1960 to 2005, billions of dollars of total U.S. personal income I can be approximated by the formula

$$I = 456.1(1.074)^t$$

where t is the number of years past 1960. (*Source:* U.S. Department of Commerce)

(a) What t-values correspond to the years 1970, 1990, and 2002?

(b) The actual total personal incomes (in billions of dollars) for the years in part (a) were as follows.

1970	1990	2002
838.8	4878.6	8881.9

What does the formula predict for these years?

(c) What does the formula predict for the total personal income in 2012?

(d) Does this formula seem to indicate that total personal income doubles almost every 10 years?

66. ***Stock shares traded*** On the New York Stock Exchange (NYSE) for 1970–2006, the average daily shares traded S (in millions of shares) can be approximated by the formula

$$S = 0.50274(1.1626)^t$$

where t is the number of years past 1950. (*Source:* New York Stock Exchange)

(a) What t-values correspond to the years 1990, 2000, and 2006?

(b) For the years in (a), the actual average millions of shares traded on the NYSE were as follows.

1990	2000	2006
156.777	1041.58	2343.16

What does the formula predict for these years?

(c) Suppose in 2015 that a stock market average (such as the Dow Jones Industrial Average) dramatically soared or tumbled; do you think this formula's predictions would be accurate, too low, or too high? Explain.

67. ***Endangered species*** The total number of endangered species y can be approximated by the formula

$$y = \frac{1883}{1 + 7.892(1.097)^{-t}}$$

where t is the number of years past 1980. (*Source:* U.S. Fish and Wildlife Service)

(a) The actual numbers of endangered species for selected years were as follows.

1990	2003	2007
442	987	1137

For each of these years, find the number of endangered species predicted by the formula. Round your answer to the nearest integer.

(b) How many more species does the formula estimate will be added to the endangered list for 2020 than the actual number given for 2007?

(c) Why do you think the answer to (b) is smaller than the number of species added from 1990 to 2003?

(d) Why is it reasonable for a formula such as this to have an upper limit that cannot be exceeded? Use large t-values in the formula to discover this formula's upper limit.

68. ***Internet users*** The percent p of U.S. households with Internet service can be approximated by the equation

$$p = \frac{73.92}{1 + 5.441(1.515)^{-t}}$$

where t is the number of years past 1995. (*Source:* U.S. Department of Commerce)

(a) The percents of U.S. households with Internet service for selected years were as follows.

2001	2004	2007
50.0%	68.8%	70.2%

For each of these years, use the equation to find the predicted percent of households with Internet service.

(b) In the three years from 2001 to 2004, the percent of households with Internet service increased by 18.8%. What percent increase does the equation predict from 2008 to 2011? Why do you think the 2008–2011 change is so different from the 2001–2004 change?

(c) Why is it reasonable for a formula such as this to have an upper limit that cannot be exceeded? Use large t-values in the formula to discover this formula's upper limit.

69. ***Health care expenditures*** The national health care expenditure H (in billions of dollars) can be modeled (that is, accurately approximated) by the formula

$$H = 30.58(1.102)^t$$

where t is the number of years past 1960. (*Source:* U.S. Department of Health and Human Services)

(a) What t-value corresponds to 1970?

(b) Approximate the national health care expenditure in 1970.

(c) Approximate the national health care expenditure in 2005.

(d) Estimate the national health care expenditure in 2015.

0.4 Radicals and Rational Exponents

Roots

A process closely linked to that of raising numbers to powers is that of extracting roots. From geometry we know that if an edge of a cube has a length of x units, its volume is x^3 cubic units. Reversing this process, we determine that if the volume of a cube is V cubic units, the length of an edge is the cube root of V, which is denoted

$$\sqrt[3]{V} \text{ units}$$

When we seek the **cube root** of a number such as 8 (written $\sqrt[3]{8}$), we are looking for a real number whose cube equals 8. Because $2^3 = 8$, we know that $\sqrt[3]{8} = 2$. Similarly, $\sqrt[3]{-27} = -3$ because $(-3)^3 = -27$. The expression $\sqrt[n]{a}$ is called a **radical,** where $\sqrt{\ }$ is the **radical sign,** n the **index,** and a the **radicand.** When no index is indicated, the index is assumed to be 2 and the expression is called a **square root;** thus $\sqrt{4}$ is the square root of 4 and represents the positive number whose square is 4.

Only one real number satisfies $\sqrt[n]{a}$ for a real number a and an odd number n; we call that number the **principal *n*th root** or, more simply, the ***n*th root.**

For an even index n, there are two possible cases:

1. If a is negative, there is no real number equal to $\sqrt[n]{a}$. For example, there are no real numbers that equal $\sqrt{-4}$ or $\sqrt[4]{-16}$ because there is no real number b such that $b^2 = -4$ or $b^4 = -16$. In this case, we say $\sqrt[n]{a}$ is not a real number.
2. If a is positive, there are two real numbers whose nth power equals a. For example, $3^2 = 9$ and $(-3)^2 = 9$. In order to have a unique nth root, we define the (principal) nth root, $\sqrt[n]{a}$, as the *positive* number b that satisfies $b^n = a$.

We summarize this discussion as follows.

nth Root of a

The **(principal) *n*th root** of a real number is defined as

$$\sqrt[n]{a} = b \quad \text{only if} \quad a = b^n$$

subject to the following conditions:

	$a = 0$	$a > 0$	$a < 0$
n even	$\sqrt[n]{a} = 0$	$\sqrt[n]{a} > 0$	$\sqrt[n]{a}$ not real
n odd	$\sqrt[n]{a} = 0$	$\sqrt[n]{a} > 0$	$\sqrt[n]{a} < 0$

When we are asked for the root of a number, we give the principal root.

EXAMPLE 1 *Roots*

Find the roots, if they are real numbers.

(a) $\sqrt[6]{64}$ (b) $-\sqrt{16}$ (c) $\sqrt[3]{-8}$ (d) $\sqrt{-16}$

Solution

(a) $\sqrt[6]{64} = 2$ because $2^6 = 64$ (b) $-\sqrt{16} = -(\sqrt{16}) = -4$ (c) $\sqrt[3]{-8} = -2$

(d) $\sqrt{-16}$ is not a real number because an even root of a negative number is not real.

Fractional Exponents

In order to perform evaluations on a calculator or to perform calculus operations, it is sometimes necessary to rewrite radicals in exponential form with fractional exponents.

We have stated that for $a \geq 0$ and $b \geq 0$,

$$\sqrt{a} = b \quad \text{only if} \quad a = b^2$$

This means that $(\sqrt{a})^2 = b^2 = a$, or $(\sqrt{a})^2 = a$. In order to extend the properties of exponents to rational exponents, it is necessary to define

$$a^{1/2} = \sqrt{a} \quad \text{so that} \quad (a^{1/2})^2 = a$$

Exponent $1/n$

For a positive integer n, we define

$$a^{1/n} = \sqrt[n]{a} \quad \text{if} \quad \sqrt[n]{a} \text{ exists}$$

Thus $(a^{1/n})^n = a^{(1/n)\cdot n} = a$.

Because we wish the properties established for integer exponents to extend to rational exponents, we make the following definitions.

Rational Exponents

For positive integer n and any integer m (with $a \neq 0$ when $m \leq 0$ and with m/n in lowest terms):

1. $a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m$
2. $a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}$ if a is nonnegative when n is even.

Throughout the remaining discussion, we assume all expressions are real.

EXAMPLE 2 *Radical Form*

Write the following in radical form and simplify.

(a) $16^{3/4}$ (b) $y^{-3/2}$ (c) $(6m)^{2/3}$

Solution

(a) $16^{3/4} = \sqrt[4]{16^3} = (\sqrt[4]{16})^3 = (2)^3 = 8$

(b) $y^{-3/2} = \dfrac{1}{y^{3/2}} = \dfrac{1}{\sqrt{y^3}}$

(c) $(6m)^{2/3} = \sqrt[3]{(6m)^2} = \sqrt[3]{36m^2}$

● EXAMPLE 3 *Fractional Exponents*

Write the following without radical signs.

(a) $\sqrt{x^3}$ (b) $\dfrac{1}{\sqrt[3]{b^2}}$ (c) $\sqrt[3]{(ab)^3}$

Solution

(a) $\sqrt{x^3} = x^{3/2}$ (b) $\dfrac{1}{\sqrt[3]{b^2}} = \dfrac{1}{b^{2/3}} = b^{-2/3}$ (c) $\sqrt[3]{(ab)^3} = (ab)^{3/3} = ab$

Our definition of $a^{m/n}$ guarantees that the rules for exponents will apply to fractional exponents. Thus we can perform operations with fractional exponents as we did with integer exponents.

● EXAMPLE 4 *Operations with Fractional Exponents*

Simplify the following expressions.

(a) $a^{1/2} \cdot a^{1/6}$ (b) $a^{3/4}/a^{1/3}$ (c) $(a^3b)^{2/3}$ (d) $(a^{3/2})^{1/2}$ (e) $a^{-1/2} \cdot a^{-3/2}$

Solution

(a) $a^{1/2} \cdot a^{1/6} = a^{1/2+1/6} = a^{3/6+1/6} = a^{4/6} = a^{2/3}$
(b) $a^{3/4}/a^{1/3} = a^{3/4-1/3} = a^{9/12-4/12} = a^{5/12}$
(c) $(a^3b)^{2/3} = (a^3)^{2/3}b^{2/3} = a^2b^{2/3}$
(d) $(a^{3/2})^{1/2} = a^{(3/2)(1/2)} = a^{3/4}$
(e) $a^{-1/2} \cdot a^{-3/2} = a^{-1/2-3/2} = a^{-2} = 1/a^2$

● ***Checkpoint***

1. Which of the following are *not* real numbers?
   (a) $\sqrt[3]{-64}$ (b) $\sqrt{-64}$ (c) $\sqrt{0}$ (d) $\sqrt[4]{1}$ (e) $\sqrt[5]{-1}$ (f) $\sqrt[8]{-1}$
2. (a) Write as radicals: $x^{1/3}, x^{2/5}, x^{-3/2}$
   (b) Write with fractional exponents: $\sqrt[4]{x^3} = x^?$, $\dfrac{1}{\sqrt{x}} = \dfrac{1}{x^?} = x^?$
3. Evaluate the following.
   (a) $8^{2/3}$ (b) $(-8)^{2/3}$ (c) $8^{-2/3}$ (d) $-8^{-2/3}$ (e) $\sqrt[15]{71}$
4. Complete the following.
   (a) $x \cdot x^{1/3} \cdot x^3 = x^?$ (b) $x^2 \div x^{1/2} = x^?$ (c) $(x^{-2/3})^{-3} = x^?$
   (d) $x^{-3/2} \cdot x^{1/2} = x^?$ (e) $x^{-3/2} \cdot x = x^?$ (f) $\left(\dfrac{x^4}{y^2}\right)^{3/2} = ?$
5. True or false:
   (a) $\dfrac{8x^{2/3}}{x^{-1/3}} = 4x$ (b) $(16x^8y)^{3/4} = 12x^6y^{3/4}$
   (c) $\left(\dfrac{x^2}{y^3}\right)^{-1/3} = \left(\dfrac{y^3}{x^2}\right)^{1/3} = \dfrac{y}{x^{2/3}}$

Operations with Radicals

We can perform operations with radicals by first rewriting in exponential form, performing the operations with exponents, and then converting the answer back to radical form. Another option is to apply directly the following rules for operations with radicals.

Rules for Radicals	**Examples**
Given that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real,*	
1. $\sqrt[n]{a^n} = (\sqrt[n]{a})^n = a$	1. $\sqrt[5]{6^5} = (\sqrt[5]{6})^5 = 6$
2. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$	2. $\sqrt[3]{2}\,\sqrt[3]{4} = \sqrt[3]{8} = \sqrt[3]{2^3} = 2$
3. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$ $(b \neq 0)$	3. $\dfrac{\sqrt{18}}{\sqrt{2}} = \sqrt{\dfrac{18}{2}} = \sqrt{9} = 3$

*Note that this means $a \geq 0$ and $b \geq 0$ if n is even.

Let us consider Rule 1 for radicals more carefully. Note that if n is even and $a < 0$, then $\sqrt[n]{a}$ is not real, and Rule 1 does not apply. For example, $\sqrt{-2}$ is not a real number, and

$$\sqrt{(-2)^2} \neq -2 \quad \text{because} \quad \sqrt{(-2)^2} = \sqrt{4} = 2 = -(-2)$$

We can generalize this observation as follows: If $a < 0$, then $\sqrt{a^2} = -a > 0$, so

$$\sqrt{a^2} = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

This means

$$\sqrt{a^2} = |a|$$

● EXAMPLE 5 *Simplifying Radicals*

Simplify:

(a) $\sqrt[3]{8^3}$ (b) $\sqrt[5]{x^5}$ (c) $\sqrt{x^2}$ (d) $\left[\sqrt[7]{(3x^2 + 4)^3}\right]^7$

Solution

(a) $\sqrt[3]{8^3} = 8$ by Rule 1 for radicals
(b) $\sqrt[5]{x^5} = x$ (c) $\sqrt{x^2} = |x|$ (d) $\left[\sqrt[7]{(3x^2 + 4)^3}\right]^7 = (3x^2 + 4)^3$

Up to now, to *simplify* a radical has meant to find the indicated root. More generally, a radical expression $\sqrt[n]{x}$ is considered simplified if x has no nth powers as factors. Rule 2 for radicals ($\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$) provides a procedure for simplifying radicals.

● EXAMPLE 6 *Simplifying Radicals*

Simplify the following radicals; assume the expressions are real numbers.

(a) $\sqrt{48x^5y^6}$ $(y \geq 0)$ (b) $\sqrt[3]{72a^3b^4}$

Solution

(a) To simplify $\sqrt{48x^5y^6}$, we first factor $48x^5y^6$ into perfect-square factors and other factors. Then we apply Rule 2.

$$\sqrt{48x^5y^6} = \sqrt{16 \cdot 3 \cdot x^4xy^6} = \sqrt{16}\sqrt{x^4}\sqrt{y^6}\sqrt{3x} = 4x^2y^3\sqrt{3x}$$

(b) In this case, we factor $72a^3b^4$ into factors that are perfect cubes and other factors.

$$\sqrt[3]{72a^3b^4} = \sqrt[3]{8 \cdot 9a^3b^3b} = \sqrt[3]{8} \cdot \sqrt[3]{a^3} \cdot \sqrt[3]{b^3} \cdot \sqrt[3]{9b} = 2ab\sqrt[3]{9b}$$

Rule 2 for radicals also provides a procedure for multiplying two roots with the same index.

● EXAMPLE 7 *Multiplying Radicals*

Multiply the following and simplify the answers, assuming nonnegative variables.

(a) $\sqrt[3]{2xy} \cdot \sqrt[3]{4x^2y}$ (b) $\sqrt{8xy^3z}\sqrt{4x^2y^3z^2}$

Solution

(a) $\sqrt[3]{2xy} \cdot \sqrt[3]{4x^2y} = \sqrt[3]{2xy \cdot 4x^2y} = \sqrt[3]{8x^3y^2} = \sqrt[3]{8} \cdot \sqrt[3]{x^3} \cdot \sqrt[3]{y^2} = 2x\sqrt[3]{y^2}$

(b) $\sqrt{8xy^3z}\sqrt{4x^2y^3z^2} = \sqrt{32x^3y^6z^3} = \sqrt{16x^2y^6z^2}\sqrt{2xz} = 4xy^3z\sqrt{2xz}$

Rule 3 for radicals ($\sqrt[n]{a}/\sqrt[n]{b} = \sqrt[n]{a/b}$) indicates how to find the quotient of two roots with the same index.

● EXAMPLE 8 *Dividing Radicals*

Find the quotients and simplify the answers, assuming nonnegative variables.

(a) $\dfrac{\sqrt[3]{32}}{\sqrt[3]{4}}$ (b) $\dfrac{\sqrt{16a^3x}}{\sqrt{2ax}}$

Solution

(a) $\dfrac{\sqrt[3]{32}}{\sqrt[3]{4}} = \sqrt[3]{\dfrac{32}{4}} = \sqrt[3]{8} = 2$ (b) $\dfrac{\sqrt{16a^3x}}{\sqrt{2ax}} = \sqrt{\dfrac{16a^3x}{2ax}} = \sqrt{8a^2} = 2a\sqrt{2}$

Rationalizing

Occasionally, we wish to express a fraction containing radicals in an equivalent form that contains no radicals in the denominator. This is accomplished by multiplying the numerator *and* the denominator by the expression that will remove the radical. This process is called **rationalizing the denominator.**

● EXAMPLE 9 *Rationalizing Denominators*

Express each of the following with no radicals in the denominator. (Rationalize each denominator.)

(a) $\dfrac{15}{\sqrt{x}}$ (b) $\dfrac{2x}{\sqrt{18xy}}$ $(x, y > 0)$ (c) $\dfrac{3x}{\sqrt[3]{2x^2}}$ $(x \neq 0)$

Solution

(a) We wish to create a perfect square under the radical in the denominator.

$$\frac{15}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{15\sqrt{x}}{x}$$

(b) $\dfrac{2x}{\sqrt{18xy}} \cdot \dfrac{\sqrt{2xy}}{\sqrt{2xy}} = \dfrac{2x\sqrt{2xy}}{\sqrt{36x^2y^2}} = \dfrac{2x\sqrt{2xy}}{6xy} = \dfrac{\sqrt{2xy}}{3y}$

(c) We wish to create a perfect cube under the radical in the denominator.

$$\frac{3x}{\sqrt[3]{2x^2}} \cdot \frac{\sqrt[3]{4x}}{\sqrt[3]{4x}} = \frac{3x\sqrt[3]{4x}}{\sqrt[3]{8x^3}} = \frac{3x\sqrt[3]{4x}}{2x} = \frac{3\sqrt[3]{4x}}{2}$$

● Checkpoint

6. Simplify:
   (a) $\sqrt[7]{x^7}$ (b) $\left[\sqrt[5]{(x^2+1)^2}\right]^5$ (c) $\sqrt{12xy^2} \cdot \sqrt{3x^2y}$
7. Rationalize the denominator of $\dfrac{x}{\sqrt{5x}}$ if $x \neq 0$.

It is also sometimes useful, especially in calculus, to *rationalize the numerator* of a fraction. For example, we can rationalize the numerator of

$$\frac{\sqrt[3]{4x^2}}{3x}$$

by multiplying the numerator and denominator by $\sqrt[3]{2x}$, which creates a perfect cube under the radical:

$$\frac{\sqrt[3]{4x^2}}{3x} \cdot \frac{\sqrt[3]{2x}}{\sqrt[3]{2x}} = \frac{\sqrt[3]{8x^3}}{3x\sqrt[3]{2x}} = \frac{2x}{3x\sqrt[3]{2x}} = \frac{2}{3\sqrt[3]{2x}}$$

● Checkpoint Solutions

1. Only *even* roots of negatives are not real numbers. Thus $\sqrt{-64}$ and $\sqrt[8]{-1}$ are not real numbers.
2. (a) $x^{1/3} = \sqrt[3]{x}$, $x^{2/5} = \sqrt[5]{x^2}$, $x^{-3/2} = \frac{1}{x^{3/2}} = \frac{1}{\sqrt{x^3}}$
   (b) $\sqrt[4]{x^3} = x^{3/4}$, $\frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}} = x^{-1/2}$
3. (a) $8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$ (b) $(-8)^{2/3} = (\sqrt[3]{-8})^2 = (-2)^2 = 4$
   (c) $8^{-2/3} = \frac{1}{8^{2/3}} = \frac{1}{4}$ (d) $-8^{-2/3} = -\left(\frac{1}{8^{2/3}}\right) = -\frac{1}{4}$
   (e) $\sqrt[15]{71} = (71)^{1/15} \approx 1.32867$
4. (a) $x \cdot x^{1/3} \cdot x^3 = x^{1+1/3+3} = x^{13/3}$ (b) $x^2 \div x^{1/2} = x^{2-1/2} = x^{3/2}$
   (c) $(x^{-2/3})^{-3} = x^{(-2/3)(-3)} = x^2$ (d) $x^{-3/2} \cdot x^{1/2} = x^{-3/2+1/2} = x^{-1}$
   (e) $x^{-3/2} \cdot x = x^{-3/2+1} = x^{-1/2}$ (f) $\left(\frac{x^4}{y^2}\right)^{3/2} = \frac{(x^4)^{3/2}}{(y^2)^{3/2}} = \frac{x^6}{y^3}$
5. (a) False. $\frac{8x^{2/3}}{x^{-1/3}} = 8x^{2/3} \cdot x^{1/3} = 8x^{2/3+1/3} = 8x$
   (b) False. $(16x^8y)^{3/4} = 16^{3/4}(x^8)^{3/4}y^{3/4} = (\sqrt[4]{16})^3x^6y^{3/4} = 8x^6y^{3/4}$
   (c) True.
6. (a) $\sqrt[7]{x^7} = x$
   (b) $\left[\sqrt[5]{(x^2+1)^2}\right]^5 = (x^2+1)^2$
   (c) $\sqrt{12xy^2} \cdot \sqrt{3x^2y} = \sqrt{36x^3y^3} = \sqrt{36x^2y^2 \cdot xy} = \sqrt{36x^2y^2}\sqrt{xy} = 6xy\sqrt{xy}$
7. $\frac{x}{\sqrt{5x}} = \frac{x}{\sqrt{5x}} \cdot \frac{\sqrt{5x}}{\sqrt{5x}} = \frac{x\sqrt{5x}}{5x} = \frac{\sqrt{5x}}{5}$

0.4 Exercises

Unless stated otherwise, assume all variables are nonnegative and all denominators are nonzero.

In Problems 1–8, find the powers and roots, if they are real numbers.

1. (a) $\sqrt{256/9}$ (b) $\sqrt{1.44}$
2. (a) $\sqrt[5]{-32^3}$ (b) $\sqrt[4]{-16^5}$
3. (a) $16^{3/4}$ (b) $(-16)^{-3/2}$
4. (a) $-27^{-1/3}$ (b) $32^{3/5}$
5. $\left(\frac{8}{27}\right)^{-2/3}$
6. $\left(\frac{4}{9}\right)^{3/2}$
7. (a) $8^{2/3}$ (b) $(-8)^{-2/3}$
8. (a) $8^{-2/3}$ (b) $-8^{2/3}$

In Problems 9 and 10, rewrite each radical with a fractional exponent, and then approximate the value with a calculator.

9. $\sqrt[9]{(6.12)^4}$
10. $\sqrt[12]{4.96}$

In Problems 11–14, replace each radical with a fractional exponent. Do not simplify.

11. $\sqrt{m^3}$
12. $\sqrt[3]{x^5}$
13. $\sqrt[4]{m^2n^5}$
14. $\sqrt[5]{x^3}$

In Problems 15–18, write in radical form. Do not simplify.

15. $x^{7/6}$
16. $y^{11/5}$
17. $-(1/4)x^{-5/4}$
18. $-x^{-5/3}$

In Problems 19–32, use the properties of exponents to simplify each expression so that only positive exponents remain.

19. $y^{1/4} \cdot y^{1/2}$ 20. $x^{2/3} \cdot x^{1/5}$ 21. $z^{3/4} \cdot z^4$
22. $x^{-2/3} \cdot x^2$ 23. $y^{-3/2} \cdot y^{-1}$ 24. $z^{-2} \cdot z^{5/3}$
25. $\dfrac{x^{1/3}}{x^{-2/3}}$ 26. $\dfrac{x^{-1/2}}{x^{-3/2}}$ 27. $\dfrac{y^{-5/2}}{y^{-2/5}}$
28. $\dfrac{x^{4/9}}{x^{1/12}}$ 29. $(x^{2/3})^{3/4}$ 30. $(x^{4/5})^3$
31. $(x^{-1/2})^2$ 32. $(x^{-2/3})^{-2/5}$

In Problems 33–38, simplify each expression by using the properties of radicals. Assume nonnegative variables.

33. $\sqrt{64x^4}$ 34. $\sqrt[3]{-64x^6y^3}$ 35. $\sqrt{128x^4y^5}$
36. $\sqrt[3]{54x^5x^8}$ 37. $\sqrt[3]{40x^8y^5}$ 38. $\sqrt{32x^5y}$

In Problems 39–46, perform the indicated operations and simplify.

39. $\sqrt{12x^3y} \cdot \sqrt{3x^2y}$ 40. $\sqrt[3]{16x^2y} \cdot \sqrt[3]{3x^2y}$
41. $\sqrt{63x^5y^3} \cdot \sqrt{28x^2y}$ 42. $\sqrt{10xz^{10}} \cdot \sqrt{30x^{17}z}$
43. $\dfrac{\sqrt{12x^3y^{12}}}{\sqrt{27xy^2}}$ 44. $\dfrac{\sqrt{250xy^7z^4}}{\sqrt{18x^{17}y^2}}$
45. $\dfrac{\sqrt[4]{32a^9b^5}}{\sqrt[4]{162a^{17}}}$ 46. $\dfrac{\sqrt[3]{-16x^3y^4}}{\sqrt[3]{128y^2}}$

In Problems 47–50, use properties of exponents and radicals to determine a value for x that makes each statement true.

47. $(A^9)^x = A$ 48. $(B^{20})^x = B$
49. $(\sqrt[7]{R})^x = R$ 50. $(\sqrt{T^3})^x = T$

In Problems 51–56, rationalize each denominator and then simplify.

51. $\sqrt{2/3}$ 52. $\sqrt{5/8}$ 53. $\dfrac{\sqrt{m^2x}}{\sqrt{mx^2}}$
54. $\dfrac{5x^3w}{\sqrt{4xw^2}}$ 55. $\dfrac{\sqrt[3]{m^2x}}{\sqrt[3]{mx^5}}$ 56. $\dfrac{\sqrt[4]{mx^3}}{\sqrt[4]{y^2z^5}}$

In calculus it is frequently important to write an expression in the form cx^n, where c is a constant and n is a rational number. In Problems 57–60, write each expression in this form.

57. $\dfrac{-2}{3\sqrt[3]{x^2}}$ 58. $\dfrac{-2}{3\sqrt[4]{x^3}}$ 59. $3x\sqrt{x}$ 60. $\sqrt{x} \cdot \sqrt[3]{x}$

In calculus problems, the answers are frequently expected to be in a simple form with a radical instead of an exponent. In Problems 61–64, write each expression with radicals.

61. $\dfrac{3}{2}x^{1/2}$ 62. $\dfrac{4}{3}x^{1/3}$ 63. $\dfrac{1}{2}x^{-1/2}$ 64. $\dfrac{-1}{2}x^{-3/2}$

APPLICATIONS

65. ***Richter scale*** The Richter scale reading for an earthquake measures its intensity (as a multiple of some minimum intensity used for comparison). The intensity I corresponding to a Richter scale reading R is given by

$$I = 10^R$$

(a) A quake measuring 8.5 on the Richter scale would be severe. Express the intensity of such a quake in exponential form and in radical form.
(b) Find the intensity of a quake measuring 8.5.
(c) The San Francisco quake that occurred during the 1989 World Series measured 7.1, and the 1906 San Francisco quake (which devastated the city) measured 8.25. Calculate the ratio of these intensities (larger to smaller).

66. ***Sound intensity*** The intensity of sound I (as a multiple of the average minimum threshold of hearing intensity) is related to the decibel level D (or loudness of sound) according to

$$I = 10^{D/10}$$

(a) Express $10^{D/10}$ using radical notation.
(b) The background noise level of a relatively quiet room has a decibel reading of 32. Find the intensity I_1 of this noise level.
(c) A decibel reading of 140 is at the threshold of pain. If I_2 is the intensity of this threshold and I_1 is the intensity found in (b), express the ratio I_2/I_1 as a power of 10. Then approximate this ratio.

67. ***Investment*** If \$1000 is invested at $r\%$ compounded annually, the future value S of the account after two and a half years is given by

$$S = 1000\left(1 + \frac{r}{100}\right)^{2.5} = 1000\left(1 + \frac{r}{100}\right)^{5/2}$$

(a) Express this equation with radical notation.
(b) Find the value of this account if the interest rate is 6.6% compounded annually.

68. ***Life span*** Life expectancy in the United States can be approximated with the equation

$$L = 29x^{0.21}$$

where x is the number of years that the birth year is past 1900. (*Source:* National Center for Health Statistics)
(a) Express this equation with radical notation.
(b) Use the equation to estimate the life expectancy for a person born in 2015.

69. ***Population*** The population P of India (in billions) for 2000–2050 can be approximated by the equation

$$P = 0.924t^{0.13}$$

where $t > 0$ is the number of years past 2000. (*Source:* United Nations)

(a) Express this equation with radical notation.
(b) Does this equation predict a greater increase from 2005 to 2010 or from 2045 to 2050? What might explain this difference?

70. ***Transportation*** The percent p of paved roads and streets in the United States can be approximated with the equation

$$p = 6.75t^{0.55}$$

where t is the number of years past 1940. (*Source:* U.S. Department of Transportation)
(a) Express this equation with radical notation.
(b) Does this equation estimate a greater percent change during the decade of the 1970s or during the decade from 2000 to 2010? What might explain this?
(c) When can you be certain this equation is no longer valid?

Half-life **In Problems 71 and 72, use the fact that the quantity of a radioactive substance after t years is given by $q = q_0(2^{-t/k})$, where q_0 is the original amount of radioactive material and k is its half-life (the number of years it takes for half the radioactive substance to decay).**

71. The half-life of strontium-90 is 25 years. Find the amount of strontium-90 remaining after 10 years if $q_0 = 98$ kg.
72. The half-life of carbon-14 is 5600 years. Find the amount of carbon-14 remaining after 10,000 years if $q_0 = 40.0$ g.

73. ***Population growth*** Suppose the formula for the growth of the population of a city for the next 10 years is given by

$$P = P_0(2.5)^{ht}$$

where P_0 is the population of the city at the present time and P is the population t years from now. If $h = 0.03$ and $P_0 = 30{,}000$, find P when $t = 10$.

74. ***Advertising and sales*** Suppose it has been determined that the sales at Ewing Gallery decline after the end of an advertising campaign, with daily sales given by

$$S = 2000(2^{-0.1x})$$

where S is in dollars and x is the number of days after the campaign ends. What are the daily sales 10 days after the end of the campaign?

75. ***Company growth*** The growth of a company can be described by the equation

$$N = 500(0.02)^{0.7^t}$$

where t is the number of years the company has been in existence and N is the number of employees.
(a) What is the number of employees when $t = 0$? (This is the number of employees the company has when it starts.)
(b) What is the number of employees when $t = 5$?

0.5 Operations with Algebraic Expressions

In algebra we are usually dealing with combinations of real numbers (such as 3, 6/7, and $-\sqrt{2}$) and letters (such as x, a, and m). Unless otherwise specified, the letters are symbols used to represent real numbers and are sometimes called **variables.** An expression obtained by performing additions, subtractions, multiplications, divisions, or extractions of roots with one or more real numbers or variables is called an **algebraic expression.** Unless otherwise specified, the variables represent all real numbers for which the algebraic expression is a real number. Examples of algebraic expressions include

$$3x + 2y, \quad \frac{x^3y + y}{x - 1}, \quad \text{and} \quad \sqrt{x} - 3$$

Note that the variable x cannot be negative in the expression $\sqrt{x} - 3$ and that $(x^3y + y)/(x - 1)$ is not a real number when $x = 1$, because division by 0 is undefined.

Any product of a real number (called the **coefficient**) and one or more variables to powers is called a **term.** The sum of a finite number of terms with nonnegative integer powers on the variables is called a **polynomial.** If a polynomial contains only one variable x, then it is called a polynomial in x.

Polynomial in x

The general form of a **polynomial in x** is

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where each coefficient a_i is a real number for $i = 0, 1, 2, \ldots, n$. If $a_n \neq 0$, the **degree** of the polynomial is n, and a_n is called the **leading coefficient.** The term a_0 is called the **constant term.**

Thus $4x^3 - 2x - 3$ is a third-degree polynomial in x with leading coefficient 4 and constant term -3. If two or more variables are in a term, the degree of the term is the sum of the exponents of the variables. The degree of a nonzero constant term is zero. Thus the degree of $4x^2y$ is $2 + 1 = 3$, the degree of $6xy$ is $1 + 1 = 2$, and the degree of 3 is 0. The **degree of a polynomial** containing one or more variables is the degree of the term in the polynomial having the highest degree. Therefore, $2xy - 4x + 6$ is a second-degree polynomial.

A polynomial containing two terms is called a **binomial,** and a polynomial containing three terms is called a **trinomial.** A single-term polynomial is a **monomial.**

Operations with Algebraic Expressions

Because monomials and polynomials represent real numbers, the properties of real numbers can be used to add, subtract, multiply, divide, and simplify polynomials. For example, we can use the Distributive Law to add $3x$ and $2x$.

$$3x + 2x = (3 + 2)x = 5x$$

Similarly, $9xy - 3xy = (9 - 3)xy = 6xy$.

Terms with exactly the same variable factors are called **like terms.** We can add or subtract like terms by adding or subtracting the coefficients of the variables.

● EXAMPLE 1 *Adding Polynomials*

Compute $(4xy + 3x) + (5xy - 2x)$.

Solution

$$\begin{aligned}(4xy + 3x) + (5xy - 2x) &= 4xy + 3x + 5xy - 2x \\ &= 9xy + x\end{aligned}$$

Subtraction of polynomials uses the Distributive Law to remove the parentheses.

● EXAMPLE 2 *Subtracting Polynomials*

Compute $(3x^2 + 4xy + 5y^2 + 1) - (6x^2 - 2xy + 4)$.

Solution

Removing the parentheses yields

$$3x^2 + 4xy + 5y^2 + 1 - 6x^2 + 2xy - 4$$

which simplifies to

$$-3x^2 + 6xy + 5y^2 - 3$$

Using the rules of exponents and the Commutative and Associative Laws for multiplication, we can multiply and divide monomials, as the following example shows.

● EXAMPLE 3 *Products and Quotients*

Perform the indicated operations.

(a) $(8xy^3)(2x^3y)(-3xy^2)$ (b) $-15x^2y^3 \div (3xy^5)$

Solution

(a) $8 \cdot 2 \cdot (-3) \cdot x \cdot x^3 \cdot x \cdot y^3 \cdot y \cdot y^2 = -48x^5y^6$

(b) $\dfrac{-15x^2y^3}{3xy^5} = -\dfrac{15}{3} \cdot \dfrac{x^2}{x} \cdot \dfrac{y^3}{y^5} = -5 \cdot x \cdot \dfrac{1}{y^2} = -\dfrac{5x}{y^2}$

Symbols of grouping are used in algebra in the same way as they are used in the arithmetic of real numbers. We have removed parentheses in the process of adding and subtracting polynomials. Other symbols of grouping, such as brackets, [ ], are treated the same as parentheses.

When there are two or more symbols of grouping involved, we begin with the innermost and work outward.

● EXAMPLE 4 *Symbols of Grouping*

Simplify $3x^2 - [2x - (3x^2 - 2x)]$.

Solution

$$\begin{aligned} 3x^2 - [2x - (3x^2 - 2x)] &= 3x^2 - [2x - 3x^2 + 2x] \\ &= 3x^2 - [4x - 3x^2] \\ &= 3x^2 - 4x + 3x^2 \\ &= 6x^2 - 4x \end{aligned}$$

By the use of the Distributive Law, we can multiply a binomial by a monomial. For example,

$$x(2x + 3) = x \cdot 2x + x \cdot 3 = 2x^2 + 3x$$

We can extend the Distributive Law to multiply polynomials with more than two terms. For example,

$$5(x + y + 2) = 5x + 5y + 10$$

● EXAMPLE 5 *Distributive Law*

Find the following products.

(a) $-4ab(3a^2b + 4ab^2 - 1)$ (b) $(4a + 5b + c)ac$

Solution

(a) $-4ab(3a^2b + 4ab^2 - 1) = -12a^3b^2 - 16a^2b^3 + 4ab$

(b) $(4a + 5b + c)ac = 4a \cdot ac + 5b \cdot ac + c \cdot ac = 4a^2c + 5abc + ac^2$

The Distributive Law can be used to show us how to multiply two polynomials. Consider the indicated multiplication $(a + b)(c + d)$. If we first treat the sum $(a + b)$ as a single quantity, then two successive applications of the Distributive Law gives

$$(a + b)(c + d) = (a + b) \cdot c + (a + b) \cdot d = ac + bc + ad + bd$$

Thus we see that the product can be found by multiplying $(a + b)$ by c, multiplying $(a + b)$ by d, and then adding the products. This is frequently set up as follows.

Product of Two Polynomials

Procedure	Example
To multiply two polynomials:	Multiply $(3x + 4xy + 3y)$ by $(x - 2y)$.
1. Write one of the polynomials above the other.	1. $3x + 4xy + 3y$ $\underline{x - 2y}$
2. Multiply each term of the top polynomial by each term of the bottom one, and write the similar terms of the product under one another.	2. $3x^2 + 4x^2y + 3xy$ $\underline{\qquad - 6xy - 8xy^2 - 6y^2}$
3. Add like terms to simplify the product.	3. $3x^2 + 4x^2y - 3xy - 8xy^2 - 6y^2$

● **EXAMPLE 6** ***The Product of Two Polynomials***

Multiply $(4x^2 + 3xy + 4x)$ by $(2x - 3y)$.

Solution

$$\begin{array}{l} 4x^2 + 3xy + 4x \\ \underline{2x \quad - 3y} \\ 8x^3 + 6x^2y + 8x^2 \\ \underline{\quad -12x^2y \qquad - 9xy^2 - 12xy} \\ 8x^3 - 6x^2y + 8x^2 - 9xy^2 - 12xy \end{array}$$

Because the multiplications we must perform often involve binomials, the following special products are worth remembering.

Special Products

A. $(x + a)(x + b) = x^2 + (a + b)x + ab$
B. $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$

It is easier to remember these two special products if we note their structure. We can obtain these products by finding the products of the First terms, Outside terms, Inside terms, and Last terms, and then adding the results. This is called the FOIL method of multiplying two binomials.

● **EXAMPLE 7** ***Products of Binomials***

Multiply the following.

(a) $(x - 4)(x + 3)$ (b) $(3x + 2)(2x + 5)$

Solution

(a) $(x - 4)(x + 3) = \overset{\text{First}}{(x^2)} + \overset{\text{Outside}}{(3x)} + \overset{\text{Inside}}{(-4x)} + \overset{\text{Last}}{(-12)} = x^2 - x - 12$

(b) $(3x + 2)(2x + 5) = (6x^2) + (15x) + (4x) + (10) = 6x^2 + 19x + 10$

Additional special products are as follows:

Additional Special Products

C. $(x + a)^2 = x^2 + 2ax + a^2$	binomial squared
D. $(x - a)^2 = x^2 - 2ax + a^2$	binomial squared
E. $(x + a)(x - a) = x^2 - a^2$	difference of two squares
F. $(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$	binomial cubed
G. $(x - a)^3 = x^3 - 3ax^2 + 3a^2x - a^3$	binomial cubed

● EXAMPLE 8 *Special Products*

Multiply the following.

(a) $(x + 5)^2$
(b) $(3x - 4y)^2$
(c) $(x - 2)(x + 2)$
(d) $(x^2 - y^3)^2$
(e) $(x + 4)^3$

Solution

(a) $(x + 5)^2 = x^2 + 2(5)x + 25 = x^2 + 10x + 25$
(b) $(3x - 4y)^2 = (3x)^2 - 2(3x)(4y) + (4y)^2 = 9x^2 - 24xy + 16y^2$
(c) $(x - 2)(x + 2) = x^2 - 4$
(d) $(x^2 - y^3)^2 = (x^2)^2 - 2(x^2)(y^3) + (y^3)^2 = x^4 - 2x^2y^3 + y^6$
(e) $(x + 4)^3 = x^3 + 3(4)(x^2) + 3(4^2)(x) + 4^3 = x^3 + 12x^2 + 48x + 64$

● ***Checkpoint***

1. Remove parentheses and combine like terms: $9x - 5x(x + 2) + 4x^2$
2. Find the following products.
   (a) $(2x + 1)(4x^2 - 2x + 1)$
   (b) $(x + 3)^2$
   (c) $(3x + 2)(x - 5)$
   (d) $(1 - 4x)(1 + 4x)$

All algebraic expressions can represent real numbers, so the techniques used to perform operations on polynomials and to simplify polynomials also apply to other algebraic expressions.

● EXAMPLE 9 *Operations with Algebraic Expressions*

Perform the indicated operations.

(a) $3\sqrt{3} + 4x\sqrt{y} - 5\sqrt{3} - 11x\sqrt{y} - (\sqrt{3} - x\sqrt{y})$
(b) $x^{3/2}(x^{1/2} - x^{-1/2})$
(c) $(x^{1/2} - x^{1/3})^2$
(d) $(\sqrt{x} + 2)(\sqrt{x} - 2)$

Solution

(a) We remove parentheses and then combine the terms containing $\sqrt{3}$ and the terms containing $x\sqrt{y}$.

$$(3 - 5 - 1)\sqrt{3} + (4 - 11 + 1)x\sqrt{y} = -3\sqrt{3} - 6x\sqrt{y}$$

(b) $x^{3/2}(x^{1/2} - x^{-1/2}) = x^{3/2} \cdot x^{1/2} - x^{3/2} \cdot x^{-1/2} = x^2 - x$
(c) $(x^{1/2} - x^{1/3})^2 = (x^{1/2})^2 - 2x^{1/2}x^{1/3} + (x^{1/3})^2 = x - 2x^{5/6} + x^{2/3}$
(d) $(\sqrt{x} + 2)(\sqrt{x} - 2) = (\sqrt{x})^2 - (2)^2 = x - 4$

In later chapters we will need to write problems in a simplified form so that we can perform certain operations on them. We can often use division of one polynomial by another to obtain the simplification, as shown in the following procedure.

Division of Polynomials

Procedure	Example
To divide one polynomial by another:	Divide $4x^3 + 4x^2 + 5$ by $2x^2 + 1$.
1. Write both polynomials in descending powers of a variable. Include missing terms with coefficient 0 in the dividend.	1. $2x^2 + 1\overline{)4x^3 + 4x^2 + 0x + 5}$
2. (a) Divide the highest power term of the divisor into the highest power term of the dividend, and write this partial quotient above the dividend. Multiply the partial quotient times the divisor, write the product under the dividend, and subtract, getting a new dividend.	2. (a) $$\begin{array}{r} 2x \\ 2x^2 + 1\overline{)4x^3 + 4x^2 + 0x + 5} \\ \underline{4x^3 + 2x } \\ 4x^2 - 2x + 5 \end{array}$$
(b) Repeat until the degree of the new dividend is less than the degree of the divisor. Any remainder is written over the divisor and added to the quotient.	(b) $$\begin{array}{r} 2x + 2 \\ 2x^2 + 1\overline{)4x^3 + 4x^2 + 0x + 5} \\ \underline{4x^3 + 2x } \\ 4x^2 - 2x + 5 \\ \underline{4x^2 + 2} \\ -2x + 3 \end{array}$$ Degree $(-2x + 3) <$ degree $(2x^2 + 1)$ Quotient: $2x + 2 + \dfrac{-2x + 3}{2x^2 + 1}$

EXAMPLE 10 *Division of Polynomials*

Divide $(4x^3 - 13x - 22)$ by $(x - 3)$, $x \neq 3$.

Solution

$$\begin{array}{r} 4x^2 + 12x + 23 \\ x - 3\overline{)4x^3 + 0x^2 - 13x - 22} \\ \underline{4x^3 - 12x^2 } \\ 12x^2 - 13x - 22 \\ \underline{12x^2 - 36x } \\ 23x - 22 \\ \underline{23x - 69} \\ 47 \end{array}$$

$0x^2$ is inserted so that each power of x is present.

The quotient is $4x^2 + 12x + 23$, with remainder 47, or

$$4x^2 + 12x + 23 + \frac{47}{x - 3}$$

Checkpoint

3. Use long division to find $(x^3 + 2x + 7) \div (x - 4)$.

Spreadsheet Note

One important use of algebraic expressions is to describe relationships among quantities. For example, the expression "one more than a number" could be written as $n + 1$, where n represents an arbitrary number. This ability to represent quantities or their interrelationships algebraically is one of the keys to using spreadsheets. For more details regarding spreadsheets, see the *Excel Guide for Finite Mathematics and Applied Calculus* by Revathi Narasimhan that accompanies this text.

Each cell in a spreadsheet has an address based on its row and column (see Table 0.3). These cell addresses can act like variables in an algebraic expression, and the "fill down" or "fill across" capabilities update this cell referencing while maintaining algebraic relationships. For example, we noted previously that if \$1000 is invested in an account that earns 10%, compounded annually, then the future value of the account after n years is given by $\$1000(1.1)^n$. We can track the future value by starting with the spreadsheet shown in Table 0.4.

TABLE 0.3

	A	B	C	D
1	cell A1	cell B1	cell C1	.
2	cell A2	cell B2	cell C2	
3	cell A3	cell B3	cell C3	
4	.			
5	.			

TABLE 0.4

	A	B
1	Year	Future value
2	1	= 1000*(1.1) ^ (A2)
3	=A2+1	

gives →

	A	B
1	Year	Future value
2	1	1100
3	2	

The use of the = sign to begin a cell entry indicates an algebraic expression whose variables are other cells. In cell A3 of Table 0.4, typing the entry =A2+1 creates a referencing scheme based on the algebraic expression $n + 1$, but with cell A2 acting as the variable. From this beginning, highlighting cell A3 and using the "fill down" command updates the referencing from cell to cell and creates a counter for the number of years (see Table 0.5).

TABLE 0.5

	A	B	C
1	Year	Future value	Interest earned
2	1	1100	100
3	2	1210	210
4	3	1331	331
5	4	1464.1	464.1
6	5	1610.51	610.51
7	6	1771.561	771.561
8	7	1948.7171	948.7171
9	8	2143.58881	1143.58881
10	9	2357.947691	1357.947691
11	10	2593.7424601	1593.74246
12	11	2853.1167061	1853.116706
13	12	3138.4283767	2138.428377
14	13	3452.2712144	2452.271214
15	14	3797.4983358	2797.498336
16	15	4177.2481694	3177.248169

The future value is given by a formula, so we use that formula in the cell but replace the variable with the cell name where a value of that variable can be found. The "fill down" command updates the referencing and hence the values used. The first 15 years of the spreadsheet begun in Table 0.4 are shown in Table 0.5, along with a new column for interest earned. Because the interest earned is found by subtracting the original investment of $1000 from the future value, cell C2 contains the entry =B2−1000, and those below it are obtained by using "fill down." The entries for future value and interest earned also could be expressed in dollars and cents by rounding appropriately. ■

● Checkpoint Solutions

1. $9x - 5x(x + 2) + 4x^2 = 9x - 5x^2 - 10x + 4x^2$
   $= -x^2 - x$

   Note that without parentheses around $9x - 5x$, multiplication has priority over subtraction.

2. (a) $4x^2 - 2x + 1$

$$\begin{array}{r@{}l} & 4x^2 - 2x + 1 \\ & 2x + 1 \\ \hline & 8x^3 - 4x^2 + 2x \\ & 4x^2 - 2x + 1 \\ \hline & 8x^3 + 1 \end{array}$$

   (b) $(x + 3)^2 = x^2 + 2(3x) + 3^2 = x^2 + 6x + 9$

   (c) $(3x + 2)(x - 5) = 3x^2 - 15x + 2x - 10 = 3x^2 - 13x - 10$

   (d) $(1 - 4x)(1 + 4x) = 1 - 16x^2$ Note that this is different from $16x^2 - 1$.

3.
$$\begin{array}{r|l} & x^2 + 4x + 18 \\ \hline x - 4 & x^3 + 0x^2 + 2x + 7 \\ & x^3 - 4x^2 \\ \hline & 4x^2 + 2x + 7 \\ & 4x^2 - 16x \\ \hline & 18x + 7 \\ & 18x - 72 \\ \hline & 79 \end{array}$$

   The answer is $x^2 + 4x + 18 + \dfrac{79}{x - 4}$.

0.5 Exercises

For each polynomial in Problems 1–4, (a) give the degree of the polynomial, (b) give the coefficient (numerical) of the highest-degree term, (c) give the constant term, and (d) decide whether it is a polynomial of one or several variables.

1. $10 - 3x - x^2$
2. $5x^4 - 2x^9 + 7$
3. $7x^2y - 14xy^3z$
4. $2x^5 + 7x^2y^3 - 5y^6$

The expressions in Problems 5 and 6 are polynomials with the form $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where n is a positive integer. Complete the following.

5. For $2x^5 - 3x^2 - 5$,
   (a) $2 = a_?$ (b) $a_3 = ?$
   (c) $-3 = a_?$ (d) $a_0 = ?$
6. For $5x^3 - 4x - 17$,
   (a) $a_3 = ?$ (b) $a_1 = ?$
   (c) $a_2 = ?$ (d) $-17 = a_?$

In Problems 7–10, evaluate each algebraic expression at the indicated values of the variables.

7. $4x - x^2$ at $x = -2$
8. $3x^2 - 4y^2 - 2xy$ at $x = 3$ and $y = -4$
9. $\dfrac{2x - y}{x^2 - 2y}$ at $x = -5$ and $y = -3$
10. $\dfrac{16y}{1 - y}$ at $y = -3$
11. Evaluate $1.98T - 1.09(1 - H)(T - 58) - 56.8$ when $T = 74.7$ and $H = 0.80$.
12. Evaluate $R\left[\dfrac{0.083i}{1 - (1 + 0.083i)^{-n}}\right]$ when $R = 100{,}000$, $i = 0.07$, $n = 360$.

In Problems 13–20, simplify by combining like terms.

13. $(16pq - 7p^2) + (5pq + 5p^2)$
14. $(3x^3 + 4x^2y^2) + (3x^2y^2 - 7x^3)$

15. $(4m^2 - 3n^2 + 5) - (3m^2 + 4n^2 + 8)$
16. $(4rs - 2r^2s - 11rs^2) - (11rs^2 - 2rs + 4r^2s)$
17. $-[8 - 4(q + 5) + q]$
18. $x^3 + [3x - (x^3 - 3x)]$
19. $x^2 - [x - (x^2 - 1) + 1 - (1 - x^2)] + x$
20. $y^3 - [y^2 - (y^3 + y^2)] - [y^3 + (1 - y^2)]$

In Problems 21–58, perform the indicated operations and simplify.

21. $(5x^3)(7x^2)$
22. $(-3x^2y)(2xy^3)(4x^2y^2)$
23. $(39r^3s^2) \div (13r^2s)$
24. $(-15m^3n) \div (5mn^4)$
25. $ax^2(2x^2 + ax + ab)$
26. $-3(3 - x^2)$
27. $(3y + 4)(2y - 3)$
28. $(4x - 1)(x - 3)$
29. $6(1 - 2x^2)(2 - x^2)$
30. $2(x^3 + 3)(2x^3 - 5)$
31. $(4x + 3)^2$
32. $(2y + 5)^2$
33. $\left(x^2 - \frac{1}{2}\right)^2$
34. $(x^3y^3 - 0.3)^2$
35. $9(2x + 1)(2x - 1)$
36. $3(5y + 2)(5y - 2)$
37. $(0.1 - 4x)(0.1 + 4x)$
38. $\left(\frac{2}{3} + x\right)\left(\frac{2}{3} - x\right)$
39. $(0.1x - 2)(x + 0.05)$
40. $(6.2x + 4.1)(6.2x - 4.1)$
41. $(x - 2)(x^2 + 2x + 4)$
42. $(a + b)(a^2 - ab + b^2)$
43. $(x^3 + 5x)(x^5 - 2x^3 + 5)$
44. $(x^3 - 1)(x^7 - 2x^4 - 5x^2 + 5)$
45. (a) $(3x - 2)^2 - 3x - 2(3x - 2) + 5$
    (b) $(3x - 2)^2 - (3x - 2)(3x - 2) + 5$
46. (a) $(2x - 3)(3x + 2) - (5x - 2)(x - 3)$
    (b) $2x - 3(3x + 2) - 5x - 2(x - 3)$
47. $(18m^2n + 6m^3n + 12m^4n^2) \div (6m^2n)$
48. $(16x^2 + 4xy^2 + 8x) \div (4xy)$
49. $(24x^8y^4 + 15x^5y - 6x^7y) \div (9x^5y^2)$
50. $(27x^2y^2 - 18xy + 9xy^2) \div (6xy)$
51. $(x + 1)^3$
52. $(x - 3)^3$
53. $(2x - 3)^3$
54. $(3x + 4)^3$
55. $(x^3 + x - 1) \div (x + 2)$
56. $(x^5 + 5x - 7) \div (x + 1)$
57. $(x^4 + 3x^3 - x + 1) \div (x^2 + 1)$
58. $(x^3 + 5x^2 - 6) \div (x^2 - 2)$

In Problems 59–66, perform the indicated operations with expressions involving fractional exponents and radicals, and then simplify.

59. $x^{1/2}(x^{1/2} + 2x^{3/2})$
60. $x^{-2/3}(x^{5/3} - x^{-1/3})$
61. $(x^{1/2} + 1)(x^{1/2} - 2)$
62. $(x^{1/3} - x^{1/2})(4x^{2/3} - 3x^{3/2})$
63. $(\sqrt{x} + 3)(\sqrt{x} - 3)$
64. $(x^{1/5} + x^{1/2})(x^{1/5} - x^{1/2})$
65. $(2x + 1)^{1/2}[(2x + 1)^{3/2} - (2x + 1)^{-1/2}]$
66. $(4x - 3)^{-5/3}[(4x - 3)^{8/3} + 3(4x - 3)^{5/3}]$

APPLICATIONS

67. ***Revenue*** A company sells its product for \$55 per unit. Write an expression for the amount of money received (revenue) from the sale of x units of the product.

68. ***Profit*** Suppose a company's revenue R (in dollars) from the sale of x units of its product is given by

$$R = 215x$$

Suppose further that the total costs C (in dollars) of producing those x units is given by

$$C = 65x + 15{,}000$$

(a) If profit is revenue minus cost, find an expression for the profit from the production and sale of x units.
(b) Find the profit received if 1000 units are sold.

69. ***Rental*** A rental truck costs \$49.95 for a day plus 49¢ per mile.
(a) If x is the number of miles driven, write an expression for the total cost of renting the truck for a day.
(b) Find the total cost of the rental if it was driven 132 miles.

70. ***Cell phones*** Cell Pro makes cell phones and has total weekly costs of \$1500 for rent, utilities, and equipment plus labor and material costs of \$18.50 for each phone it makes.
(a) If x represents the number of phones produced and sold, write an expression for Cell Pro's weekly costs.
(b) If Cell Pro sells the phones to dealers for \$45.50 each, write an expression for the weekly total revenue for the phones.
(c) Cell Pro's weekly profit is the total revenue minus the total cost. Write an expression for Cell Pro's weekly profit.

71. ***Investments*** Suppose that you have \$4000 to invest, and you invest x dollars at 10% and the remainder at 8%. Write expressions in x that represent
(a) the amount invested at 8%,
(b) the interest earned on the x dollars at 10%,
(c) the interest earned on the money invested at 8%,
(d) the total interest earned.

72. ***Medications*** Suppose that a nurse needs 10 cc (cubic centimeters) of a 15.5% solution (that is, a solution that is 15.5% ingredient) of a certain medication, which must be obtained by mixing x cc of a 20% solution and y cc of a 5% solution. Write expressions involving x for
(a) y, the amount of 5% solution,
(b) the amount of ingredient in the x cc of 20% solution,
(c) the amount of ingredient in the 5% solution,
(d) the total amount of ingredient in the mixture.