9.6

OBJECTIVES

- To use the Chain Rule to differentiate functions
- To use the Power Rule to differentiate functions

The Chain Rule and the Power Rule

Application Preview

The demand x for a product is given by

$$x = \frac{98}{\sqrt{2p + 1}} - 1$$

where p is the price per unit. To find how fast demand is changing when price is \$24, we take the derivative of x with respect to p. If we write this function with a power rather than a radical, it has the form

$$x = 98(2p + 1)^{-1/2} - 1$$

The formulas learned so far cannot be used to find this derivative. We use a new formula, the **Power Rule,** to find this derivative. (See Example 7.) In this section we will discuss the **Chain Rule** and the Power Rule, which is one of the results of the Chain Rule, and we will use these formulas to solve applied problems.

Composite Functions

Recall from Section 1.2, "Functions," that if f and g are functions, then the composite functions g of f (denoted $g \circ f$) and f of g (denoted $f \circ g$) are defined as follows:

$$(g \circ f)(x) = g(f(x)) \quad \text{and} \quad (f \circ g)(x) = f(g(x))$$

EXAMPLE 1 *Composite Function*

If $f(x) = 3x^2$ and $g(x) = 2x - 1$, find $F(x) = f(g(x))$.

Solution

Substituting $g(x) = 2x - 1$ for x in $f(x)$ gives

$$f(g(x)) = f(2x - 1) = 3(2x - 1)^2$$

Thus $F(x) = 3(2x - 1)^2$.

Chain Rule

We could find the derivative of the function $F(x) = 3(2x - 1)^2$ by expanding the expression $3(2x - 1)^2$. Then

$$F(x) = 3(4x^2 - 4x + 1) = 12x^2 - 12x + 3$$

so $F'(x) = 24x - 12$. But we can also use a very powerful rule, called the **Chain Rule,** to find derivatives of composite functions. If we write the composite function $y = f(g(x))$ in the form $y = f(u)$, where $u = g(x)$, we state the Chain Rule as follows.

Chain Rule

If f and g are differentiable functions with $y = f(u)$, and $u = g(x)$, then y is a differentiable function of x, and

$$\frac{dy}{dx} = f'(u) \cdot g'(x)$$

or, equivalently,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Note that dy/du represents the derivative of $y = f(u)$ *with respect to u* and du/dx represents the derivative of $u = g(x)$ *with respect to x*. For example, if $y = 3(2x - 1)^2$, then the outside function, f, is the squaring function, and the inside function, g, is $2x - 1$, so we may write $y = f(u) = 3u^2$, where $u = g(x) = 2x - 1$. Then the derivative is

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 6u \cdot 2 = 12u$$

To write this derivative in terms of x, we substitute $2x - 1$ for u. Thus

$$\frac{dy}{dx} = 12(2x - 1) = 24x - 12$$

Note that we get the same result by using the Chain Rule as we did by expanding $f(x) = 3(2x - 1)^2$. The Chain Rule is important because it is not always possible to rewrite the function as a polynomial. Consider the following example.

● EXAMPLE 2 *Chain Rule*

If $y = \sqrt{x^2 - 1}$, find $\dfrac{dy}{dx}$.

Solution

If we write this function as $y = f(u) = \sqrt{u} = u^{1/2}$, when $u = x^2 - 1$, we can find the derivative.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2} \cdot u^{-1/2} \cdot 2x = u^{-1/2} \cdot x = \frac{1}{\sqrt{u}} \cdot x = \frac{x}{\sqrt{u}}$$

To write this derivative in terms of x alone, we substitute $x^2 - 1$ for u. Then

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2 - 1}}$$

Note that we could not find the derivative of a function like that of Example 2 by the methods learned previously.

● EXAMPLE 3 *Chain Rule*

If $y = \dfrac{1}{x^2 + 3x + 1}$, find $\dfrac{dy}{dx}$.

Solution

If we let $u = x^2 + 3x + 1$, we can write $y = f(u) = \dfrac{1}{u}$, or $y = u^{-1}$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -u^{-2}(2x + 3) = \frac{-2x - 3}{u^2}$$

Substituting for u gives

$$\frac{dy}{dx} = \frac{-2x - 3}{(x^2 + 3x + 1)^2}$$

Note in Example 3 that we also could have found the derivative with the Quotient Rule and obtained the same result.

$$\frac{dy}{dx} = \frac{(x^2 + 3x + 1)(0) - (1)(2x + 3)}{(x^2 + 3x + 1)^2}$$
$$= \frac{-(2x + 3)}{(x^2 + 3x + 1)^2} = \frac{-2x - 3}{(x^2 + 3x + 1)^2}$$

EXAMPLE 4 *Allometric Relationships*

The relationship between the length L (in meters) and weight W (in kilograms) of a species of fish in the Pacific Ocean is given by $W = 10.375L^3$. The rate of growth in length is given by $\frac{dL}{dt} = 0.36 - 0.18L$, where t is measured in years.

(a) Determine a formula for the rate of growth in weight $\frac{dW}{dt}$ in terms of L.

(b) If a fish weighs 30 kilograms, approximate its rate of growth in weight using the formula found in part (a).

Solution

(a) The rate of change uses the Chain Rule, as follows:

$$\frac{dW}{dt} = \frac{dW}{dL} \cdot \frac{dL}{dt} = 31.125L^2(0.36 - 0.18L) = 11.205L^2 - 5.6025L^3$$

(b) From $W = 10.375L^3$ and $W = 30$ kg, we can find L by solving

$$30 = 10.375L^3$$
$$\frac{30}{10.375} = L^3 \quad \text{so} \quad L = \sqrt[3]{\frac{30}{10.375}} \approx 1.4247 \text{ m}$$

Hence, the rate of growth in weight is

$$\frac{dW}{dt} = 11.205(1.4247)^2 - 5.6025(1.4247)^3 = 6.542 \text{ kilograms/year}$$

Power Rule

The Chain Rule is very useful and will be extremely important with functions that we will study later. A special case of the Chain Rule, called the **Power Rule,** is useful for the algebraic functions we have studied so far, composite functions where the outside function is a power.

Power Rule

If $y = u^n$, where u is a differentiable function of x, then

$$\frac{dy}{dx} = nu^{n-1} \cdot \frac{du}{dx}$$

EXAMPLE 5 *Power Rule*

(a) If $y = (x^2 - 4x)^6$, find $\frac{dy}{dx}$. (b) If $p = \frac{4}{3q^2 + 1}$, find $\frac{dp}{dq}$.

Solution

(a) This has the form $y = u^n = u^6$, with $u = x^2 - 4x$. Thus, by the Power Rule,

$$\frac{dy}{dx} = nu^{n-1} \cdot \frac{du}{dx} = 6u^5(2x - 4)$$

Substituting $x^2 - 4x$ for u gives

$$\frac{dy}{dx} = 6(x^2 - 4x)^5(2x - 4) = (12x - 24)(x^2 - 4x)^5$$

(b) We can use the Power Rule to find dp/dq if we write the equation in the form

$$p = 4(3q^2 + 1)^{-1}$$

Then

$$\frac{dp}{dq} = 4[-1(3q^2 + 1)^{-2}(6q)] = \frac{-24q}{(3q^2 + 1)^2}$$

The derivative of the function in Example 5(b) can also be found by using the Quotient Rule, but the Power Rule provides a more efficient method.

● EXAMPLE 6 *Power Rule with a Radical*

Find the derivatives of (a) $y = 3\sqrt[3]{x^2 - 3x + 1}$ and (b) $g(x) = \dfrac{1}{\sqrt{(x^2 + 1)^3}}$.

Solution

(a) Because $y = 3(x^2 - 3x + 1)^{1/3}$, we can make use of the Power Rule with $u = x^2 - 3x + 1$.

$$y' = 3\left(nu^{n-1}\frac{du}{dx}\right) = 3\left[\frac{1}{3}u^{-2/3}(2x - 3)\right]$$
$$= (x^2 - 3x + 1)^{-2/3}(2x - 3)$$
$$= \frac{2x - 3}{(x^2 - 3x + 1)^{2/3}}$$

(b) Writing $g(x)$ as a power gives

$$g(x) = (x^2 + 1)^{-3/2}$$

Then

$$g'(x) = -\frac{3}{2}(x^2 + 1)^{-5/2}(2x) = -3x \cdot \frac{1}{(x^2 + 1)^{5/2}} = \frac{-3x}{\sqrt{(x^2 + 1)^5}}$$

● Checkpoint

1. (a) If $f(x) = (3x^4 + 1)^{10}$, does $f'(x) = 10(3x^4 + 1)^9$?
   (b) If $f(x) = (2x + 1)^5$, does $f'(x) = 10(2x + 1)^4$?
   (c) If $f(x) = \dfrac{[u(x)]^n}{c}$, where c is a constant, does $f'(x) = \dfrac{n[u(x)]^{n-1} \cdot u'(x)}{c}$?
2. (a) If $f(x) = \dfrac{12}{2x^2 - 1}$, find $f'(x)$ by using the Power Rule (not the Quotient Rule).
   (b) If $f(x) = \dfrac{\sqrt{x^3 - 1}}{3}$, find $f'(x)$ by using the Power Rule (not the Quotient Rule).

EXAMPLE 7 *Demand (Application Preview)*

The demand for x hundred units of a product is given by

$$x = 98(2p + 1)^{-1/2} - 1$$

where p is the price per unit in dollars. Find the rate of change of the demand with respect to price when $p = 24$.

Solution

The rate of change of demand with respect to price is

$$\frac{dx}{dp} = 98\left[-\frac{1}{2}(2p + 1)^{-3/2}(2)\right] = -98(2p + 1)^{-3/2}$$

When $p = 24$, the rate of change is

$$\left.\frac{dx}{dp}\right|_{p=24} = -98(48 + 1)^{-3/2} = -98 \cdot \frac{1}{49^{3/2}}$$

$$= -98 \cdot \frac{1}{343} = -\frac{2}{7}$$

This means that when the price is \$24, demand is changing at the rate of $-2/7$ hundred units per dollar, or if the price changes by \$1, demand will change by about $-200/7$ units.

Checkpoint Solutions

1. (a) No, $f'(x) = 10(3x^4 + 1)^9(12x^3)$. (b) Yes (c) Yes
2. (a) $f(x) = 12(2x^2 - 1)^{-1}$

$$f'(x) = -12(2x^2 - 1)^{-2}(4x) = \frac{-48x}{(2x^2 - 1)^2}$$

(b) $f(x) = \frac{1}{3}(x^3 - 1)^{1/2}$

$$f'(x) = \frac{1}{6}(x^3 - 1)^{-1/2}(3x^2) = \frac{x^2}{2\sqrt{x^3 - 1}}$$

9.6 Exercises

In Problems 1–4, find $\frac{dy}{du}$, $\frac{du}{dx}$, and $\frac{dy}{dx}$.

1. $y = u^3$ and $u = x^2 + 1$
2. $y = u^4$ and $u = x^2 + 4x$
3. $y = u^4$ and $u = 4x^2 - x + 8$
4. $y = u^{10}$ and $u = x^2 + 5x$

Differentiate the functions in Problems 5–20.

5. $f(x) = (3x^5 - 2)^{20}$
6. $g(x) = (3 - 2x)^{10}$
7. $h(x) = \frac{3}{4}(x^5 - 2x^3 + 5)^8$
8. $k(x) = \frac{5}{7}(2x^3 - x + 6)^{14}$
9. $g(x) = (x^4 - 5x)^{-2}$
10. $p = (q^3 + 1)^{-5}$
11. $f(x) = \dfrac{3}{(2x^5 + 1)^4}$
12. $g(x) = \dfrac{1}{4x^3 + 1}$
13. $g(x) = \dfrac{1}{(2x^3 + 3x + 5)^{3/4}}$
14. $y = \dfrac{1}{(3x^3 + 4x + 1)^{3/2}}$
15. $y = \sqrt{3x^2 + 4x + 9}$
16. $y = \sqrt{x^2 + 3x}$
17. $y = \dfrac{11(x^3 - 7)^6}{9}$
18. $y = \dfrac{5\sqrt{1 - x^3}}{6}$
19. $y = \dfrac{(3x + 1)^5 - 3x}{7}$
20. $y = \dfrac{\sqrt{2x - 1} - \sqrt{x}}{2}$

At the indicated point, for each function in Problems 21–24, find
(a) the slope of the tangent line, and
(b) the instantaneous rate of change of the function.
A graphing utility's numerical derivative feature can be used to check your work.

21. $y = (x^3 + 2x)^4$ at $x = 2$
22. $y = \sqrt{5x^2 + 2x}$ at $x = 1$
23. $y = \sqrt{x^3 + 1}$ at $(2, 3)$
24. $y = (4x^3 - 5x + 1)^3$ at $(1, 0)$

In Problems 25–28, write the equation of the line tangent to the graph of each function at the indicated point. As a check, graph both the function and the tangent line you found to see whether it looks correct.

25. $y = (x^2 - 3x + 3)^3$ at $(2, 1)$
26. $y = (x^2 + 1)^3$ at $(2, 125)$
27. $y = \sqrt{3x^2 - 2}$ at $x = 3$
28. $y = \left(\dfrac{1}{x^3 - x}\right)^3$ at $x = 2$

In Problems 29 and 30, complete the following for each function.
(a) Find $f'(x)$.
(b) Check your result in part (a) by graphing both it and the numerical derivative of the function.
(c) Find x-values for which the slope of the tangent is 0.
(d) Find points (x, y) where the slope of the tangent is 0.
(e) Use a graphing utility to graph the function and locate the points found in part (d).

29. $f(x) = (x^2 - 4)^3 + 12$
30. $f(x) = 10 - (x^2 - 2x - 8)^2$

In Problems 31 and 32, do the following for each function $f(x)$.
(a) Find $f'(x)$.
(b) Graph both $f(x)$ and $f'(x)$ with a graphing utility.
(c) Determine x-values where $f'(x) = 0$, $f'(x) > 0$, $f'(x) < 0$.
(d) Determine x-values for which $f(x)$ has a maximum or minimum point, where the graph is increasing, and where it is decreasing.

31. $f(x) = 12 - 3(1 - x^2)^{4/3}$
32. $f(x) = 3 + \dfrac{1}{16}(x^2 - 4x)^4$

In Problems 33 and 34, find the derivative of each function.

33. (a) $y = \dfrac{2x^3}{3}$ (b) $y = \dfrac{2}{3x^3}$
(c) $y = \dfrac{(2x)^3}{3}$ (d) $y = \dfrac{2}{(3x)^3}$

34. (a) $y = \dfrac{3}{(5x)^5}$ (b) $y = \dfrac{3x^5}{5}$
(c) $y = \dfrac{3}{5x^5}$ (d) $y = \dfrac{(3x)^5}{5}$

APPLICATIONS

35. ***Ballistics*** Ballistics experts are able to identify the weapon that fired a certain bullet by studying the markings on the bullet. Tests are conducted by firing into a bale of paper. If the distance s, in inches, that the bullet travels into the paper is given by

$$s = 27 - (3 - 10t)^3$$

for $0 \le t \le 0.3$ second, find the velocity of the bullet one-tenth of a second after it hits the paper.

36. ***Population of microorganisms*** Suppose that the population of a certain microorganism at time t (in minutes) is given by

$$P = 1000 - 1000(t + 10)^{-1}$$

Find the rate of change of population.

37. ***Revenue*** The revenue from the sale of a product is

$$R = 1500x + 3000(2x + 3)^{-1} - 1000 \quad \text{dollars}$$

where x is the number of units sold. Find the marginal revenue when 100 units are sold. Interpret your result.

38. ***Revenue*** The revenue from the sale of x units of a product is

$$R = 15(3x + 1)^{-1} + 50x - 15 \quad \text{dollars}$$

Find the marginal revenue when 40 units are sold. Interpret your result.

39. ***Pricing and sales*** Suppose that the weekly sales volume y (in thousands of units sold) depends on the price per unit of the product according to

$$y = 32(3p + 1)^{-2/5}, \quad p > 0$$

where p is in dollars.
(a) What is the rate of change in sales volume when the price is \$21?
(b) Interpret your answer to part (a).

40. ***Pricing and sales*** A chain of auto service stations has found that its monthly sales volume y (in thousands of dollars) is related to the price p (in dollars) of an oil change according to

$$y = \frac{90}{\sqrt{p + 5}}, \quad p > 10$$