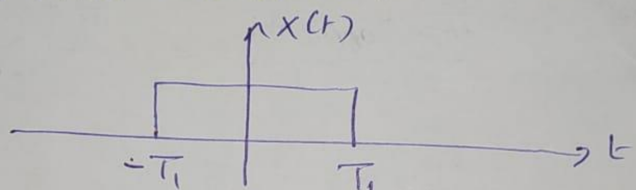
Fourier Transform

We find that periodic signals can be represented as a fourier series. So we write $X(t)$ as a sum of complex exponentials.

$X(t)$ is non-periodic signal, Can we write $X(t)$ as a sum of complex exponentials.

Given $X(t)$ as below;



Can we represent this signal as sum of complex exponentials. To do so we create a signal $X_p(t)$ {periodic signal from $X(t)$}. Such that $X(t)$ is one period of $X_p(t)$.

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Then $x(t)$ Can be written as:

$$x(t) = \lim_{T_0 \to \infty} x_p(t)$$

$x_p(t)$ is periodic So it Can be written as a Fourier Series

$$x_p(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$ multiply by T_0

$$\Rightarrow T_0 x_p(t) = \sum_{n=-\infty}^{\infty} T_0 D_n e^{jn\omega_0 t}.$$

with $$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_p(t) e^{-jn\omega_0 t} dt.$$

$$\Rightarrow T_0 D_n = \int_{-T_0/2}^{T_0/2} x_p(t) e^{-jn\omega_0 t} dt.$$

Lets call $$T_0 D_n = X(jn\omega_0) = \int_{-T_0/2}^{T_0/2} x_p(t) e^{-jn\omega_0 t} dt$$

Then $x_p(t)$ Can be written as:

$$x_p(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} X(jn\omega_0) e^{jn\omega_0 t}.$$

③

$$T_0 = \frac{2\pi}{\omega_0} \Rightarrow \frac{1}{T_0} = \frac{\omega_0}{2\pi}$$

$$\Rightarrow X_p(t) = \frac{\omega_0}{2\pi} \sum_{n=-\infty}^{\infty} X(jn\omega_0) e^{jn\omega_0 t}$$

$$X_p(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} X(jn\omega_0) e^{jn\omega_0 t} \omega_0.$$

as $T_0 \to \infty$ $\quad X_p(t) \longrightarrow X(t)$

$\omega_0 = \frac{2\pi}{T_0} \quad \omega_0 \to 0.$

$n\omega_0 \longrightarrow \omega$ (continous function).

and summation approches to $\int$

So as $T_0 \longrightarrow \infty$:-

$$\boxed{\begin{aligned} X(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} \, d\omega \\ X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt \end{aligned}}$$

This is called Fourier Transform pair.

$X(j\omega)$ is called the Fourier Transform of $x(t)$

In general $X(j\omega)$ is a complex function of frequency ω so it can be written as:

$$X(j\omega) = |X(j\omega)|\, e^{j\theta(j\omega)}$$

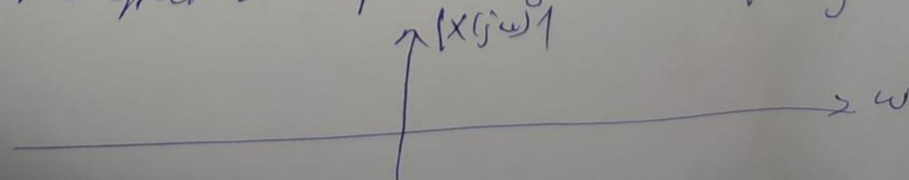
$X(j\omega)$ is the frequency domain representation of $x(t)$

So in general $x(t)$ can be represented in two ways:

① Time domain representation of the signal which is to represent the signal as a function of time

② Frequency domain representation of $x(t)$ which is to represent the signal as a frequency spectrum. So the signal can be represented as two spectrums:

① Amplitude spectrum: which a graph that shows all frequency components of the signal and the effect or amplitude of each frequency.



c) phase spectrum: which shows all frequency components of the signal and the phase of each frequency.

For a real valued $x(t)$:

$$X(j\omega) = X^*(-j\omega)$$

$$|X(j\omega)| = |X(-j\omega)|$$

$$\theta(j\omega) = -\theta(j\omega).$$

Ex Find the Fourier transform of

$$x(t) = \delta(t).$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

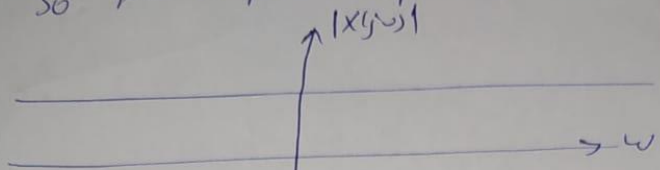
Ex $x(t) = \delta(t-1)$

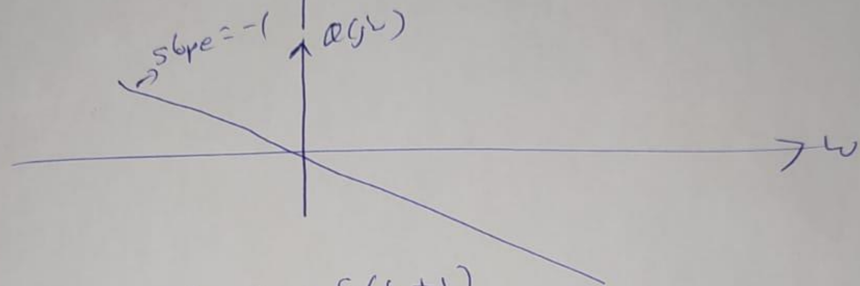
$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t-1) e^{-j\omega t} dt = e^{-j\omega}.$$

$$|X(j\omega)| = |e^{-j\omega}| = 1$$

$$\theta(j\omega) = -\omega.$$

So the Amplitude Spectrum.

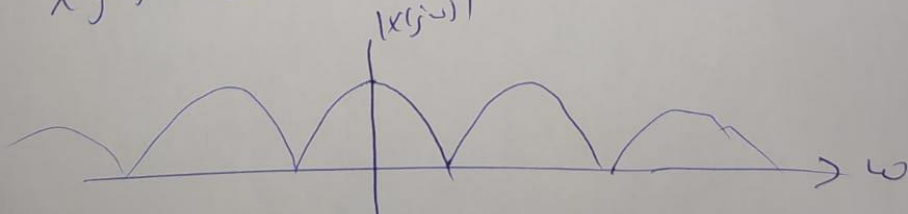




EX $x(t) = \delta(t-1) + \delta(t+1)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t-1)\, e^{-j\omega t}\, dt + \int_{-\infty}^{\infty} \delta(t+1)\, e^{-j\omega t}\, dt$$

$$X(j\omega) = e^{-j\omega} + e^{j\omega} = 2\cos\omega$$



$\theta(j\omega) = 0$.

Ex. $x(t) = e^{-at} u(t) \quad a > 0$

$$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} \, dt = \int_0^{\infty} e^{-t(a+j\omega)} \, dt$$

$$X(j\omega) = \frac{-1}{a+j\omega} \left[e^{-t(a+j\omega)} \Big|_0^{\infty} \right] = \frac{1}{a+j\omega}$$

amplitude spectrum:

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

phase spectrum

$$\theta(j\omega) = -\tan^{-1} \frac{\omega}{a}$$

at $\omega = 0$

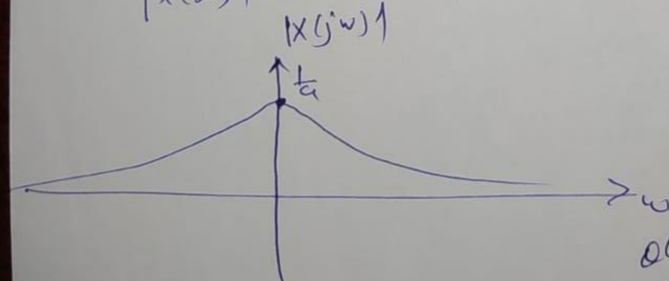
$$|X(0)| = \frac{1}{\sqrt{a^2}} = \frac{1}{a}$$

at $\omega = \infty$

$$|X(\infty)| = 0$$

at $\omega = -\infty$

$$|X(-\infty)| = 0$$



at $\omega = 0$

$\theta(0) = -\tan^{-1} 0 = 0$

at $\omega = a$

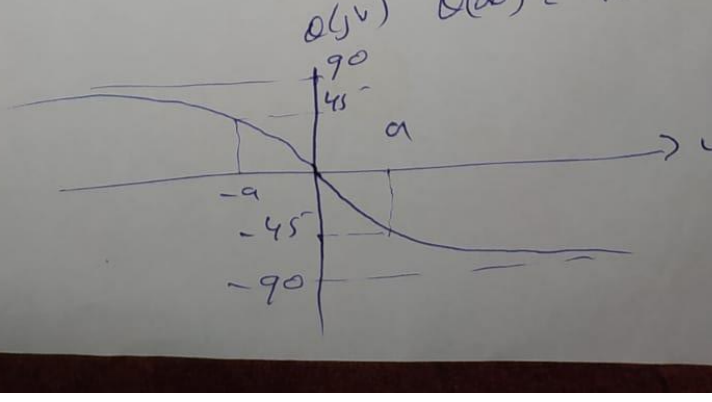
$\theta(a) = -\tan^{-1} 1 = -45^\circ$

at $\omega = -a$

$\theta(-a) = -\tan^{-1} -1 = 45^\circ$

at $\omega = \infty$

$\theta(\infty) = 90^\circ$



Ex 1 (20)

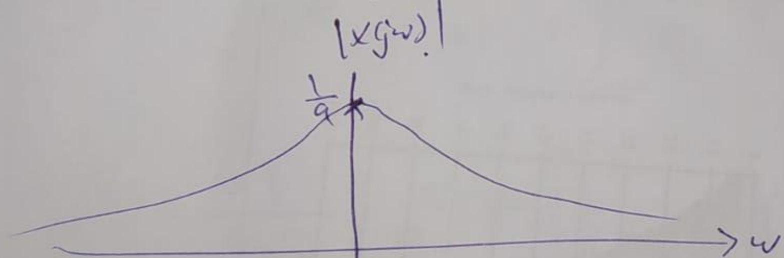
$$x(t) = e^{at} u(-t) \qquad a > 0. \qquad (1)$$

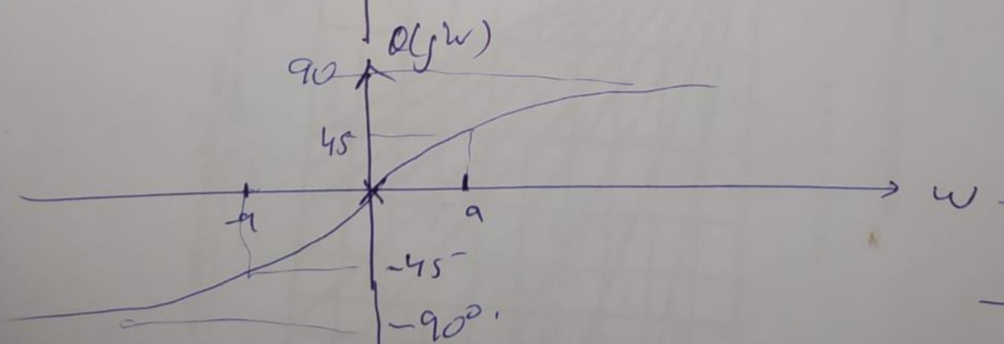
$$X(j\omega) = \int_{-\infty}^{0} e^{at} e^{-j\omega t} \, dt.$$

$$X(j\omega) = \int_{-\infty}^{0} e^{t(a-j\omega)} \, dt = \frac{1}{a-j\omega}\left[e^{t(a-j\omega)} \Big|_{-\infty}^{0} \right]$$

$$X(j\omega) = \frac{1}{a-j\omega}.$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \qquad \theta(j\omega) = \tan^{-1}\frac{\omega}{a}.$$





Ex 2 Find the Fourier Transform of:

$$x(t) = e^{-a|t|} \qquad a>0.$$

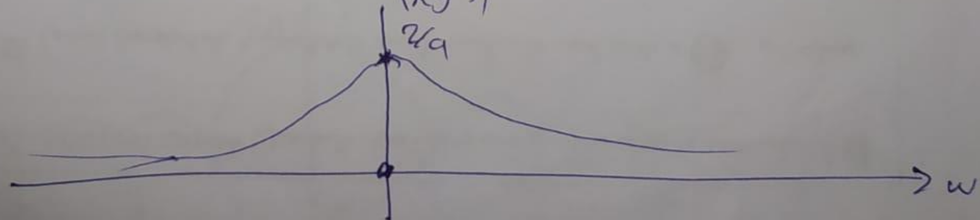
$$x(t) = \begin{cases} e^{-at} & t \geq 0. \\ e^{at} & t < 0. \end{cases}$$

$$X(j\omega) = \int_{0}^{\infty} e^{-at} e^{-j\omega t} \, dt + \int_{-\infty}^{0} e^{at} e^{-j\omega t} \, dt.$$

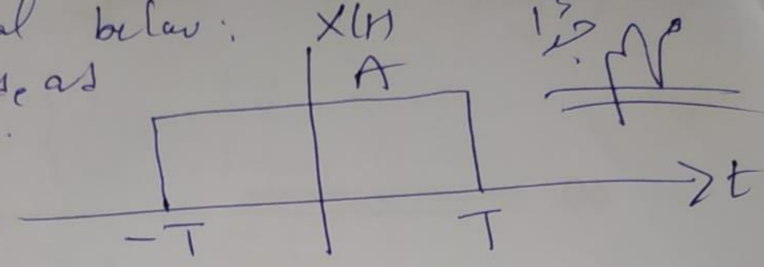
$$X(j\omega) = \int_{0}^{\infty} e^{-t(a+j\omega)} \, dt + \int_{-\infty}^{0} e^{t(a-j\omega)} \, dt$$

$$X(j\omega) = \frac{-1}{a+j\omega} \left[e^{-t(a+j\omega)} \Big|_{0}^{\infty} \right] + \frac{1}{a+j\omega} \left[e^{t(a+j\omega)} \Big|_{-\infty}^{0} \right].$$

$$X(j\omega) = \frac{1}{a+j\omega} + \frac{1}{a-j\omega} = \frac{2a}{a^2+\omega^2}.$$



EX3 Find the Fourier Transform of the signal below: draw the amplitude and phase spectrum.



$$X(j\omega) = \int_{-T}^{T} A e^{-j\omega t}\, dt = \frac{A}{-j\omega}\left[e^{-j\omega t} \Big|_{-T}^{T} \right].$$

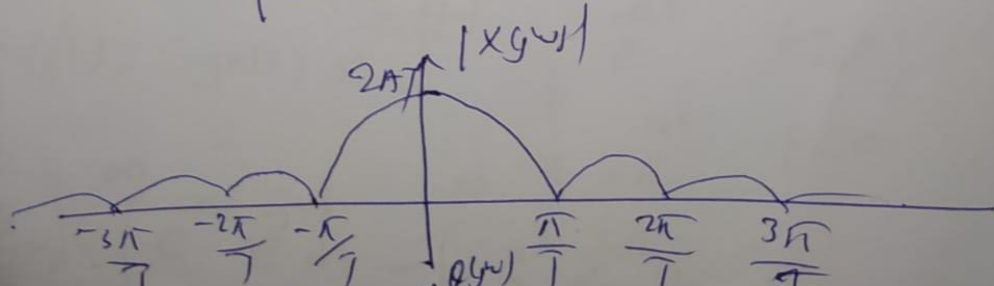
$$X(j\omega) = \frac{-A}{j\omega}\left[e^{-j\omega T} - e^{j\omega T} \right].$$

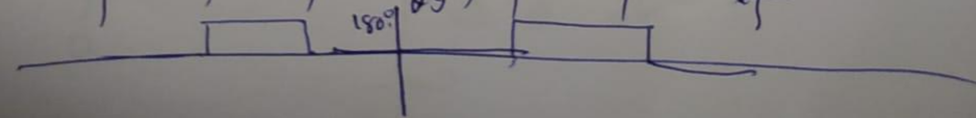
$$X(j\omega) = \frac{2A}{2j\omega}\left[e^{j\omega T} - e^{-j\omega T} \right] = \frac{2A}{\omega} \sin \omega T.$$

$$X(j\omega) = 2AT \frac{\sin \omega T}{\omega T} = 2AT \operatorname{Sinc} \omega T.$$

Sinc

$$|X(j\omega)| = 2AT\,|\operatorname{Sinc} \omega T|.$$





(4)

Ex 4 $x(t) = \text{Sgn}(t)$. find $X(j\omega)$.

$$\text{Sgn}(t) = \begin{cases} 1 & t \geq 0 \\ -1 & t < 0. \end{cases}$$

$$X(j\omega) = \int_{0}^{\infty} e^{-j\omega t}\,dt + \int_{-\infty}^{0} -e^{-j\omega t}$$

not integrable.

$\Rightarrow$ To find the fourier trasform of $x(t)$ we do the following: multiply $x(t)$ be $e^{-a|t|}$.

$$y(t) = e^{-a|t|}\,x(t)$$

$$\Rightarrow x(t) = \lim_{a \to 0} y(t).$$

$\Rightarrow$ find the fourier trasform for $y(t)$ and take the limit as $a \to 0$.

$$y(t) = \text{Sgn}(t)\,e^{-a|t|} = \begin{cases} e^{-at} & t \geq 0. \\ -e^{at} & t < 0. \end{cases}$$

$$Y(j\omega) = \int_{0}^{\infty} e^{-at}\,e^{-j\omega t}\,dt - \int_{-\infty}^{0} e^{at}\,e^{-j\omega t}\,dt$$

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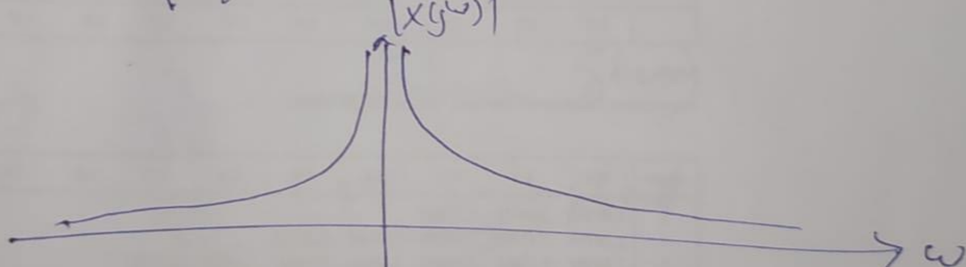
$$Y(j\omega) = \frac{1}{a+j\omega} - \frac{1}{a-j\omega}.$$

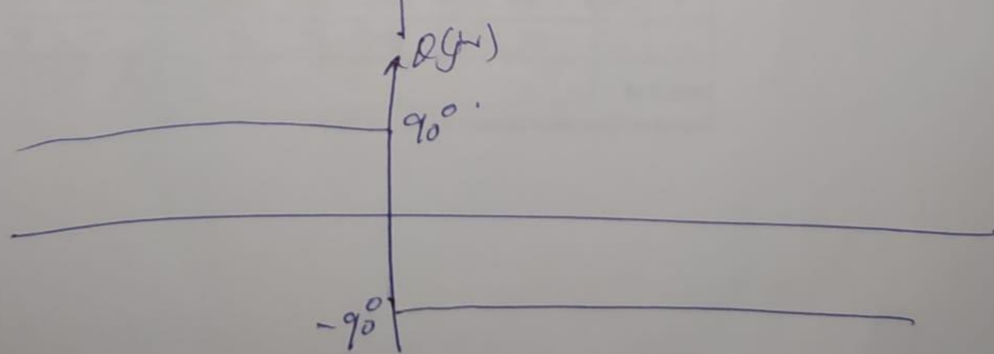
$$X(j\omega) = \lim_{a \to 0} Y(j\omega).$$

$$X(j\omega) = \frac{1}{j\omega} + \frac{1}{j\omega} = \frac{2}{j\omega}.$$

$$|X(j\omega)| = \frac{2}{|\omega|} \qquad X(j\omega) = \frac{-2j}{\omega}.$$

$$\theta(j\omega) = \begin{cases} -90^\circ & \omega > 0 \\ 90^\circ & \omega < 0. \end{cases}$$





Ex5 $x(t) = 1$

$X(j\omega) = ??$

Lets take the following signal.

$$Y(j\omega) = \delta(\omega).$$

Then:

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t}\, d\omega.$$

$$y(t) = \frac{1}{2\pi}$$

$$\Rightarrow \quad \frac{1}{2\pi} \xrightarrow{FT} \delta(\omega).$$

$$1 \xrightarrow{FT} 2\pi\delta(\omega)$$

$$\Rightarrow \boxed{X(j\omega) = 2\pi\,\delta\omega}$$

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Ex 6 $x(t) = u(t)$ find $X(j\omega)$.

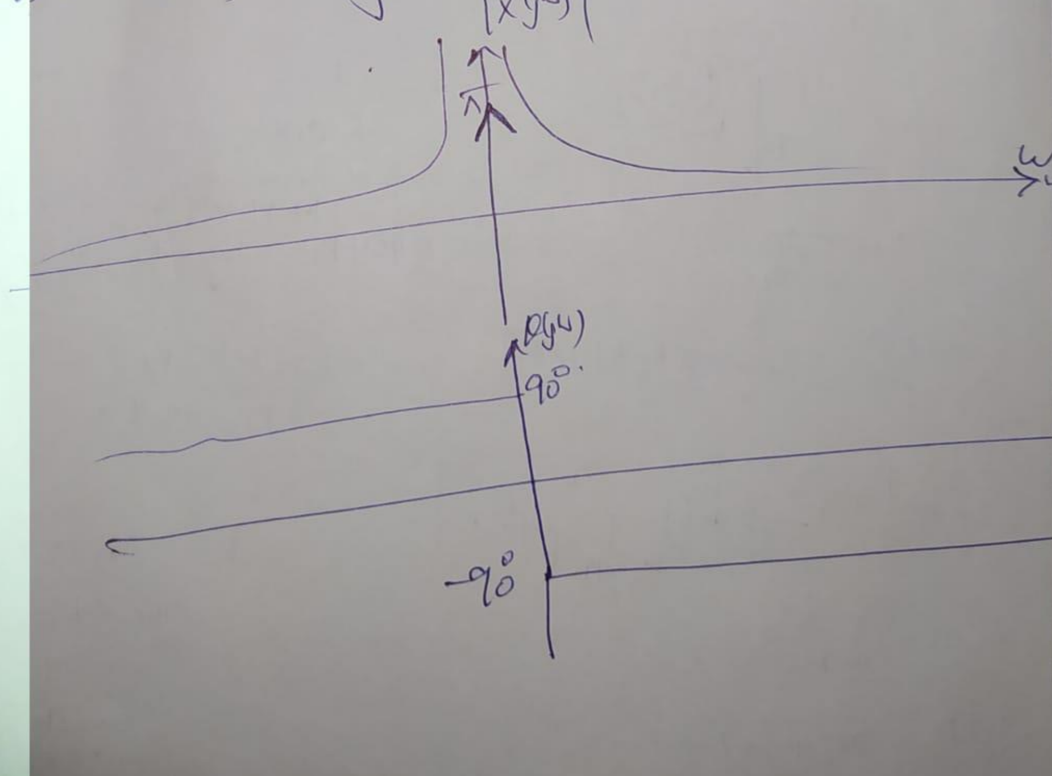
$x(t) = u(t)$ can be written as:

$$x(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t)$$

$$X(j\omega) = F\left\{\frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t)\right\}.$$

$$X(j\omega) = \frac{1}{2}F\{1\} + \frac{1}{2}F\{\mathrm{sgn}(t)\}.$$

$$X(j\omega) = \pi\delta(\omega) + \frac{1}{j\omega}.$$



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Properties of Fourier Transform:

1 Linearity.

$$x_1(t) \xrightarrow{FT} X_1(j\omega)$$
$$x_2(t) \xrightarrow{FT} X_2(j\omega).$$

Then the Fourier transform of the Signal $x(t)$:

$$x(t) = A\,x_1(t) + B\,x_2(t)$$

is $$X(j\omega) = A\,X_1(j\omega) + B\,X_2(j\omega).$$

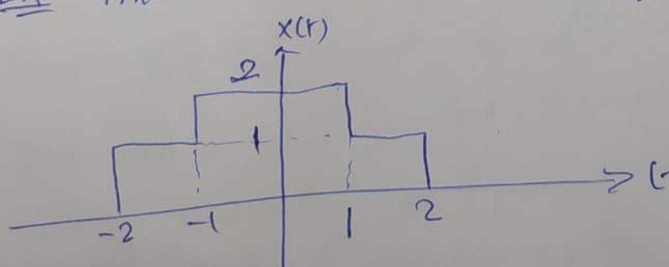
Proof : The Fourier Transform of $x(t)$ is:

$$X(j\omega) = \int_{-\infty}^{\infty} \left(A\,x_1(t) + B\,x_2(t)\right) e^{-j\omega t}\, dt.$$

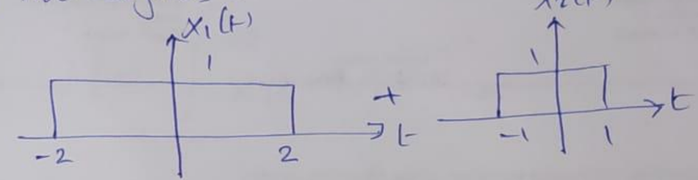
$$X(j\omega) = \int_{-\infty}^{\infty} A\,x_1(t)\,e^{-j\omega t}\, dt + \int_{-\infty}^{\infty} B\,x_2(t)\, e^{-j\omega t}\, dt.$$

$$X(j\omega) = A\,X_1(j\omega) + B\,X_2(j\omega).$$

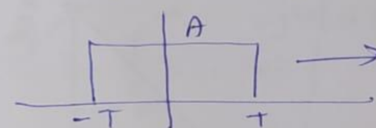
EX Find the Fourier Transform of the following Signal.



This signal can be written as the sum of two signals. (2)

$X(t) =$ 

we know that the fourier transform of the signal.



$\longrightarrow 2AT\,\mathrm{Sinc}\,\omega T$.

$\Rightarrow X_1(j\omega) = 2 \times 1 \times 2\,\mathrm{Sinc}\,2\omega$.

$X_2(j\omega) = 2 \times 1 \times 1\,\mathrm{Sinc}\,\omega$.

$\Rightarrow X(j\omega) = X_1(j\omega) + X_2(j\omega)$

$X(j\omega) = 4\,\mathrm{Sinc}\,2\omega + 2\,\mathrm{Sinc}\,\omega$.

(2) Time shifting Property.

$x(t) \xrightarrow{FT} X(j\omega)$

Then $x(t-t_0) \xrightarrow{FT} e^{-j\omega t_0} X(j\omega)$.

This means shift in time domain multiplication by an exponential in frequency domain.

The amplitude spectrum is the same.

$|X(j\omega)| = |e^{-j\omega t_0} X(j\omega)| = |X(j\omega)|$.

$|e^{-j\omega t_0}| = 1$.

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Proof The Fourier Transform of the Signal $x(t-t_0)$ is:

$$F\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j\omega t}\, dt.$$

Let $t - t_0 = u \Rightarrow t = u + t_0$.
$dt = du$.

$$\Rightarrow F\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(u)\, e^{-j\omega(u+t_0)}\, du.$$

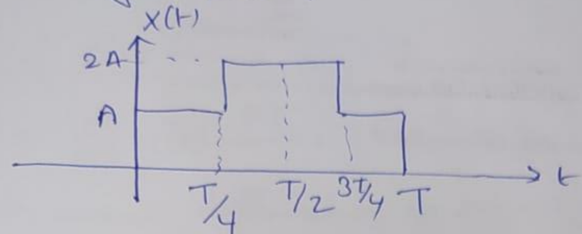
$$= \int_{-\infty}^{\infty} x(u)\, e^{-j\omega u}\, e^{j\omega t_0}\, du.$$

$$= e^{-j\omega t_0} \underbrace{\int_{-\infty}^{\infty} x(u)\, e^{-j\omega u}\, du}_{X(j\omega)}$$

$$\boxed{\Rightarrow F\{x(t-t_0)\} = e^{-j\omega t_0} X(j\omega)}.$$

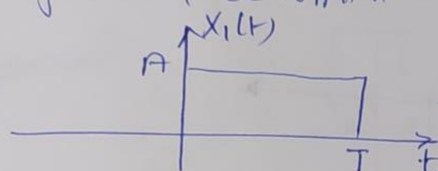
Ex

Ex Find The Fourier Transform of the Signal below; (4)



This signal can be written as:

$x(t) =$



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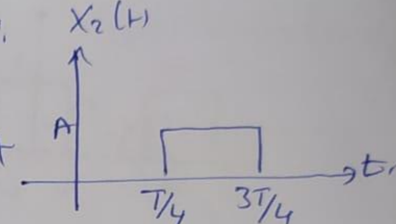
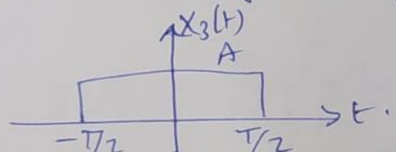
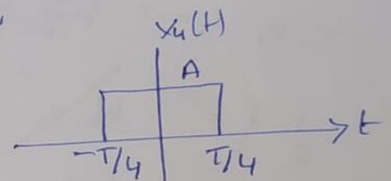


Lets take the following two signal





$X_4(j\omega) = 2AT/4 \, Sinc \, \omega T/4$

$X_3(j\omega) = 2 \times A \times T/2 \, Sinc \, \omega T/2$

$x_2(t) = x_4(t - T/2)$

$x_1(t) = x_3(t - T/2)$

$X_2(j\omega) = X_4(j\omega) \, e^{-j\omega T/2}$

$\Rightarrow X_1(j\omega) = X_3(j\omega) \, e^{-j\omega T/2}$

$\Rightarrow X(j\omega) = X_1(j\omega) + X_2(j\omega)$

$= \left[AT \, Sinc \, \omega T/2 + \frac{AT}{4} Sinc(\omega T/4) \right] e^{-j\omega T/2}$

(3) Differentiation in time domain. (5)

$$x(t) \xrightarrow{FT} X(j\omega)$$

Then $\frac{d}{dt}x(t) \xrightarrow{FT} j\omega X(j\omega)$.

This means differentiation in time $\Rightarrow$ multiplication by $j\omega$ in frequency domain.

In general:-

$$\frac{d^n x(t)}{dt^n} \xrightarrow{FT} (j\omega)^n X(j\omega).$$

Proof in general we find that:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega.$$

Then:

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\frac{d}{dt} e^{j\omega t}\, d\omega.$$

Then

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)\, e^{+j\omega t}\, d\omega.$$

$$\Rightarrow F\left\{\frac{dx(t)}{dt}\right\} = j\omega X(j\omega).$$

(6)

④ Differentiation in frequency domain:

$$x(t) \xrightarrow{FT} X(j\omega).$$

Then $t\,x(t) \xrightarrow{FT} j\frac{d}{d\omega} X(j\omega).$

Proof

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$$

Then:

$$\frac{d}{d\omega} X(j\omega) = \int_{-\infty}^{\infty} x(t) \frac{d}{dt} e^{-j\omega t}\, dt.$$

$$= \int_{-\infty}^{\infty} -jt\, x(t)\, e^{-j\omega t}\, dt.$$

divide by $-j$

$$\frac{1}{-j} \frac{d}{d\omega} X(j\omega) = \int_{-\infty}^{\infty} t\,x(t)\, e^{-j\omega t}\, dt.$$

$$j\frac{d}{d\omega} X(j\omega) = \int_{-\infty}^{\infty} t\,x(t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow t\,x(t) \xrightarrow{FT} j\frac{d}{d\omega} X(j\omega)$$

(5) Time integral:

$$x(t) \xrightarrow{FT} X(j\omega).$$

Then $\int_{-\infty}^{t} x(\tau)d\tau \xrightarrow{FT} \frac{1}{j\omega} X(j\omega).$

This is true when $X(j\omega)\Big|_{\omega=0} = 0.$

if $X(j\omega)\Big|_{\omega=0} \neq 0 \Rightarrow$

$$\int_{-\infty}^{t} x(\tau)\, d\tau \xrightarrow{FT} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

$$X(0) = X(j\omega)\Big|_{\omega=0}.$$

Proof $\quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega.$

Integrate both sides.

$$\int_{-\infty}^{t} x(\tau)d\tau = \left\{ \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) \int_{-\infty}^{t} e^{j\omega\tau} d\tau \right.$$

(6). Frequency shifting property.

$$x(t) \xrightarrow{FT} X(j\omega).$$

Then

$$x(t)e^{j\omega_0 t} \longleftrightarrow X(j(\omega-\omega_0)).$$

This means multiplication by an exponential in time domain results in a shift by ω_0 in frequency.

Proof HW.

EX (1) find the Fourier Transform of

$$x(t) = e^{j\omega_0 t}.$$

Soln: $x(t)$ can be written as:

$$x(t) = 1 \times e^{j\omega_0 t}.$$

$$1 \xrightarrow{FT} 2\pi\delta(\omega).$$

Then $1 . e^{j\omega_0 t} \xrightarrow{FT} 2\pi\delta(\omega-\omega_0)$.

$$\Rightarrow X(j\omega) = 2\pi\delta(\omega-\omega_0).$$

(2) Find the Fourier Transform of:

$$x(t) = e^{-j\omega_0 t}$$

$$x(t) = 1 . e^{-j\omega_0 t}$$

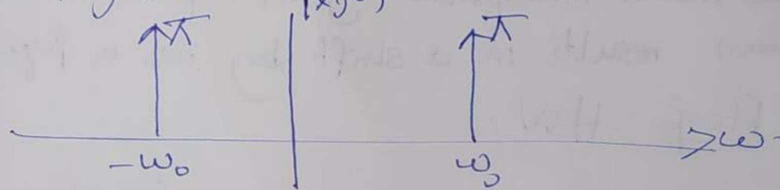
$$\Rightarrow X(j\omega) = 2\pi\delta(\omega+\omega_0).$$

5 (3) Find The Fourier Transform of

$x(t) = \cos\omega_0 t$.

Soln $x(t) = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$.

$X(j\omega) = \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$.

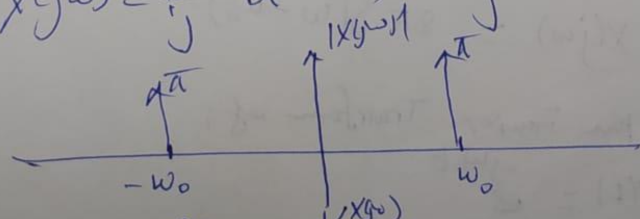


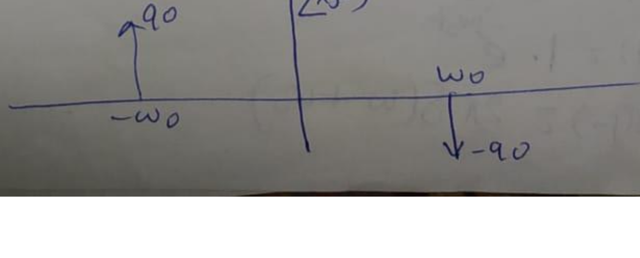
(4) $x(t) = \sin\omega_0 t$.

$x(t) = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$.

$x(t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$.

$X(j\omega) = \frac{\pi}{j}\delta(\omega-\omega_0) - \frac{\pi}{j}\delta(\omega+\omega_0)$.





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(5) $y(t) = x(t)\cos\omega_0 t$.

$$y(t) = \frac{1}{2}x(t)e^{-j\omega_0 t} + \frac{1}{2}x(t)e^{j\omega_0 t}.$$

$$\Rightarrow Y(j\omega) = \frac{1}{2}X(j(\omega-\omega_0)) + \frac{1}{2}X(j(\omega+\omega_0)),$$

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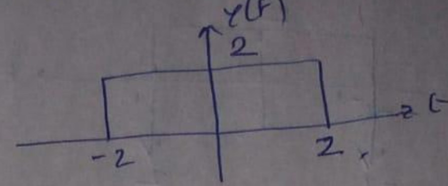
EX Find the Fourier transform of the following signal and draw the amplitude spectrum. (1)

$$x(t) = y(t)\cos 10t$$



Soln

$$x(t) = \frac{1}{2}y(t)\left[e^{j10t} + e^{-j10t}\right].$$

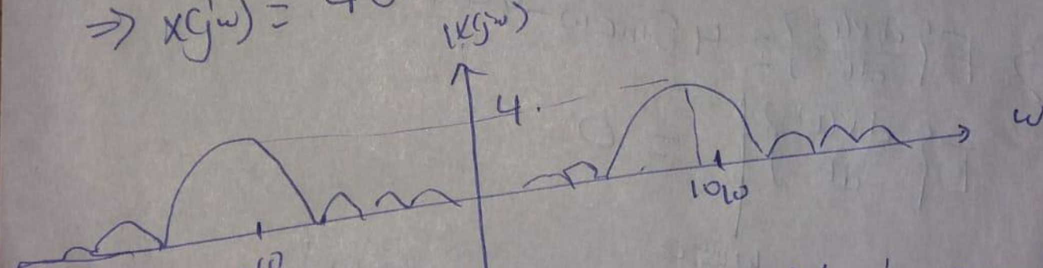
$$x(t) = \frac{1}{2}y(t)e^{j10t} + \frac{1}{2}y(t)e^{-j10t}.$$

$$X(j\omega) = \frac{1}{2}Y(j(\omega-10)) + \frac{1}{2}Y(j(\omega+10)).$$

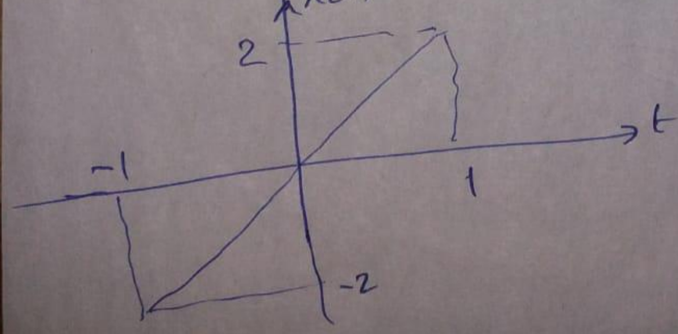
$$Y(j\omega) = 2\times2\times2\ \mathrm{Sinc}\,2\omega$$

$$= 8\ \mathrm{Sinc}\,2\omega.$$

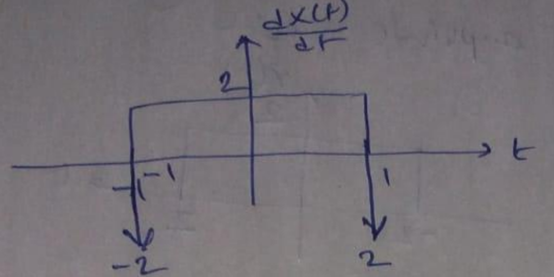
$$\Rightarrow X(j\omega) = 4\,\mathrm{Sinc}(2(\omega-\omega_0)) + 4\,\mathrm{Sinc}(2(\omega+\omega_0))$$



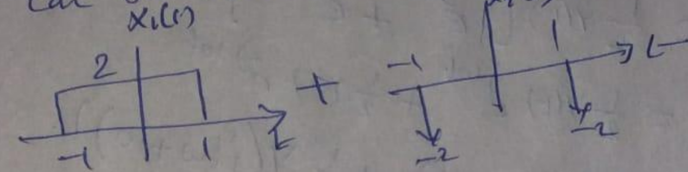
EX Find the Fourier Transform of the following signal



To solve this problem: first find $\frac{dx(t)}{dt}$.



So $\frac{dx}{dt}$ can be resolved into two signals.

$\frac{dx}{dt} =$ 

$X_1(j\omega) = 2 \times 2 \times 1 \, \mathrm{Sinc}\,\omega = 4\,\mathrm{Sinc}\,\omega$

$x_2(t) = -2\delta(t+1) - 2\delta(t-1)$

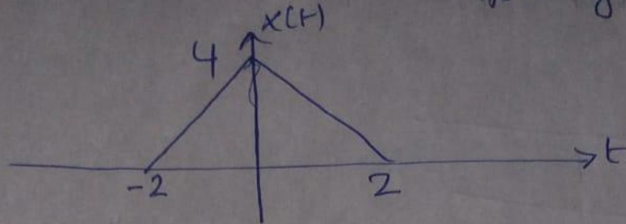
$X_2(j\omega) = -2e^{-j\omega} - 2e^{j\omega} = -4\cos\omega$

$\Rightarrow F\left\{\frac{dx(t)}{dt}\right\} = 4\,\mathrm{Sinc}\,\omega - 4\cos\omega$

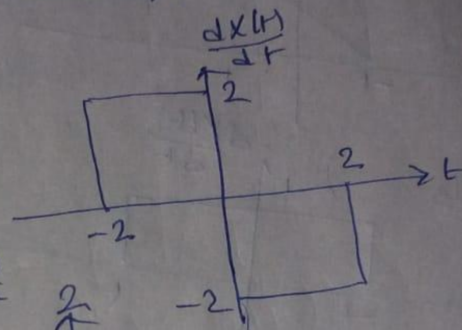
$F\left\{\frac{dx(t)}{dt}\right\}\Big|_{\omega=0} = 0$

$\Rightarrow X(j\omega) = \frac{1}{j\omega}\left(4\,\mathrm{Sinc}\,\omega - 4\cos\omega\right)$

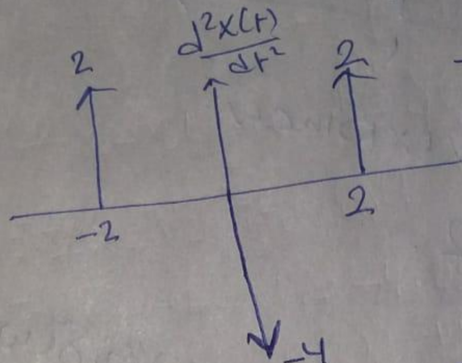
Ex Find the Fourier Transform of $x(t)$ (3)



Solu first find $\frac{dx(t)}{dt}$



$\frac{d^2x(t)}{dt^2} =$



$$\frac{d^2x(t)}{dt^2} = 2\delta(t+2) - 4\delta(t) + 2\delta(t-2)$$

$$F\left\{\frac{d^2x(t)}{dt^2}\right\} = 2e^{j2\omega} - 4 + 2e^{-j2\omega}.$$

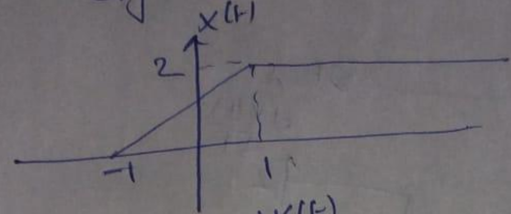
$$= 4\cos 2\omega - 4.$$

$$F\left\{\frac{d^2x(t)}{dt^2}\right\}\Big|_{\omega=0} = 0.$$

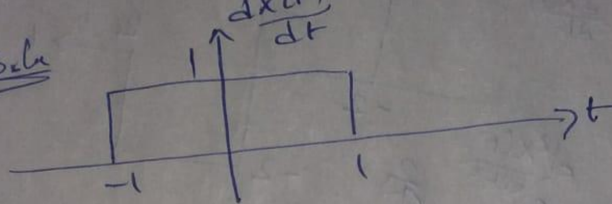
$$\Rightarrow F\left\{\frac{dx(t)}{dt}\right\} = \frac{1}{j\omega}(4\cos 2\omega - 4).$$

$$X(j\omega) = \frac{1}{j\omega}\frac{1}{j\omega}(4\cos 2\omega - 4).$$

Ex Find the Fourier Transform of the following signal. (4)



Soln



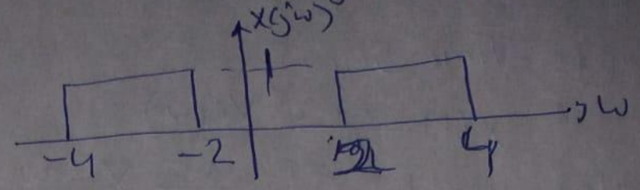
$F\left\{\frac{dx(t)}{dt}\right\} = 2 \times 1 \times 1 \,\mathrm{Sinc}\,\omega.$

$F\left\{\frac{dx(t)}{dt}\right\}\Big|_{\omega=0} = 2.$

$\Rightarrow X(j\omega) = \frac{1}{j\omega} 2\,\mathrm{Sinc}\,\omega + \pi X(0)\delta(\omega)$

$= \frac{2}{j\omega}\,\mathrm{Sinc}\,\omega + 2\pi\delta(\omega).$

EX Given the Fourier Transform of $x(t)$ as below: (5)



a) Draw the amplitude spectrum of

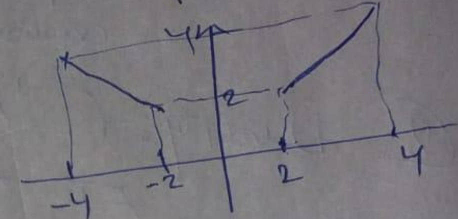
$y(t) = \frac{dx(t)}{dt}$.

$Y(j\omega) = j\omega X(j\omega)$

$\Rightarrow |Y(j\omega)| = |\omega| |X(j\omega)|$

$= |\omega| \times 1$

$= |\omega|$

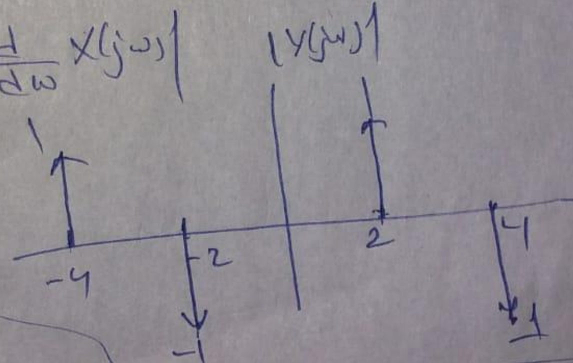


b) Draw the Amplitude Spectrum for

$y(t) = t\,x(t)$.

$Y(j\omega) = j\frac{d}{d\omega} X(j\omega)$

$|Y(j\omega)| = \left|\frac{d}{d\omega} X(j\omega)\right|$

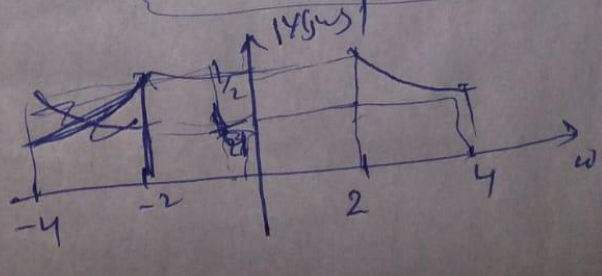


c) $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$.

$Y(j\omega) = \frac{1}{j\omega} X(j\omega)$

$|Y(j\omega)| = \frac{1}{|\omega|} |X(j\omega)|$

$= \frac{1}{|\omega|}$



(7) Time Scaling

$$x(t) \xrightarrow{FT} X(j\omega)$$

Then $x(at) \xrightarrow{FT} \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$.

This means compression in Time
expansion in frequency.

(8) Duality: (مهم جداً).

$$x(t) \xrightarrow{FT} X(j\omega)$$

Then $X(t) \xrightarrow{FT} 2\pi\, x(-j\omega)$.

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⑦ Time Scaling property

$$x(t) \xrightarrow{FT} X(j\omega)$$

Then $x(at) \xrightarrow{FT} \frac{1}{|a|} X(j\frac{\omega}{a})$.

This means if $x(t)$ is compressed in time
$\Rightarrow$ expansion in frequency.

⑧ Duality :

$$x(t) \xrightarrow{FT} X(j\omega)$$

$$X(t) \xrightarrow{FT} \cancel{2\pi x(j\omega)}$$

$$2\pi\, x(-j\omega).$$

EX The Fourier Transform of the signal $x(t) = e^{-a|t|}$ is $X(j\omega) = \frac{2a}{a^2+\omega^2}$.

find the Fourier Transform of $y(t) = \frac{2a}{a^2+t^2}$.

Solu : Using duality.

$$e^{-a|t|} \longrightarrow \frac{2a}{a^2+\omega^2}$$

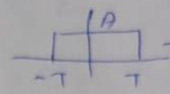
$$\frac{2a}{a^2+t^2} \longrightarrow 2\pi e^{-a|-\omega|}$$

$$Y(j\omega) = 2\pi e^{-a|\omega|}.$$

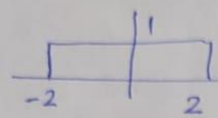
③ ②

Find the fourier Transform of

$x(t) = 2\,\mathrm{Sinc}\,2t$.



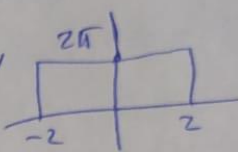
Soln using duality.



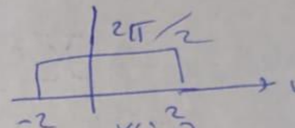
$\longrightarrow 2 \times 1 \times 2\,\mathrm{Sinc}\,2\omega$.

~~4 Sinc 2t~~

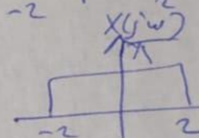
$4\,\mathrm{Sinc}\,2t \longrightarrow$



$\frac{4\,\mathrm{Sinc}\,2t}{2} \xrightarrow{FT}$



$\Rightarrow 2\,\mathrm{Sinc}\,2t \xrightarrow{FT}$



EX $x(t) = \frac{2}{t^2+1}$ find $X(j\omega)$.

using duality

$e^{-|t|} \longrightarrow \frac{2}{\omega^2+1}$

$\frac{2}{t^2+1} \longrightarrow 2\pi e^{-|-\omega|}$

$\Rightarrow X(j\omega) = 2\pi e^{-|\omega|}$.

9 Time reversal property.

$$x(t) \xrightarrow{FT} X(j\omega)$$

$$\Rightarrow \quad x(-t) \xrightarrow{FT} X(-j\omega).$$

<u>EX</u> Find the Fourier Transform of

$$x(t) = u(-t)$$

<u>soln</u>

$$F\{u(t)\} = \pi\,\delta(\omega) + \frac{1}{j\omega}.$$

Then $X(j\omega) = \pi\,\delta(\omega) - \dfrac{1}{j\omega}.$

10 Convolution property :-

For a LTI system we find that

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) \qquad y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)$$

$$y(t) = x(t) \circledast h(t).$$

in frequency domain:

$$Y(j\omega) = X(j\omega) \times H(j\omega).$$

Convolution in time $\Rightarrow$ multiplication in frequency.

11) Parsevals Theorem :-

(4)

In time domain we find the energy of the signal $x(t)$ as:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt.$$

Parsevals Theorem says that the Total energy of a signal can be found in frequency domain as:

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(jw)|^2 dw.$$

(12) Modulation property.

$$x(t) \xrightarrow{FT} X(jw).$$

$$p(t) \xrightarrow{FT} P(jw).$$

Then:

$$x(t)\ p(t) \xrightarrow{FT} \frac{1}{2\pi} X(jw) \circledast P(jw).$$

multiplication in time $\Rightarrow$ Convolution in frequency.

5

Ex Find the Fourier Transform of the following signal,

$$x(t) = \frac{d}{dt}\left(e^{-2t}u(t) \circledast e^{-5t}u(t)\right)$$

Sol.

$$e^{-2t}u(t) \xrightarrow{FT} \frac{1}{2+j\omega}$$

$$e^{-5t}u(t) \xrightarrow{FT} \frac{1}{5+j\omega},$$

$$\Rightarrow X(j\omega) = j\omega\left(\frac{1}{2+j\omega} \times \frac{1}{5+j\omega}\right).$$

Ex Find the Fourier Transform of.

$$x(t) = e^{-3t}u(t-1)$$

Solu

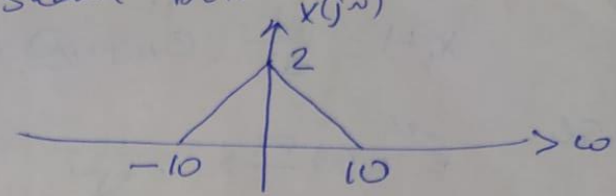
$$e^{-3t}u(t) \longrightarrow \frac{1}{3+j\omega}.$$

$$\Rightarrow e^{-3(t-1)}u(t-1) \rightarrow \frac{e^{-j\omega}}{3+j\omega}$$ time shifting property

$$\Rightarrow e^{-3t}e^{3}u(t-1) \rightarrow \frac{e^{-j\omega}}{3+j\omega}.$$

$$e^{-3t}u(t-1) \rightarrow \frac{1}{e^{3}}\,\frac{e^{-j\omega}}{3+j\omega}.$$

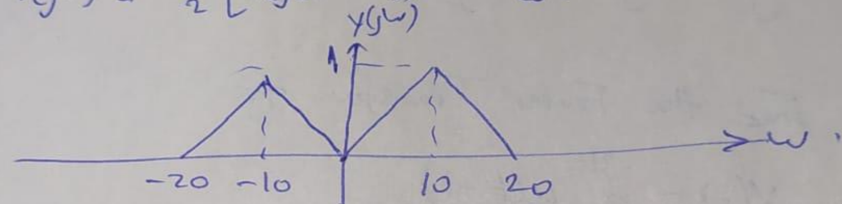
Ex Given the Fourier Transform of the signal $x(t)$ as shown below



find the Fourier Transform of

$y(t) = x(t) \cos 10t$ and draw its spectrum.

$$Y(j\omega) = \frac{1}{2}\left[X(j(\omega-10)) + X(j(\omega+10))\right]$$



(25) ①

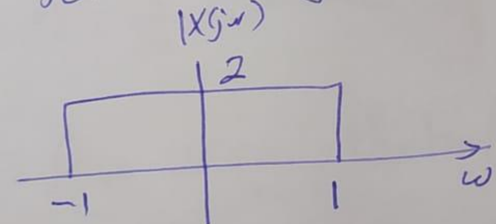
The inverse Fourier Transform:

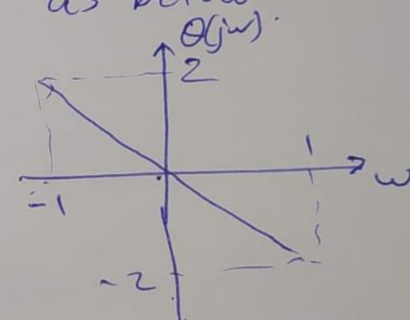
Given $X(j\omega)$ then $X(t)$ can be found in two ways:-

① By the inverse FT formula, that is you can find $X(t)$ as:

$$X(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega .$$

EX Given the amplitude and phase spectrum for the signal $X(t)$ as below:





find $X(t)$.

$$X(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$X(j\omega) = |X(j\omega)|\, e^{j\theta(j\omega)}$$

$\theta(j\omega) = -2\omega$.

$|X(j\omega)| = 1$

$$\Rightarrow X(j\omega) = 1 \times e^{-j2\omega} .$$

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$Y(j\omega) \Rightarrow \quad x(t) = \frac{1}{2\pi}\int_{-1}^{1} e^{-j2\omega} e^{j\omega t}\, d\omega.$ ②

and

$$x(t) = \frac{1}{2\pi}\int_{-1}^{1} e^{j\omega(-2+t)}\, d\omega.$$

$\Rightarrow$

$$x(t) = \frac{1}{2\pi}\frac{1}{j(-2+t)} e^{j\omega(-2+t)}\Big|_{-1}^{1}$$

$$x(t) = \frac{1}{2\pi}\frac{1}{j(-2+t)}\left[e^{j(-2+t)} - e^{-j(-2+t)}\right]$$

$$x(t) = \frac{1}{\pi}\frac{1}{(-2+t)}\sin(-2+t).$$

$$x(t) = \frac{1}{\pi}\operatorname{Sinc}(t-2).$$

② By using Partial fraction.

To simplify the fourier Transform $X(j\omega)$ to some simple function that we know its inverse Fourier Transform, using Partial fraction.

$$X(j\omega) = \frac{j\omega+2}{(j\omega)^2+4j\omega+3}$$

find $x(t)$.

(3)

$X(j\omega)$ can be written as:

$$X(j\omega) = \frac{j\omega+2}{(j\omega+3)(j\omega+1)}$$ Using Partial frac.

this can be written as:

$$\frac{j\omega+2}{(j\omega+3)(j\omega+1)} = \frac{A}{1+j\omega} + \frac{B}{3+j\omega}.$$

$$\Rightarrow j\omega+2 = A(3+j\omega) + B(1+j\omega).$$

for $j\omega = -1$

$$1 = 2A \Rightarrow A = 1/2.$$

for $j\omega = -3$

$$-1 = -2B \Rightarrow B = 1/2.$$

so $X(j\omega)$ can be written as:

$$X(j\omega) = \frac{1/2}{1+j\omega} + \frac{1/2}{3+j\omega},$$

$$\Rightarrow x(t) = \tfrac{1}{2}e^{-t}u(t) + \tfrac{1}{2}e^{-3t}u(t).$$

(4)

For a LTI system we know that the relation between input and output is $y(t) = x(t) \circledast h(t)$.

and in frequency domain:

$$Y(j\omega) = X(j\omega)\, H(j\omega)$$

$H(j\omega)$ is the Fourier Transform of the impulse response

$\Rightarrow$ $H(j\omega)$ can be written as:-

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

is called the transfer function of the system.

$H(j\omega)$ is the Fourier Transform of $h(t)$.

<u>EX</u> Using Fourier Transform find $x(t)$

$$x(t) = \frac{d}{dt}\left[e^{-2t}u(t) \circledast e^{-5t}u(t)\right].$$

Let $x_1(t) = e^{-2t}u(t) \circledast e^{-5t}u(t)$

$$\Rightarrow \quad X_1(j\omega) = \frac{1}{2+j\omega} \times \frac{1}{5+j\omega}$$

$$\Rightarrow X(j\omega) = \frac{j\omega}{(2+j\omega)(5+j\omega)}$$

$$X(j\omega) = \frac{A}{j\omega+2} + \frac{B}{j\omega+5}$$ (5)

$\Rightarrow \quad j\omega = A(j\omega+5) + B(j\omega+2)$

for $j\omega = -2$

$-2 = 3A \Rightarrow A = -2/3$

for $j\omega = -5$

$-5 = -3B \Rightarrow B = 5/3$

$$\Rightarrow X(j\omega) = \frac{-2/3}{j\omega+2} + \frac{5/3}{j\omega+5}$$

$$x(t) = \frac{5}{3} e^{-5t} u(t) - \frac{2}{3} e^{-2t} u(t)$$

Ex A LTI system have a transfer fuch.

$$H(j\omega) = \frac{1}{3+j\omega}.$$

for a particular input $x(t)$ it produces an output $y(t) = e^{-3t} u(t) + e^{-4t} u(t)$

find $x(t)$..

6

$$Y(j\omega) = \frac{1}{3+j\omega} - \frac{1}{4+j\omega} = \frac{4+j\omega-3-j\omega}{(3+j\omega)(4+j\omega)}$$

and we know that

$$Y(j\omega) = \frac{1}{(3+j\omega)(4+j\omega)}$$

$$Y(j\omega) = X(j\omega)H(j\omega).$$

$$\Rightarrow X(j\omega) = \frac{Y(j\omega)}{H(j\omega)} =$$

$$X(j\omega) = \frac{\frac{1}{\cancel{(3+j\omega)}(4+j\omega)}}{\frac{1}{\cancel{3+j\omega}}} = \frac{1}{4+j\omega}.$$

$$\Rightarrow x(t) = e^{-4t}u(t).$$

EX The relation between the input and output for a LTI system is given by.

$$5\frac{dy(t)}{dt} + 10y(t) = x(t).$$

~~find y(t)~~. find the impulse response of the system.

Soln. The impulse response of the system means that the output when the input is a unit impulse.

Take the Fourier Transform of the D.E. (7)

$$5j\omega Y(j\omega) + 10Y(j\omega) = X(j\omega)$$

$$\Rightarrow Y(j\omega) = \frac{X(j\omega)}{10+5j\omega} = \frac{X(j\omega)}{5(2+j\omega)}$$

$$x(t) = \delta(t) \Rightarrow X(j\omega) = 1$$

$$\Rightarrow Y(j\omega) = \frac{1}{5(2+j\omega)} = \frac{0.2}{2+j\omega}.$$

$\Rightarrow y(t) = 0.2e^{-2t}u(t)$ is the impulse response of the system.

for the same example find the step response of the system.

step response of the system is the output when the input is unit step.

that is: $x(t) = u(t)$

in general we find that:

$$Y(j\omega) = \frac{X(j\omega)}{5(2+j\omega)},$$

if $x(t) = u(t) \Rightarrow X(j\omega) = \pi\delta(\omega) + \frac{1}{j\omega}.$

$$Y(j\omega) = \frac{\pi\delta(\omega) + \frac{1}{j\omega}}{5(2+j\omega)}$$

$$Y(j\omega) = \frac{\pi\delta(\omega)}{5(2+j\omega)} + \frac{1}{5j\omega(2+j\omega)}$$

$$Y(j\omega) = \frac{\pi\delta(\omega)}{5(2+0)} + \frac{0.2}{j\omega(2+j\omega)}$$

$$Y(j\omega) = 0.1\pi\delta(\omega) + \frac{0.2}{j\omega(2+j\omega)}$$

$$\boxed{\pi\delta(\omega)X(j\omega) = \pi\delta(\omega)X(0)}$$ in general.

$$\boxed{\delta(t)X(t) = \delta(t)X(0)}$$

$$\frac{0.2}{j\omega(2+j\omega)} = \frac{A}{j\omega} + \frac{B}{j\omega+2}$$

$$0.2 = A(j\omega+2) + Bj\omega$$

when $j\omega = 0$

$0.2 = 2A \Rightarrow A = 0.1$

when $j\omega = -2$

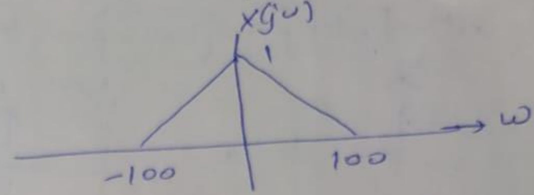
$0.2 = -2B \Rightarrow B = -0.1$

$$Y(j\omega) = \underbrace{0.1\pi\delta(\omega) + \frac{0.1}{j\omega}}_{0.1u(t)} - \frac{0.1}{2+j\omega}$$

$$Y(t) = 0.1u(t) - 0.1e^{-2t}u(t)$$ Step response

(26)

EX The Fourier Transform of $x(t)$ is shown below:



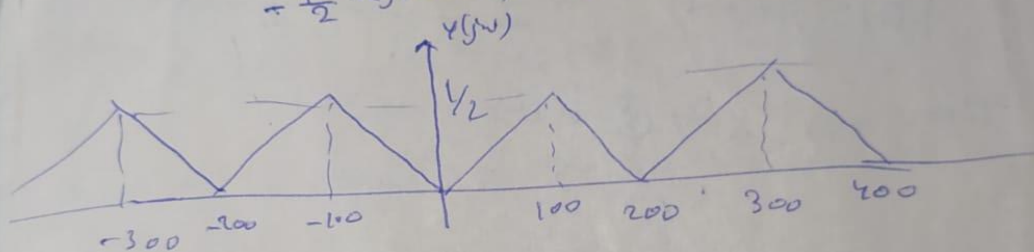
Find and draw the Amplitude spectrum of

a) $y(t) = x(t)[\cos 300t - \cos 100t]$.

$y(t) = x(t)\cos 300t - x(t)\cos 100t$

$Y(j\omega) = \frac{1}{2}X(j(\omega - 300)) + \frac{1}{2}X(j(\omega + 300))$.

$- \frac{1}{2}X(j(\omega - 300)) - \frac{1}{2}X(j\omega + 300)$

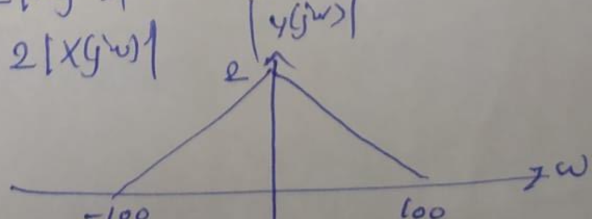


b) $y(t) = 2x(t-2)$

$Y(j\omega) = 2X(j\omega)e^{-j2\omega}$

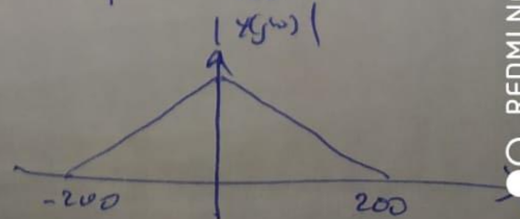
$|Y(j\omega)| = 2|X(j\omega)||e^{-j2\omega}|$

$= 2|X(j\omega)|$



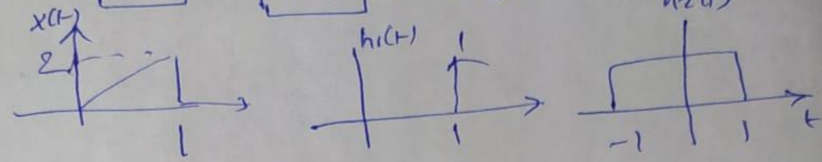
c) $y(t) = 2x(2t)$

$Y(j\omega) = 2\frac{1}{2}X(j\frac{\omega}{2})$



Ex For the system below find and draw $y(t)$ (2)

$x(t) \to \boxed{h_1(t)} \to \boxed{h_2(t)} \to y(t)$.



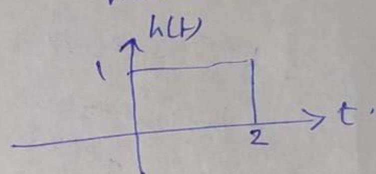
Solu This system is equivalent to.

$x(t) \to \boxed{h(t)} \to y(t)$

$h(t) = h_1(t) \circledast h_2(t)$

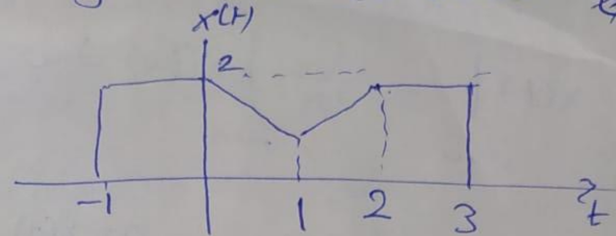
$h(t) = \delta(t-1) * h_2(t)$

$h(t) = h_2(t-1)$



Then $y(t) = x(t) \circledast h(t)$.

Ex Given An signal $x(t)$ as below. 3



a) Find $X(j\omega)$ evaluated at $\omega=0$.

in general we can find $X(j\omega)$ as follows,

$$X(j\omega)=\int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt.$$

at $\omega=0$

$$X(j0)=\int_{-\infty}^{\infty} x(t)\,e^{-j0t}\,dt. = \int_{-\infty}^{\infty} x(t)\,dt.$$

and this is equal to the area under $x(t)$.

$$\text{area} = 4\times 2 - 1 = \underline{\underline{7}}$$

So $X(j\omega)\Big|_{\omega=0} = 7.$

b) Find $\int_{-\infty}^{\infty} X(j\omega)\,d\omega$.

in general we can find $x(t)$ as follows:

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\,e^{j\omega t}\,d\omega.$$

So, at $t=0$

(5) $t = 0$.

(4)

$$x(t)\Big|_{t=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, d\omega .$$

$$\Rightarrow \int_{-\infty}^{\infty} X(j\omega)\, d\omega = 2\pi X(0) = 2\pi \times 2 = 4\pi$$

c) find $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$.

In general we find that the Total Energy of a Signal is:

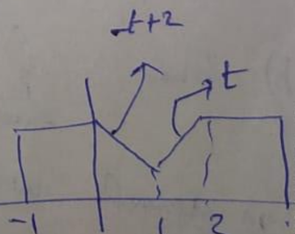
$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

in time domain

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

$$\Rightarrow \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} |x(t)|^2 dt .$$

$$\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = 2\pi\left[\int_{-1}^{0} 4\, dt + \int_{0}^{1} (2-t)^2 dt + \int_{1}^{2} t^2 dt + \int_{2}^{3} 4\, dt\right.$$



Ex for the system below find $y(t)$.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$x(t) = \cos(2\pi t) + \sin 6\pi t$.

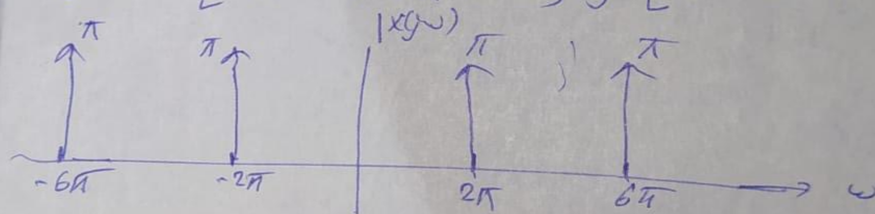
$h(t) = 4\,\mathrm{Sinc}(4\pi t)$.

Soln $Y(j\omega) = X(j\omega) \times H(j\omega)$.

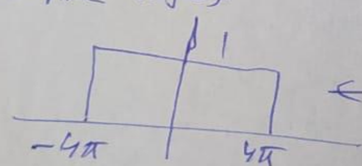
$$\cos 2\pi t \overset{FT}{\longleftrightarrow} \pi\left[\delta(\omega-2\pi)+\delta(\omega+2\pi)\right]$$

$$\sin 6\pi t \overset{FT}{\longleftrightarrow} -j\pi\left[\delta(\omega+6\pi)-\delta(\omega-6\pi)\right].$$

$$X(j\omega) = \pi\left[\delta(\omega-2\pi)+\delta(\omega+2\pi)\right] - j\pi\left[\delta(\omega+6\pi)-\delta(\omega-6\pi)\right.$$



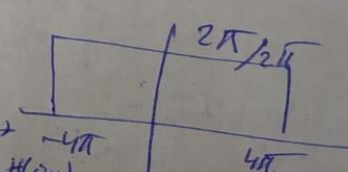
To find $H(j\omega)$.



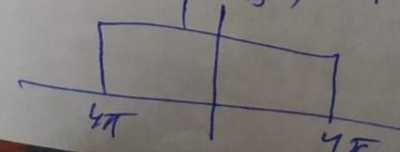
$\longleftrightarrow 2 \times 1 \times 4\pi\,\mathrm{Sinc}\,4\pi\omega$.

$8\pi\,\mathrm{Sinc}(4\pi\omega)$.

$\frac{8\pi}{2\pi}\,\mathrm{Sinc}(4\pi t) \longrightarrow$

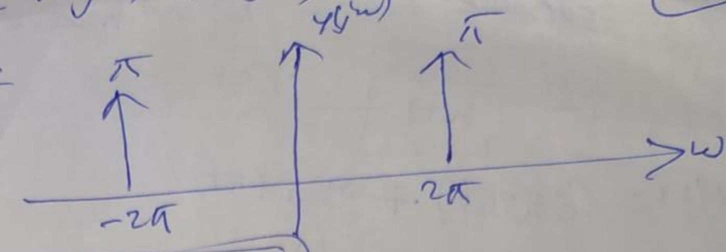


$\Rightarrow$



$$Y(j\omega) = H(j\omega) \times X(j\omega)$$

$=$



$$\boxed{y(t) = \cos 2\pi t}$$

EX The relation between the input and output for a LTI system is given by:

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = 6x(t).$$

find the step response of the system.

Soln Take the Fourier Transform of the equation

$$(j\omega)^2 Y(j\omega) + 5j\omega\, Y(j\omega) + 6Y(j\omega) = 6X(j\omega).$$

$$\Rightarrow Y(j\omega) = \frac{6X(j\omega)}{(j\omega)^2 + 5j\omega + 6}.$$

$$Y(j\omega) = \frac{6X(j\omega)}{(j\omega+3)(j\omega+2)}$$