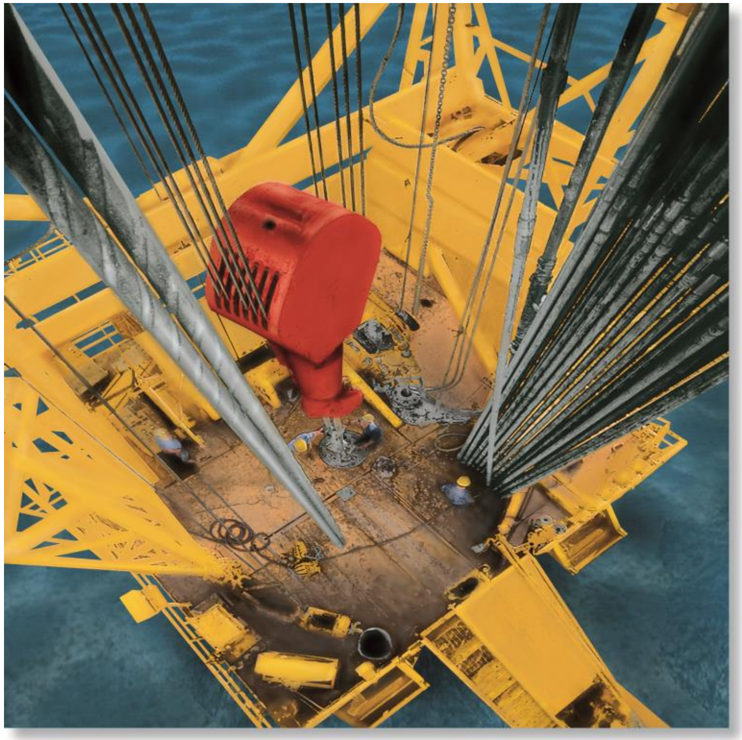
CHAPTER OBJECTIVES



- Determine deformation of axially loaded members
- Develop a method to find support reactions when it cannot be determined from equilibrium equations (statically indeterminated problem)
- Analyze the effects of thermal stress and stress concentrations.

CHAPTER OUTLINE

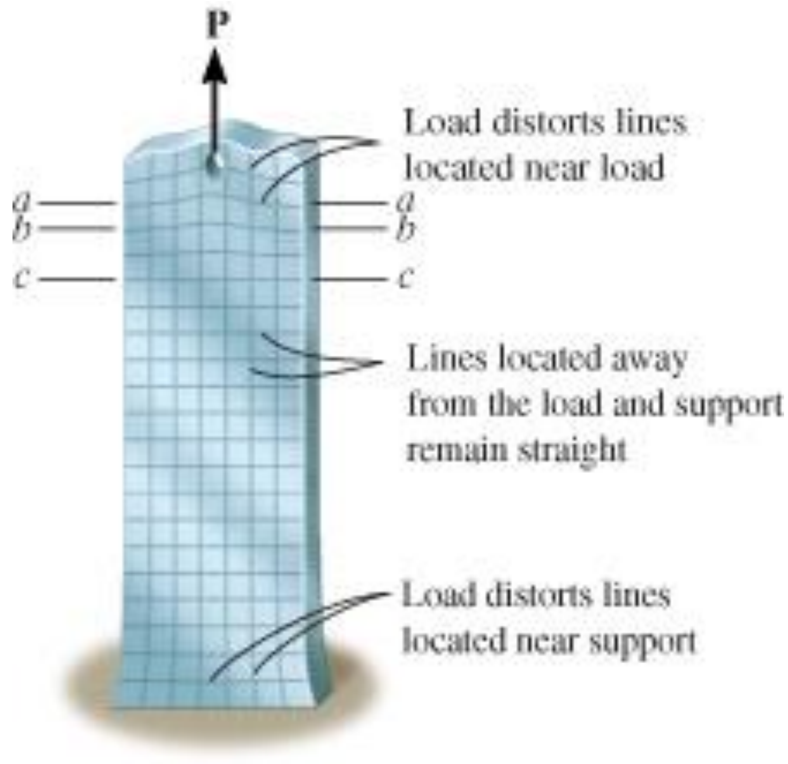
1. Saint-Venant's Principle
2. Elastic Deformation of an Axially Loaded Member
3. Principle of Superposition
4. Statically Indeterminate Axially Loaded Member
5. Force Method of Analysis for Axially Loaded Member
6. Thermal Stress
7. Stress Concentrations

4.1 SAINT-VENANT'S PRINCIPLE

- Localized deformation occurs at each end, and the deformations decrease as measurements are taken further away from the ends

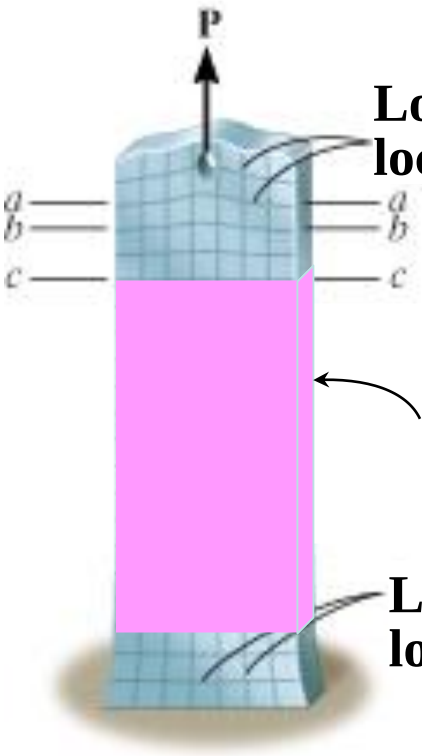
- At section *c-c*, stress reaches almost uniform value as compared to *a-a*, *b-b*

- *c-c* is sufficiently far enough away from **P** so that localized deformation "vanishes", i.e., minimum distance



(a)

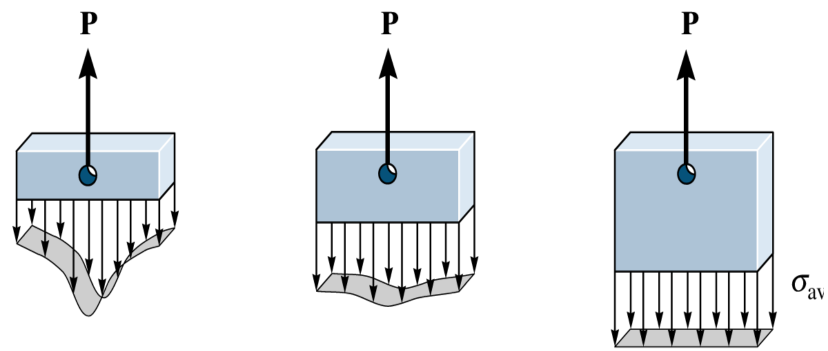
4.1 SAINT-VENANT'S PRINCIPLE



Load distorts line located near load

$\sigma_{avg} = \dfrac{P}{A}$

Load distorts line located near support



Section *a-a*

$\sigma_{avg} \neq \dfrac{P}{A}$

Section *b-b*

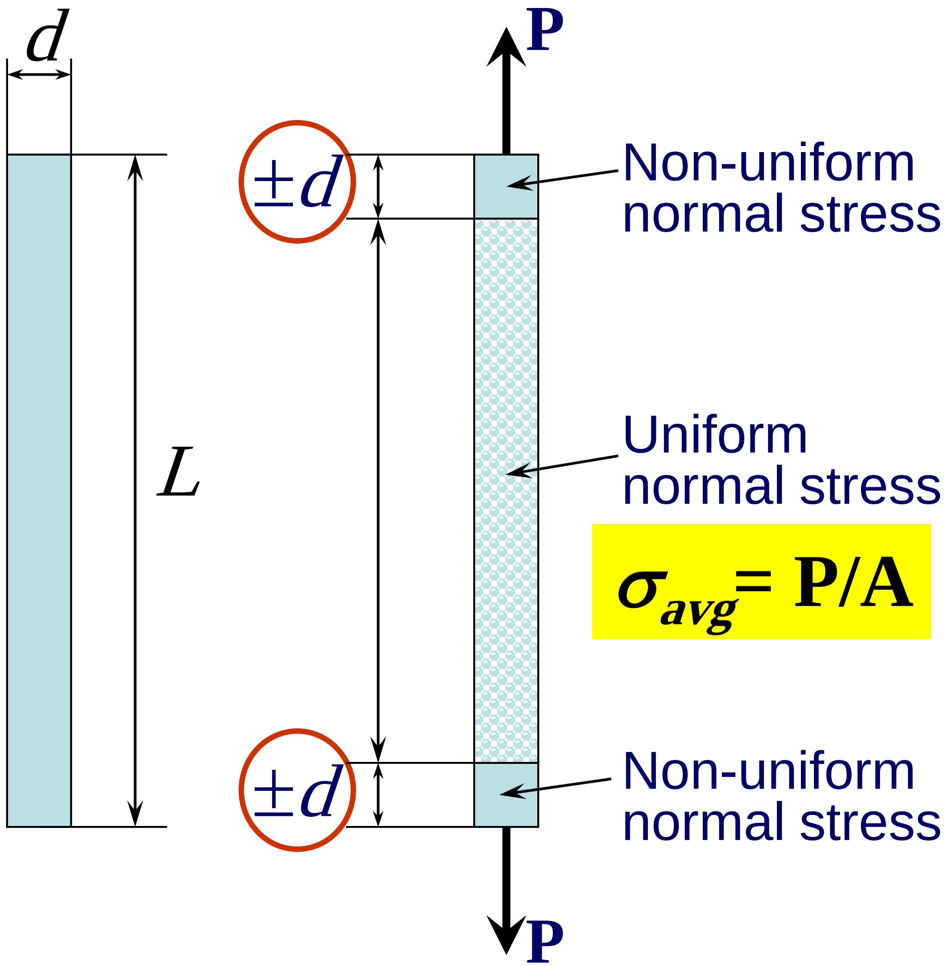
$\sigma_{avg} \neq \dfrac{P}{A}$

Section *c-c*

$\sigma_{avg} = \dfrac{P}{A}$

The pink area is the area of the uniform average normal stress, or $\sigma_{avg} = P/A$

4.1 SAINT-VENANT'S PRINCIPLE



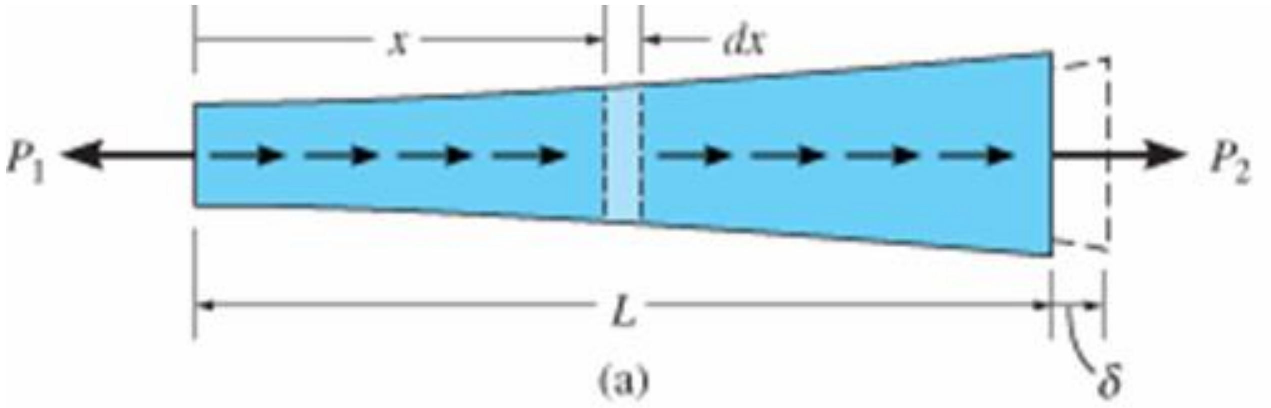
$$\sigma_{avg} = P/A$$

A long bar/rod, L >> d, is subjected to a tensile force acting at its centroidal axis

This behavior discovered by Barré de Saint-Venant in 1855, this the name of the principle

Saint-Venant Principle states that *localized effects* caused by any load acting on the body, will *dissipate/smooth out* within regions that are *sufficiently removed* from location of load

4.2 ELASTIC DEFORMATION OF AN AXIALLY LOADED MEMBER



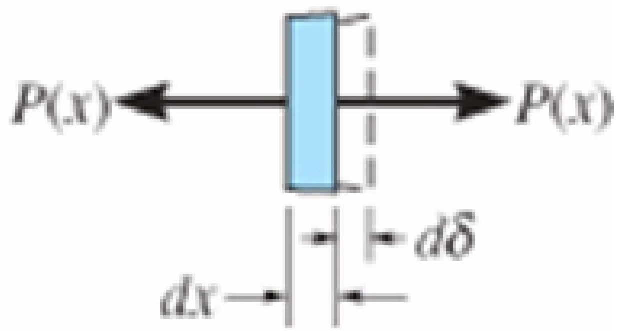
- **Relative displacement (δ) of one end of bar with respect to other end caused by this loading**

- **Applying Saint-Venant's Principle, ignore localized deformations at points of concentrated loading and where cross-section suddenly changes**

4.2 ELASTIC DEFORMATION OF AN AXIALLY LOADED MEMBER

Use method of sections,
and draw free-body diagram

$$\sigma = \frac{P(x)}{A(x)} \qquad \varepsilon = \frac{d\delta}{dx}$$



(b)

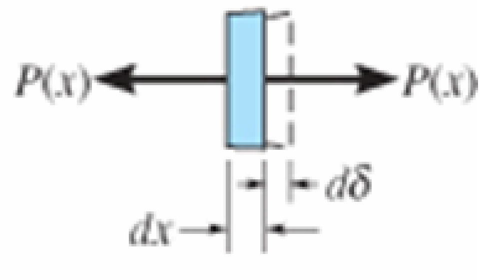
- Assume proportional limit not exceeded, thus apply Hooke's Law

$$\sigma = E\varepsilon$$

$$\frac{P(x)}{A(x)} = E\left(\frac{d\delta}{dx}\right) \Rightarrow d\delta = \frac{P(x)\,dx}{A(x)\,E}$$

4.2 ELASTIC DEFORMATION OF AN AXIALLY LOADED MEMBER

$$\delta = \int_0^L \frac{P(x)\,dx}{A(x)\,E}$$



(b)

δ = displacement of one point relative to another point

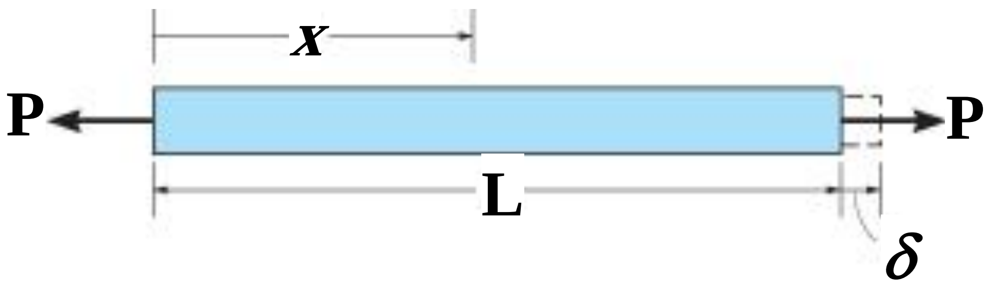
L = distance between the two points

$P(x)$ = internal axial force at the section, located a distance x from one end

$A(x)$ = cross-sectional area of the bar, expressed as a function of x

E = modulus of elasticity for material

Constant load and cross-sectional area

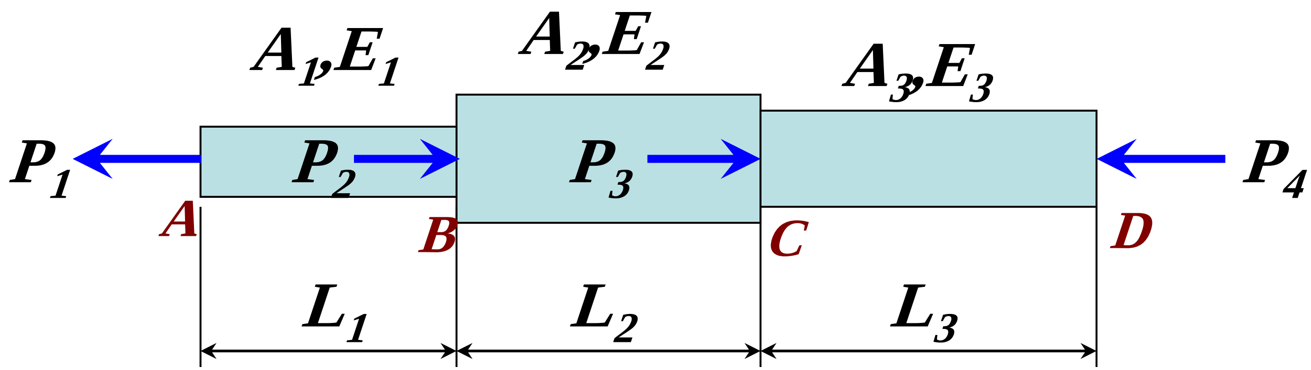


$$\delta = \int_0^L \frac{P(x)\,dx}{A(x)\,E}$$

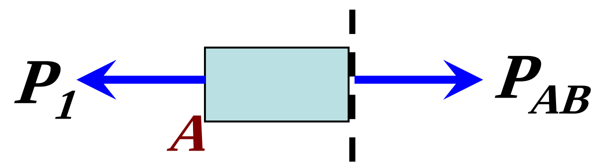
***P, A*, and *E* are constant**

$$\delta = \frac{PL}{AE}$$

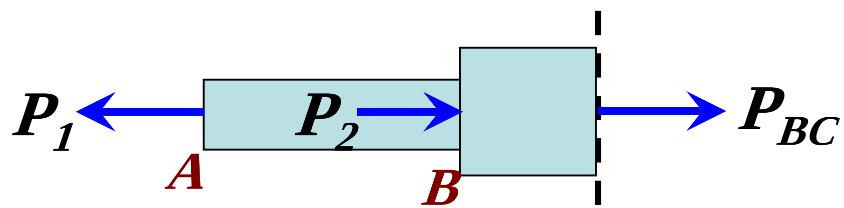
4.2 ELASTIC DEFORMATION OF AN AXIALLY LOADED MEMBER



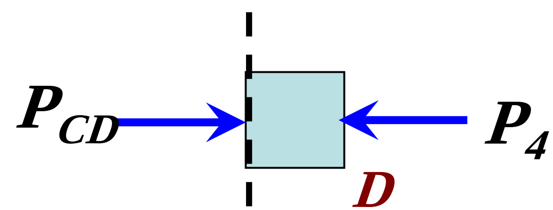
Internal force in segment *AB*:



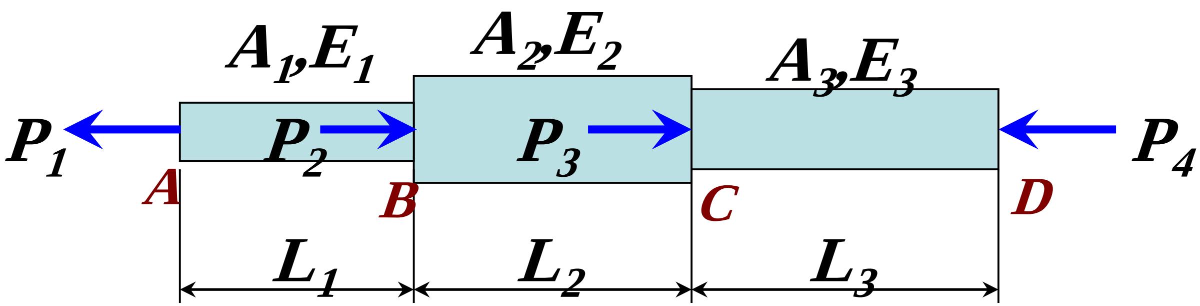
Internal force in segment *BC*:



Internal force in segment *CD*:



4.2 ELASTIC DEFORMATION OF AN AXIALLY LOADED MEMBER



Displacement in segment *AB*: $\delta_{AB} = \delta_1 = \dfrac{P_{AB} L_1}{A_1 E_1}$

Displacement in segment *BC*: $\delta_{BC} = \delta_2 = \dfrac{P_{BC} L_2}{A_2 E_2}$

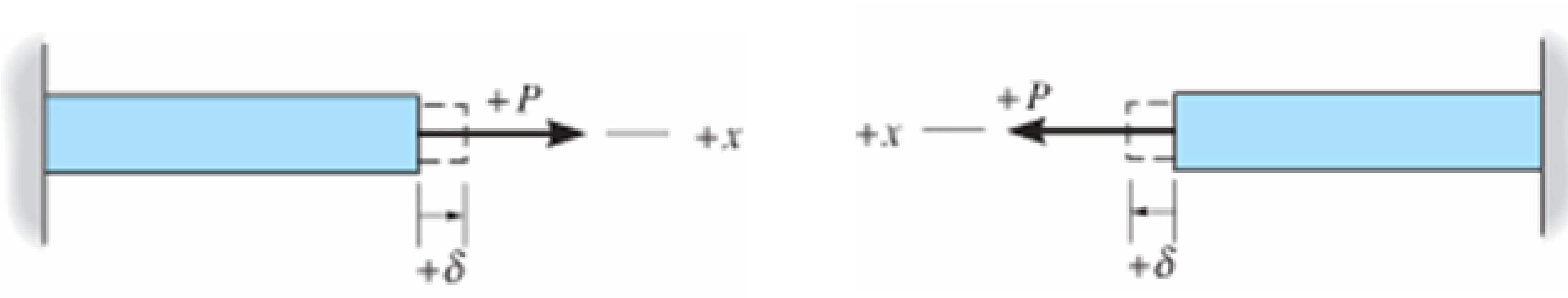
Displacement in segment *CD*: $\delta_{CD} = \delta_3 = \dfrac{P_{CD} L_3}{A_3 E_3}$

The displacement of one end of the bar with respect to the other is

$$\delta = \sum \frac{PL}{AE} = \delta_1 + \delta_2 + \delta_3$$

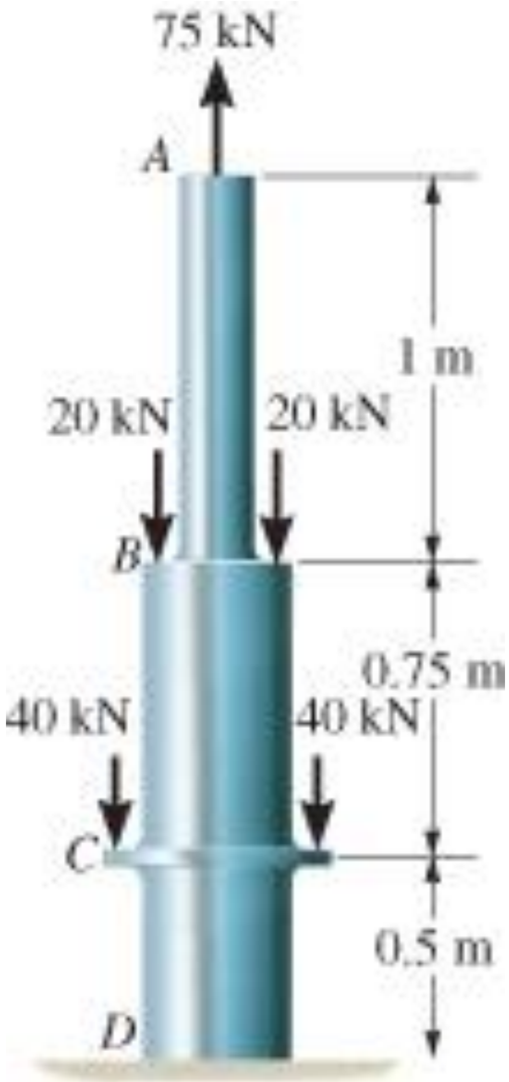
4.2 ELASTIC DEFORMATION OF AN AXIALLY LOADED MEMBER

Sign convention



Sign	Forces	Displacement
Positive (+)	Tension	Elongation
Negative (−)	Compression	Contraction

EXAMPLE 4-1



(a)

Composite A-36 steel bar shown made from two segments AB and BD. Area $A_{AB} = 600\ \text{mm}^2$ and $A_{BD} = 1200\ \text{mm}^2$.

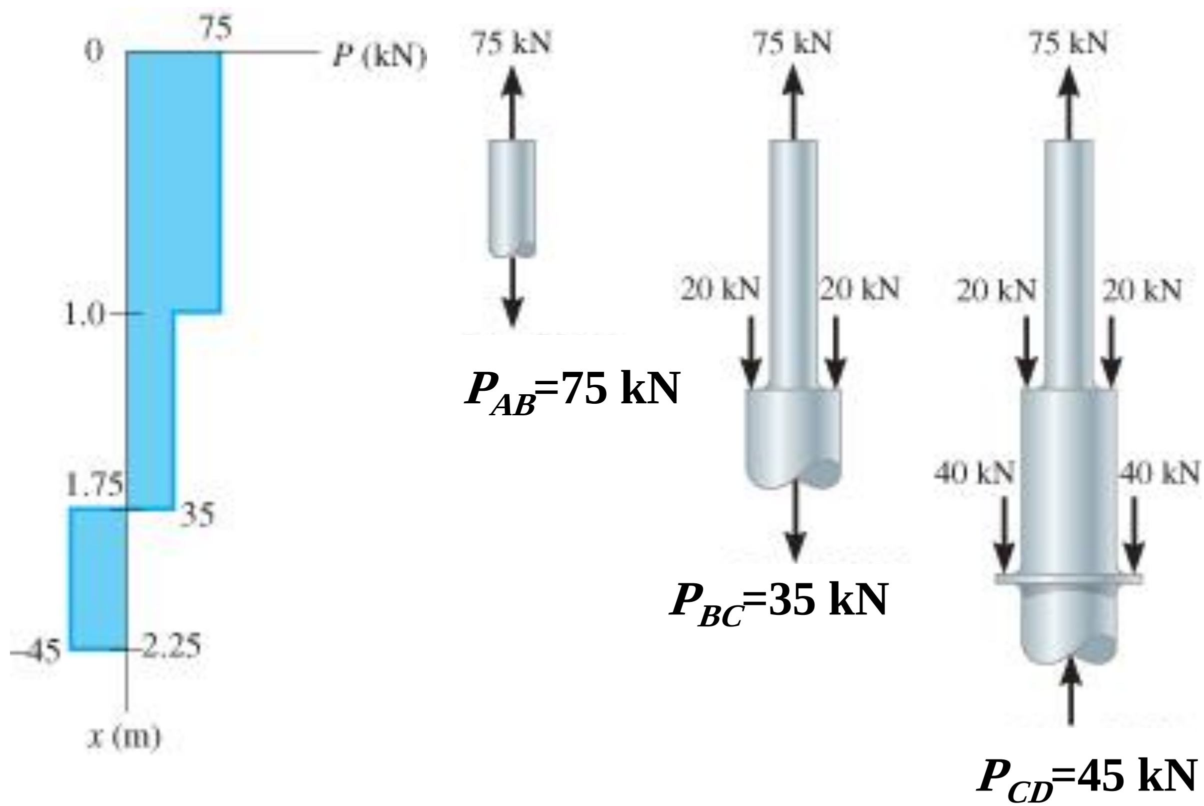
Determine the vertical displacement of end A and displacement of B relative to C.

EXAMPLE 4-1

Internal force

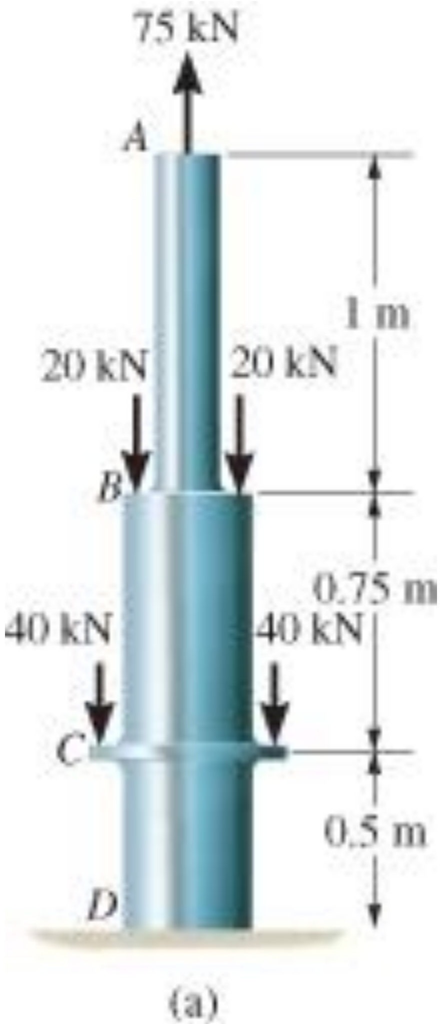
Due to external loadings, internal axial forces in segments *AB*, *BC* and *CD* are different.

Apply method of sections and equation of vertical force equilibrium as shown. Variation is also plotted.



(c)

EXAMPLE 4-1



(a)

Displacement

From tables, $E_{st} = 210(10^3)$ MPa

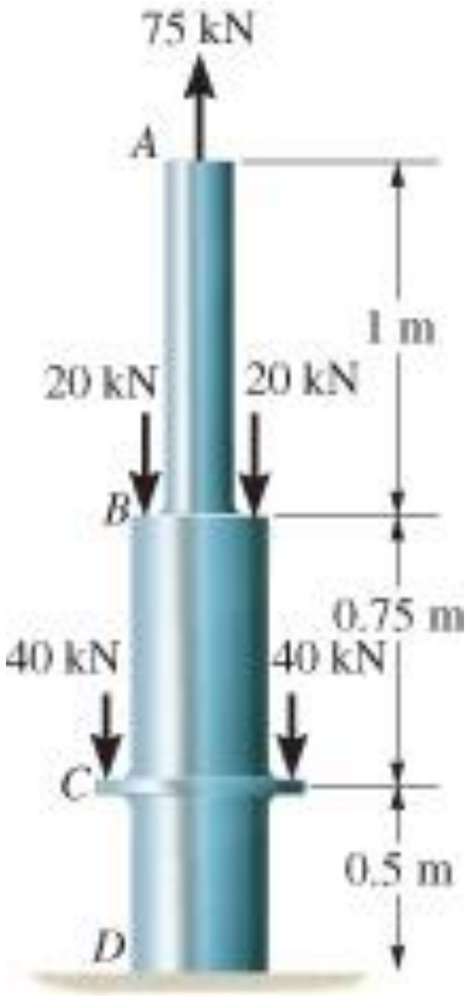
Vertical displacement of *A* relative to fixed support *D* is

$$\delta_{A/D} = \sum \frac{PL}{AE} = \frac{P_{AB} L_{AB}}{A_{AB} E} + \frac{P_{BC} L_{BC}}{A_{BC} E} - \frac{P_{CD} L_{CD}}{A_{CD} E}$$

Substituting the appropriate value into the above equation, we have

$\Rightarrow$ $\delta_{A/D} = +0.61$ mm

EXAMPLE 4-1



(a)

Since the result is positive, the bar elongates and, therefore, the displacement at *A* is upward

Displacement between *B* and *C*,

$$\delta_{B/C} = \frac{P_{BC} L_{BC}}{A_{BC} E}$$

$$= \frac{[+35 \text{ kN}](0.75 \text{ m})(10^6)}{[1200 \text{ mm}^2 (210)(10^3) \text{ kN/m}^2]}$$

$$= +0.104 \text{ mm}$$

Here, *B* moves away from *C*, since segment elongates

4.3 PRINCIPLE OF SUPERPOSITION

- After subdividing the load into components, the ***principle of superposition*** states that the resultant stress or displacement at the point can be determined by first finding the stress or displacement caused by each component load *acting separately* on the member.

- Resultant stress/displacement determined algebraically by adding the contributions of each component

4.3 PRINCIPLE OF SUPERPOSITION

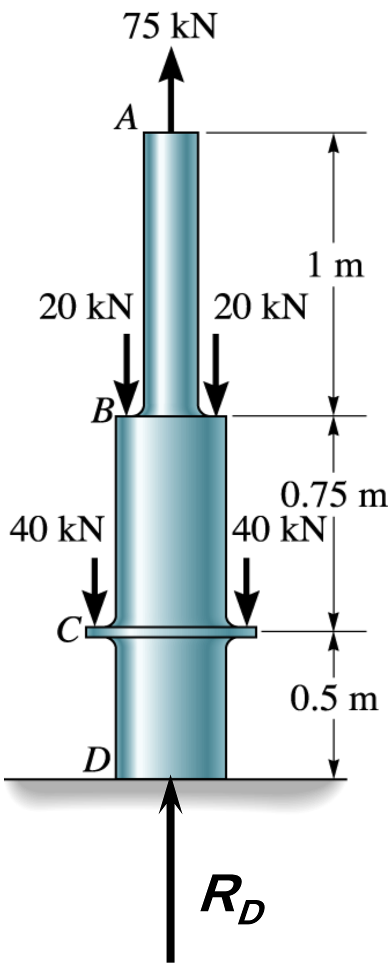
Conditions

1. The loading must be linearly related to the stress or displacement that is to be determined.
2. The loading must not significantly change the original geometry or configuration of the member

When to ignore deformations?

- Most loaded members will produce deformations so small that change in position and direction of loading will be insignificant and can be *neglected*
- Exception to this rule is a column carrying axial load, discussed in Chapter 13

4.4 STATICALLY INDETERMINATE AXIALLY LOADED MEMBER



Statically determinate : When the force equilibrium equation applied on a structure/bar is sufficient to find the reaction force at the support

We can find R_D using the force equilibrium equation.

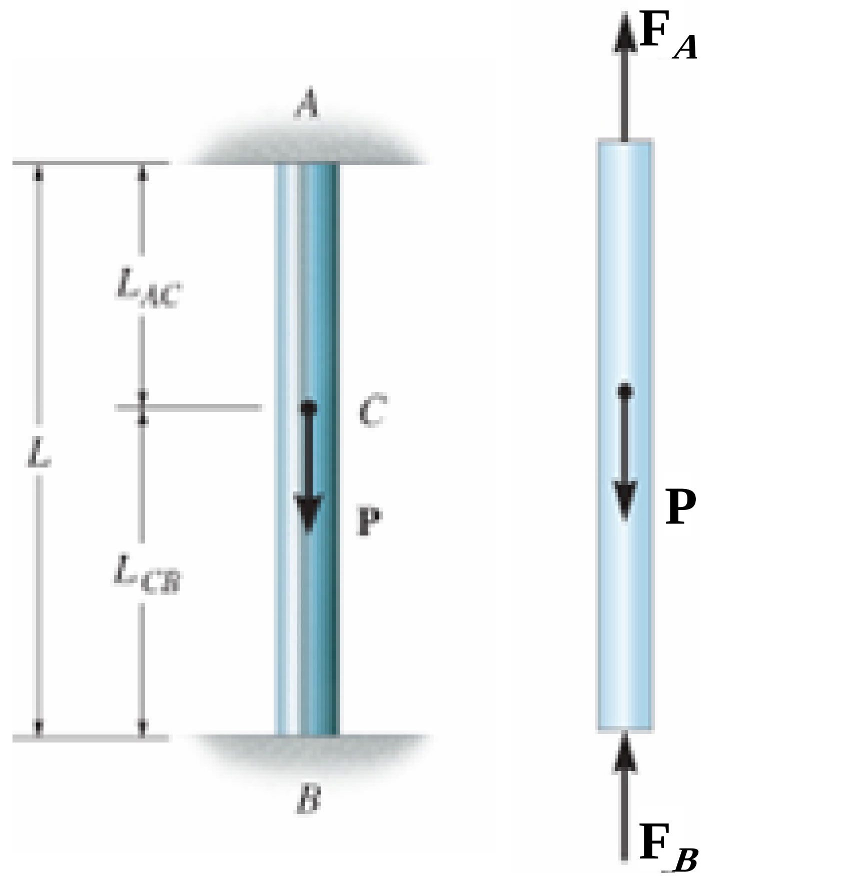
$$\uparrow + \sum F_y = 0, \quad \Rightarrow \quad R_D = 45 \text{ kN}$$

4.4 STATICALLY INDETERMINATE AXIALLY LOADED MEMBER

If bar is fixed at *both ends*, then two unknown axial reactions occur, and the bar is ***statically indeterminate***

$$+\uparrow \sum F = 0;$$

$$F_B + F_A - P = 0 \qquad (a)$$



Free body diagram

We cannot find the value of F_A and F_B.

4.4 STATICALLY INDETERMINATE AXIALLY LOADED MEMBER

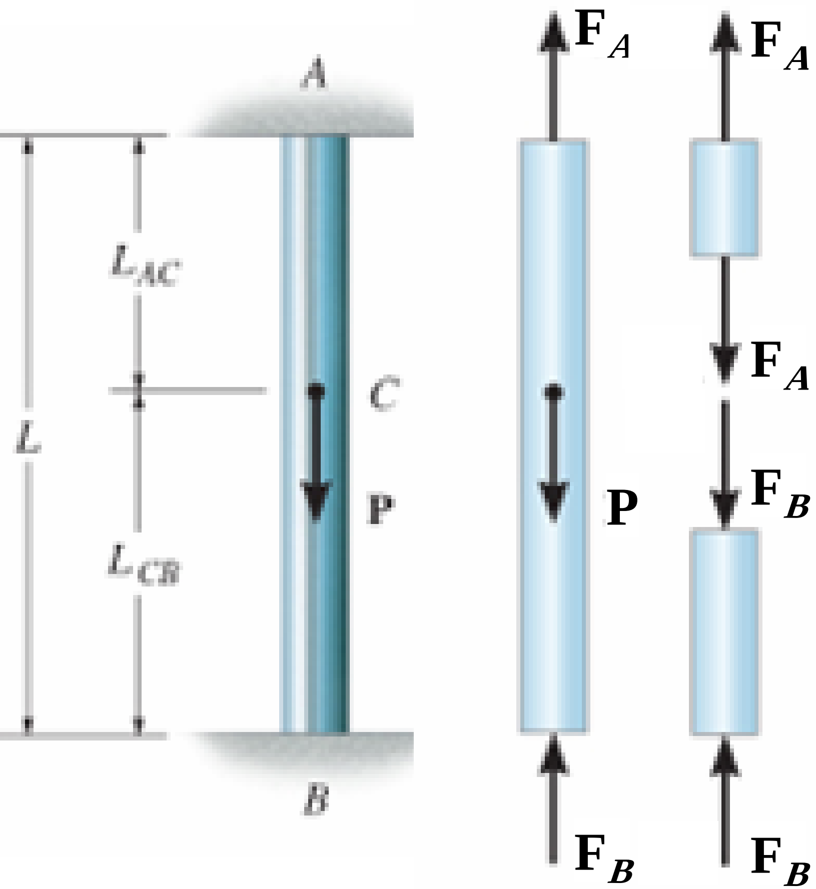
- **To establish addition equation, consider geometry of deformation. Such an equation is referred to as a compatibility or kinematic condition**

- **Since the end supports fixed are fixed, the compatibility condition is**

$$\delta_{A/B} = 0 \quad \Rightarrow \quad \delta_{A/B} = \sum \frac{PL}{AE} = 0$$

- **This equation can be expressed in terms of applied loads using a *load-displacement relationship*, which depends on the material behavior**

4.4 STATICALLY INDETERMINATE AXIALLY LOADED MEMBER



$$F_B + F_A - P = 0 \qquad (a)$$

For linear elastic behavior, compatibility equation can be written as

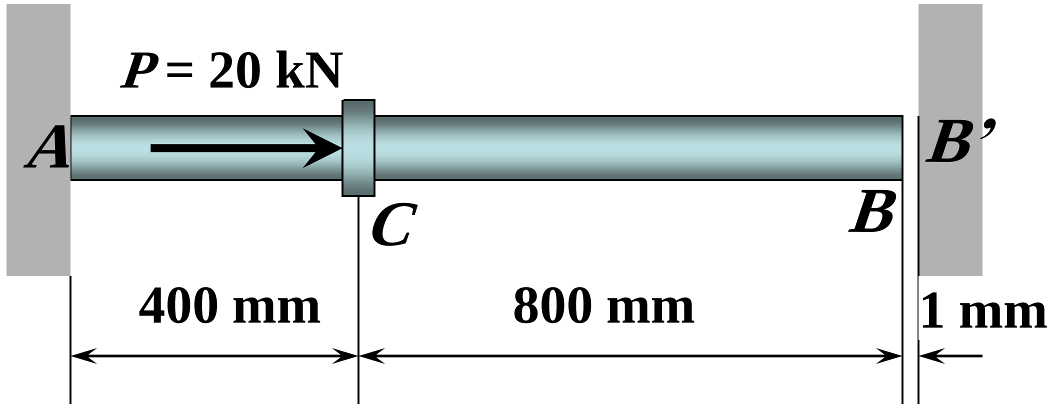
$$\delta_{AC} - \delta_{CB} = 0$$

$$\frac{F_A L_{AC}}{AE} - \frac{F_B L_{CB}}{AE} = 0 \qquad (b)$$

Assume *AE* is constant, solve Eqs.(*a*) & (*b*) simultaneously,

$$F_A = P\left(\frac{L_{CB}}{L}\right) \qquad F_B = P\left(\frac{L_{AC}}{L}\right)$$

EXAMPLE 4.2

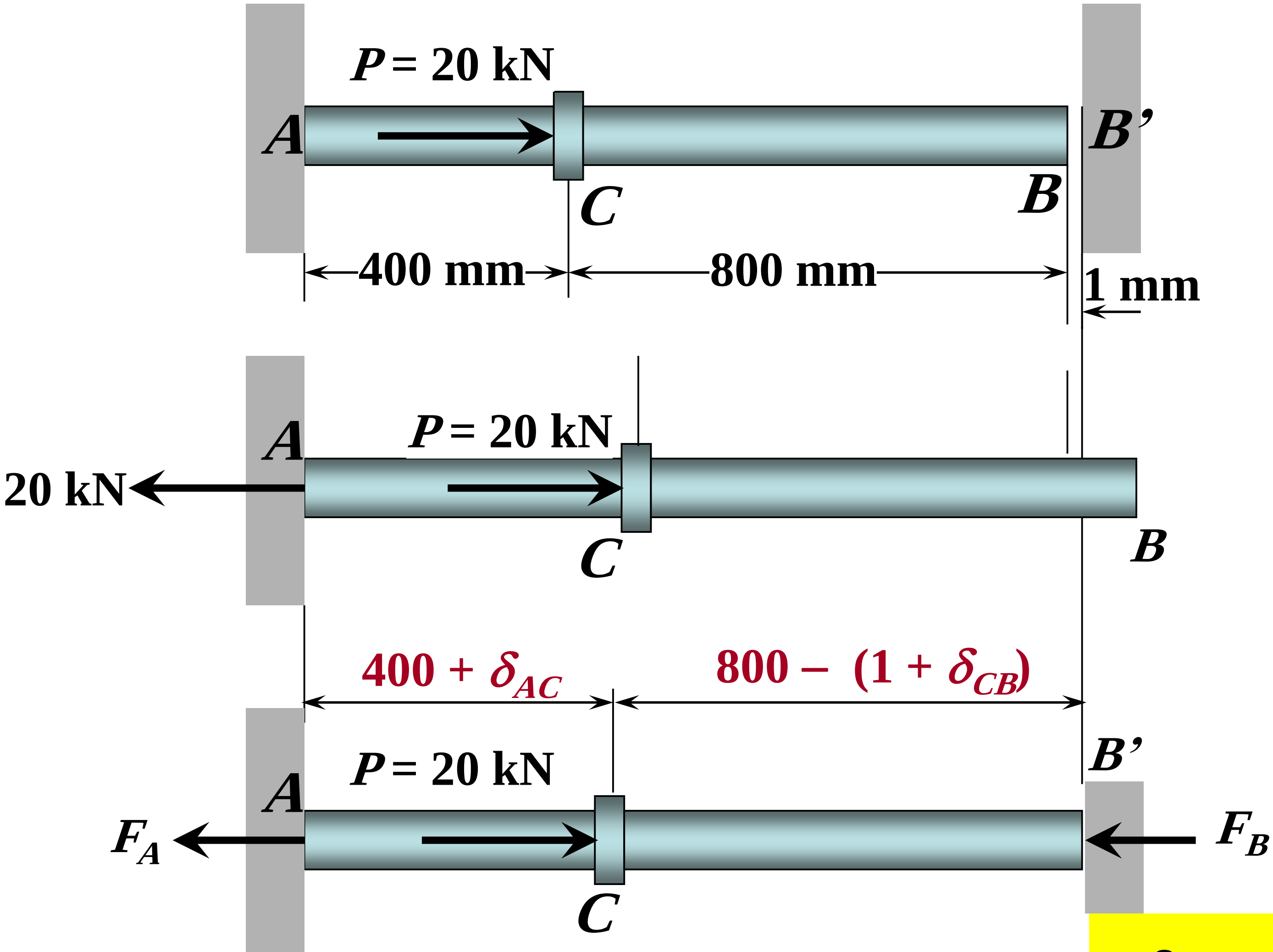


Steel rod shown has diameter of 5 mm. Attached to fixed wall at A, and before it is loaded, there is a gap between the wall at B' and the rod of 1 mm.

Determine reactions at A and B' if rod is subjected to axial force of $P = 20$ kN.

Neglect size of collar at C. Take $E_{st} = 200$ GPa

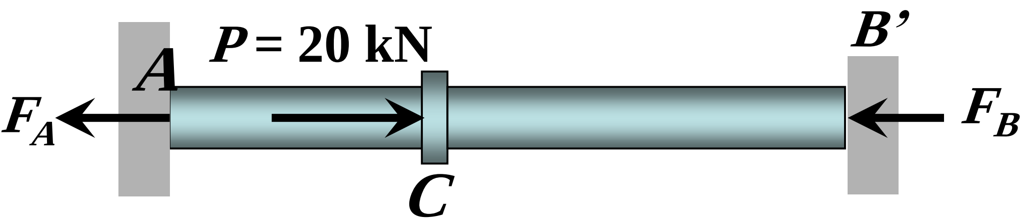
EXAMPLE 4.2



Compatibility equation:

$$\delta_{B/A} = 0.001 = \delta_{AC} + \delta_{CB}$$

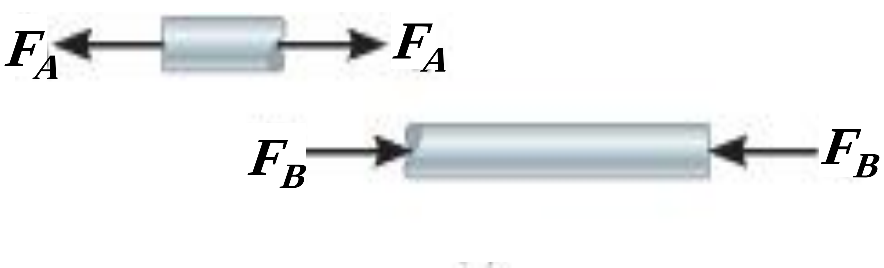
EXAMPLE 4.2 (SOLN)



Equilibrium

Assume force **P** large enough to cause rod's end B to contact wall at B'. Equilibrium requires

$$-F_A - F_B + 20(10^3)\ \text{N} = 0 \qquad (a)$$



Compatibility

Compatibility equation:

$$\delta_{B/A} = \delta_{AC} + \delta_{CB}$$

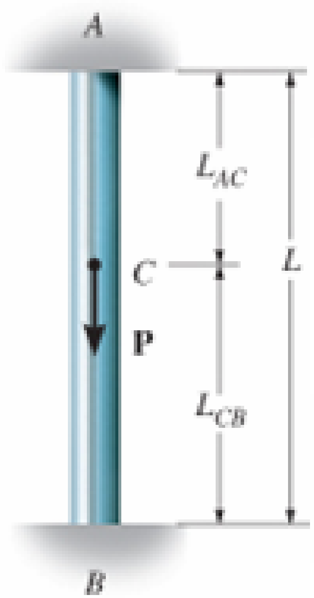
$$0.001\ \text{m} = \frac{F_A L_{AC}}{AE} - \frac{F_B L_{CB}}{AE} \qquad (b)$$

Solving Eq.(*a*) & (*b*) yields, $F_A = 16.6\ \text{kN} \qquad F_B = 3.39\ \text{kN}$

4.5 FORCE METHOD OF ANALYSIS FOR AXIALLY LOADED MEMBERS

- Used to also solve statically indeterminate problems by using superposition of the forces acting on the free-body diagram
- First, choose any one of the two supports as “redundant” and remove its effect on the bar
- Thus, the bar becomes statically determinate
- Apply principle of superposition and solve the equations simultaneously

4.5 FORCE METHOD OF ANALYSIS FOR AXIALLY LOADED MEMBERS



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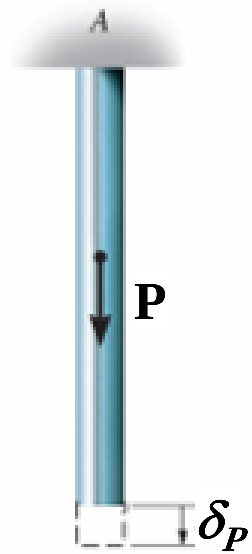

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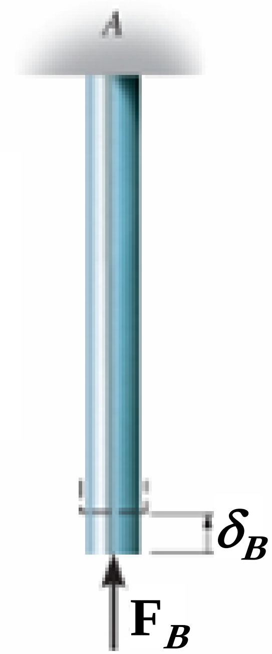


δ_P is displacement at *B* when redundant force at *B* is removed

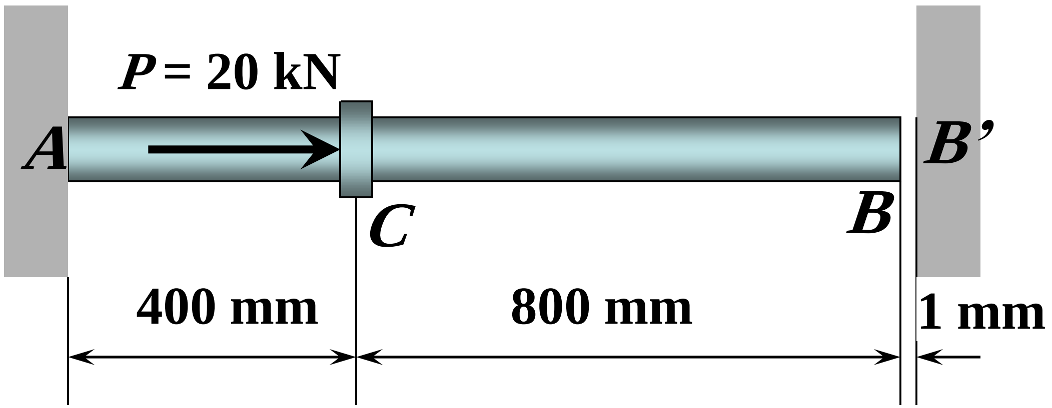
$$\delta_P = \frac{PL_{AC}}{AE}$$

δ_B is displacement at *B* only when redundant force at *B* is applied

$$\delta_B = \frac{F_B L}{AE}$$

$$\delta_P - \delta_B = 0$$

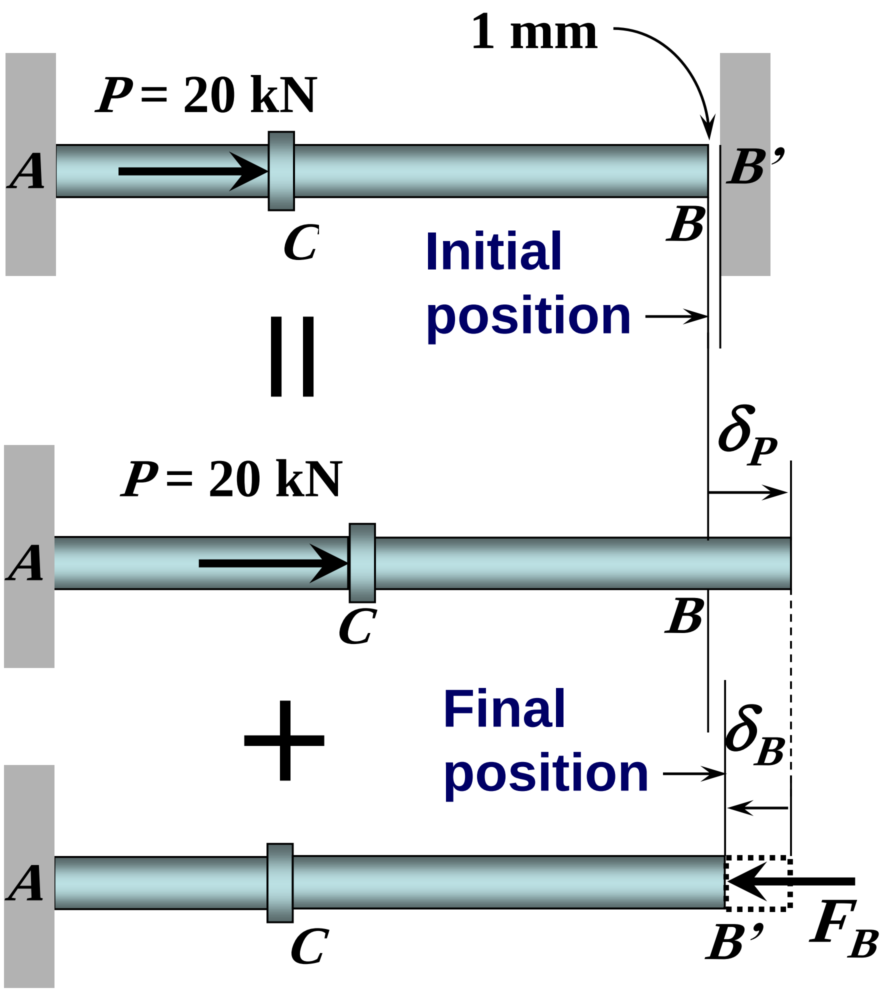
EXAMPLE 4-3



A-36 steel rod shown has diameter of 5 mm. It's attached to fixed wall at *A*, and before it is loaded, there's a gap between wall at *B'* and rod of 1 mm.

Determine reactions at *A* and *B'*.

EXAMPLE 4-3



Compatibility

Consider support at *B'* as redundant.

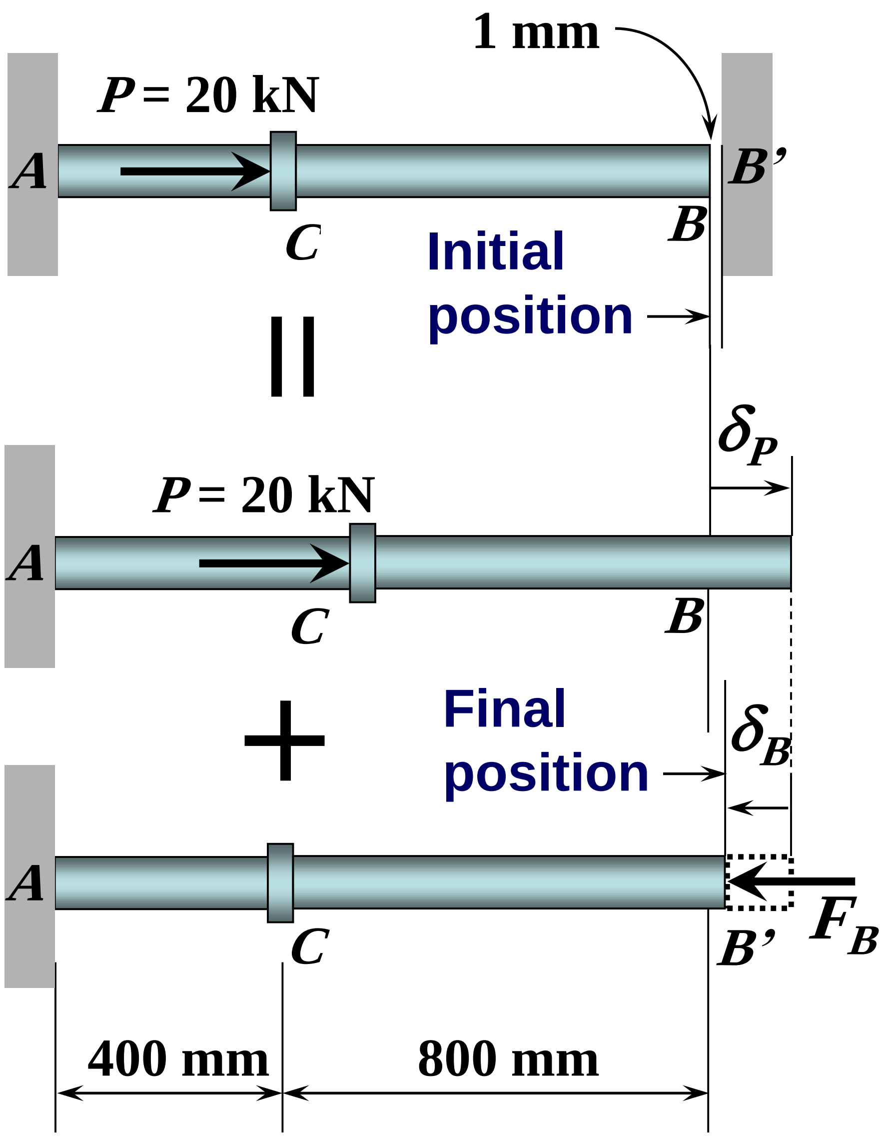
Use principle of superposition,

$(\xrightarrow{+})\quad \delta_P = \text{positive}$

$\delta_B = \text{negative}$

Compatibility equation: $\Rightarrow$ $0.001\ \text{m} = \delta_P - \delta_B$ **Eq. 1**

EXAMPLE 4-3



Compatibility equation:

$$0.001 \text{ m} = \delta_P - \delta_B \qquad \text{Eq. 1}$$

Displacement due to *P*, or δ_P

$$\delta_P = \frac{PL_{AC}}{AE} = \ldots = 0.002037 \text{ m}$$

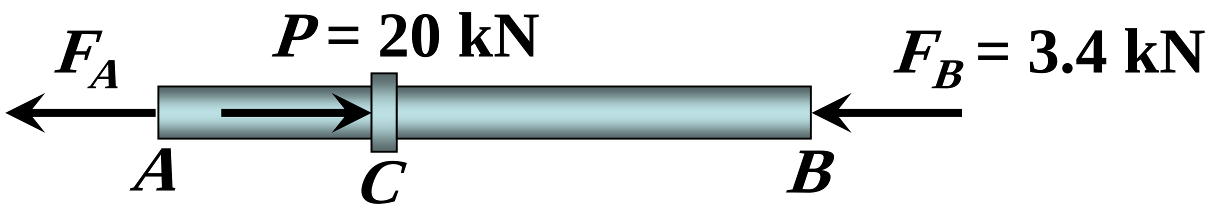
Displacement due to F_B, or δ_B

$$\delta_B = \frac{F_B L_{AB}}{AE} = \ldots$$

$$= 0.3056(10^{-6}) F_B$$

Subst δ_P & δ_B yields: $F_B = 3.40$ kN

EXAMPLE 4-3



Equilibrium

From free-body diagram

$$\xrightarrow{+} \quad \Sigma F_x = 0; \qquad -F_A + 20\text{ kN} - 3.40\text{ kN} = 0$$

$$F_A = 16.6\text{ kN}$$

4.6 THERMAL STRESS

- Expansion or contraction of material is linearly related to temperature increase or decrease that occurs (for homogenous and isotropic material)
- From experiment, deformation of a member having length L is

$$\delta_T = \alpha \Delta T L$$

δ_T = algebraic change in length of member

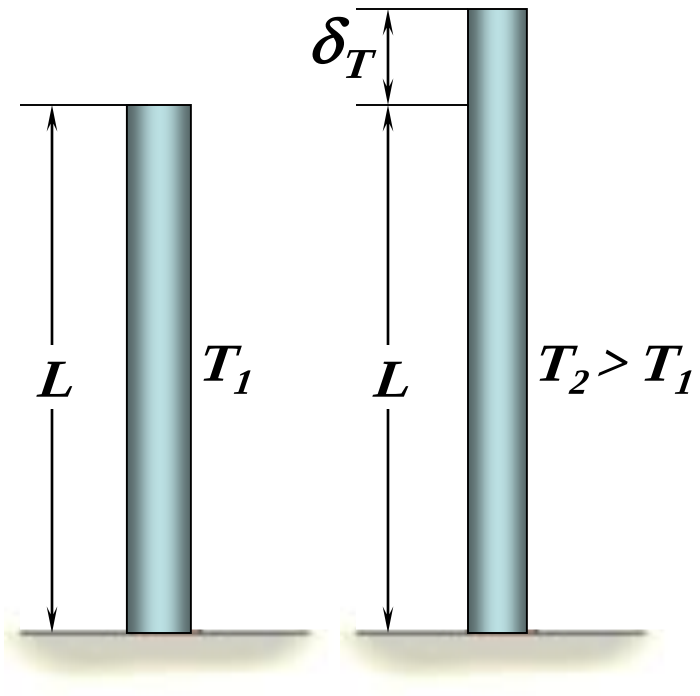
α = liner coefficient of thermal expansion. Unit measure strain per degree of temperature: $1/^{o}C$ (Celsius) or $1/^{o}K$ (Kelvin)

ΔT = algebraic change in temperature of member

4.6 THERMAL STRESS

- For a ***statically indeterminate*** member, the thermal displacements can be constrained by the supports, producing thermal stresses that must be considered in design.

4.6 THERMAL STRESS



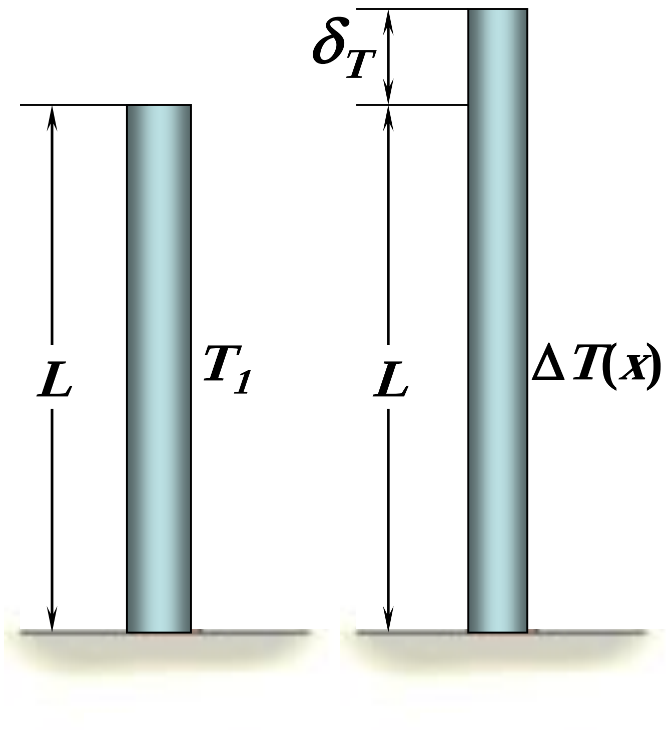
A bar has initial length L and temperature T_1.

When the temperature is increased to T_2, the change in length of the beam is

$$\delta_T = \alpha \Delta T L = \alpha (T_2 - T_1) L$$

No thermal stress produces in the bar because thermal stress will occur when the expansion of the bar is constrained

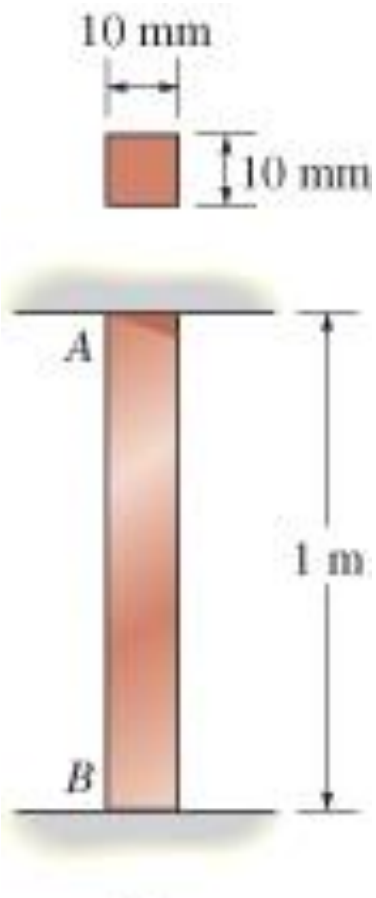
4.6 THERMAL STRESS



If the change in temperature varies throughout the length of the bar, i.e., $\Delta T = \Delta T(x)$, or it varies along the length, then the change in length is

$$\delta_T = \int_0^L \alpha \Delta T \, dx$$

EXAMPLE 4-3

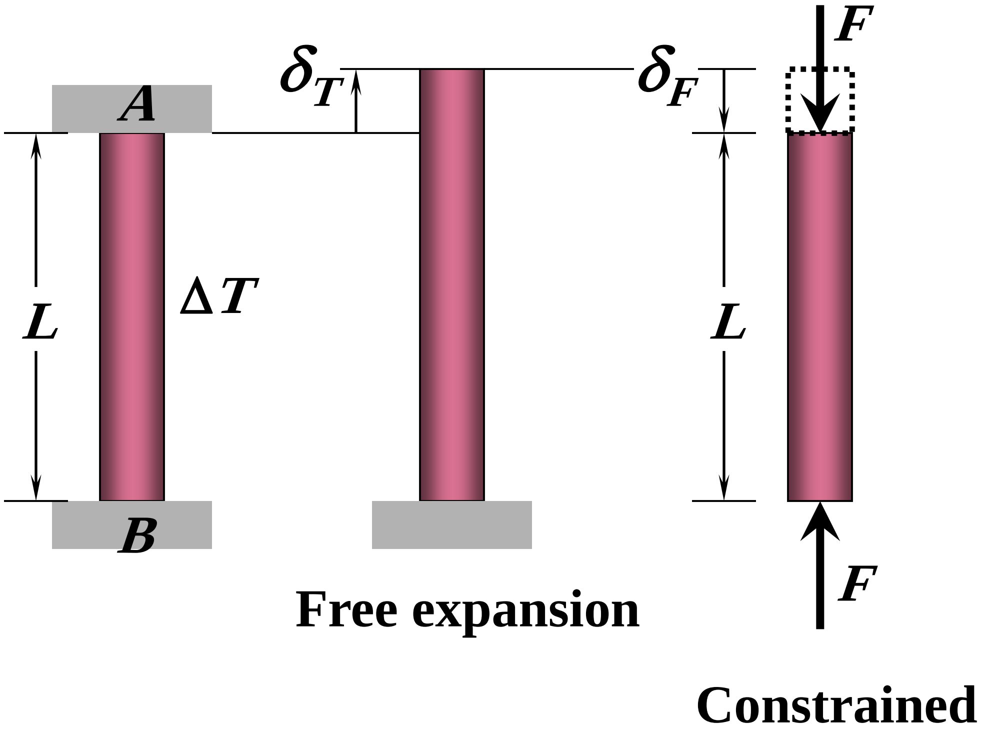


(a)

A-36 steel bar shown is constrained to just fit between two fixed supports when $T_1 = 30^{o}$C.

If temperature is raised to $T_2 = 60^{o}$C, determine the average normal thermal stress developed in the bar.

EXAMPLE 4-3



Free expansion,

$$\delta_T = \alpha(T_2 - T_1)L$$

Constrained, $\delta_F = \dfrac{FL}{AE}$

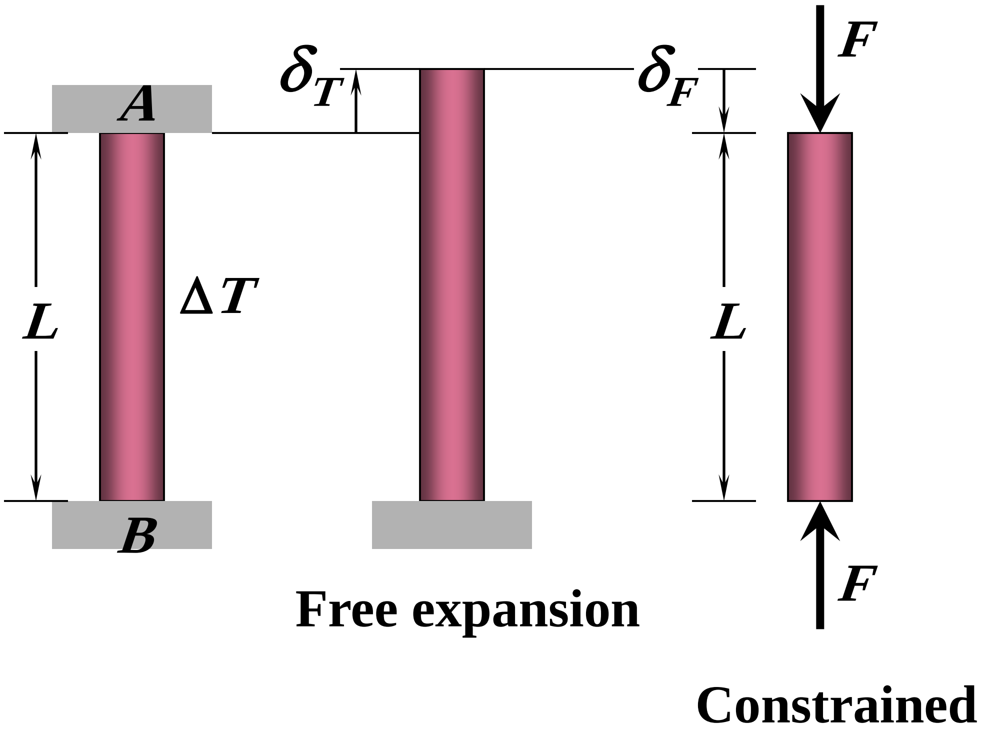
Compatibility condition,

$$\delta_{A/B} = 0 = \delta_T - \delta_F$$

Substituting the appropriate relation,

$$0 = \alpha(T_2 - T_1)L - \frac{FL}{AE}$$

EXAMPLE 4-3



From compatibility condition,

$$0 = \alpha(T_2 - T_1)L - \frac{FL}{AE}$$

Solving the above equation for force F,

$$F = \alpha(T_2 - T_1)AE$$

Data from inside back cover,

$$\alpha_{steel} = 12(10^{-6})\ {}^{o}C^{-1}$$

Substituting all values into the eq., we get $F = 7.2\text{ kN}$

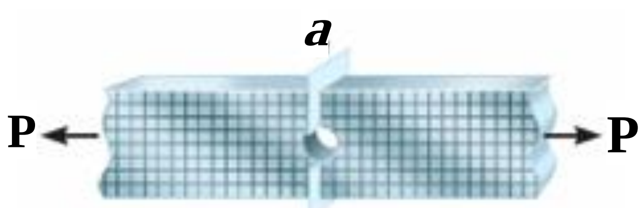
The average thermal stress is then, $\sigma = \dfrac{F}{A} = \ldots = 72\text{ MPa}$

4.7 STRESS CONCENTRATIONS

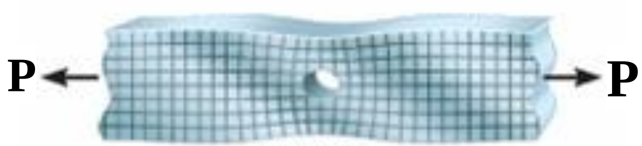
- *Force equilibrium* requires magnitude of resultant force developed by the stress distribution to be equal to P. In other words,

$$P = \int_A \sigma \, dA$$

- This integral represents graphically the *volume* under each of the stress-distribution diagrams shown.



Undistorted



Distorted

Actual stress distribution

Average stress distribution

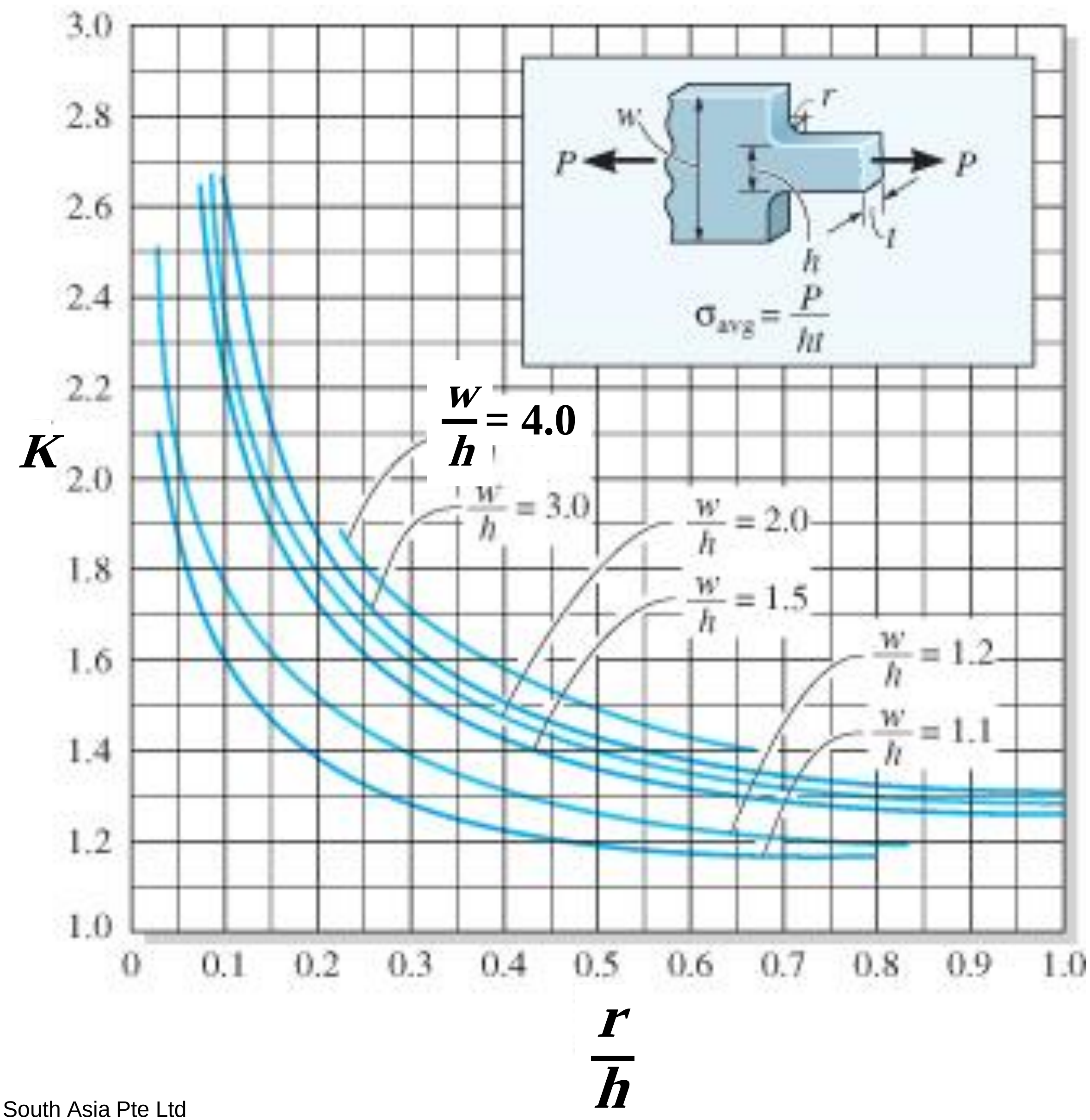
4.7 STRESS CONCENTRATIONS

- In engineering practice, actual stress distribution not needed, only ***maximum stress*** at these sections must be known. Member is designed to resist this stress when axial load **P** is applied.

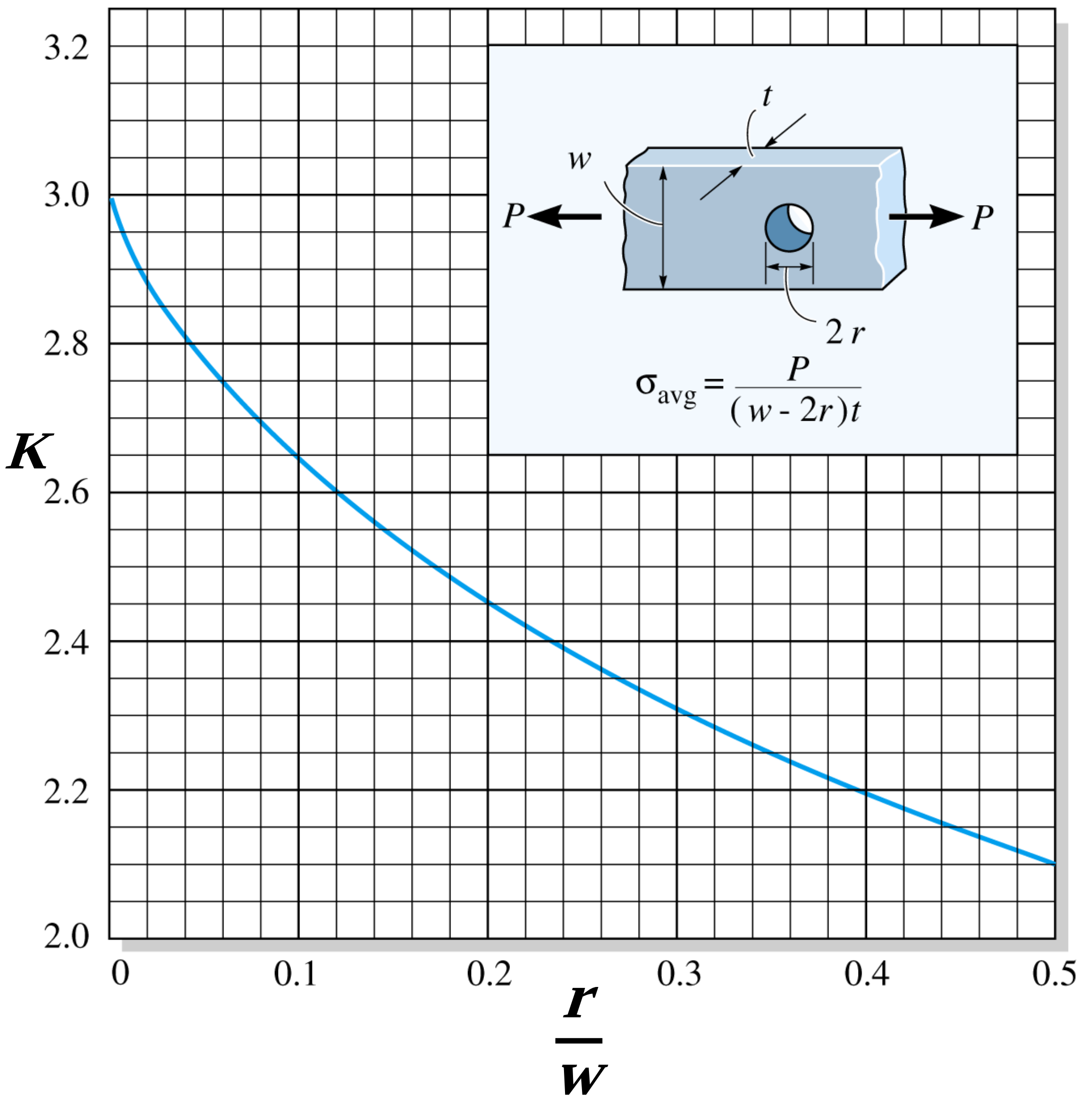
- ***K*** is defined as a ratio of the maximum stress to the average stress acting at the smallest cross section:

$$K = \frac{\sigma_{max}}{\sigma_{avg}}$$

4.7 STRESS CONCENTRATIONS



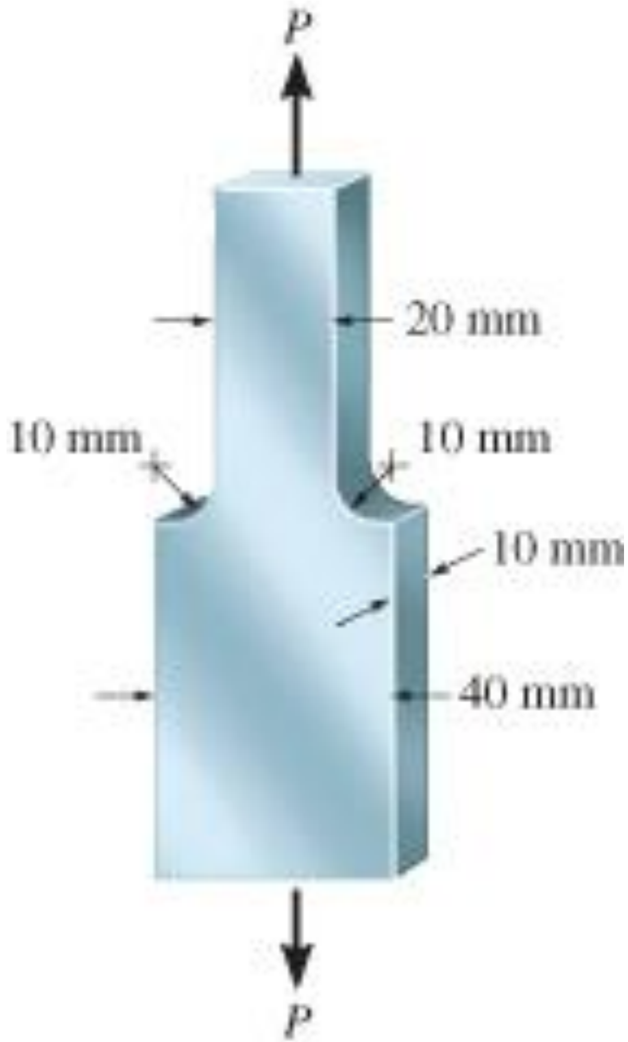
4.7 STRESS CONCENTRATIONS



4.7 STRESS CONCENTRATIONS

- ***K*** is independent of the bar's geometry and the type of discontinuity, only on the bar's geometry and the type of discontinuity.
- As size ***r*** of the discontinuity is decreased, stress concentration is increased.
- It is important to use stress-concentration factors in design when using brittle materials, but not necessary for ductile materials
- Stress concentrations also cause failure structural members or mechanical elements subjected to *fatigue loadings*

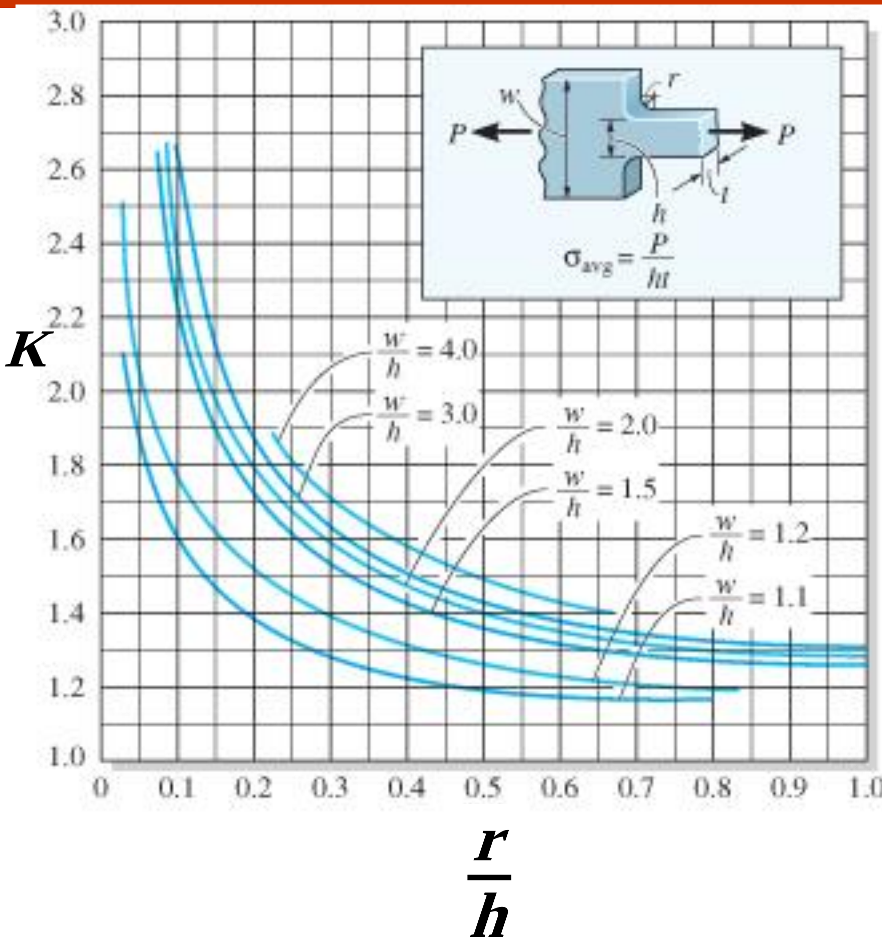
EXAMPLE 4-4

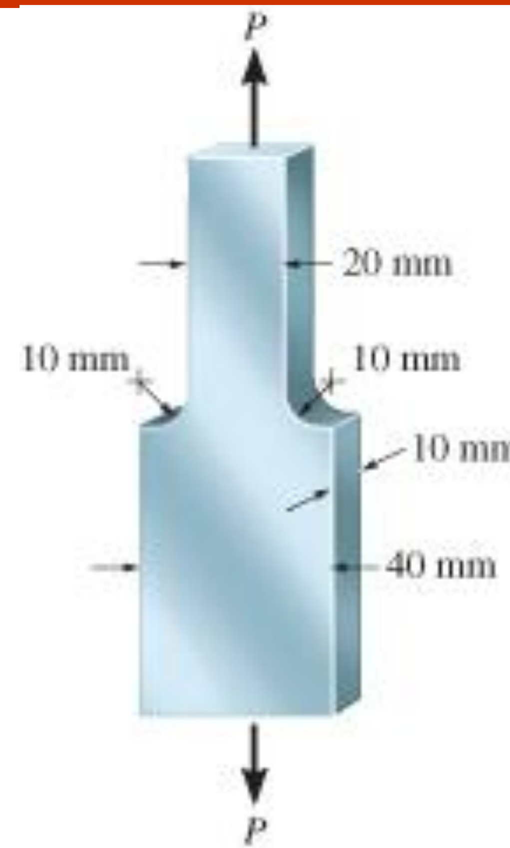


Steel bar shown below has allowable stress, σ_{allow} = 115 MPa.

Determine largest axial force **P** that the bar can carry.

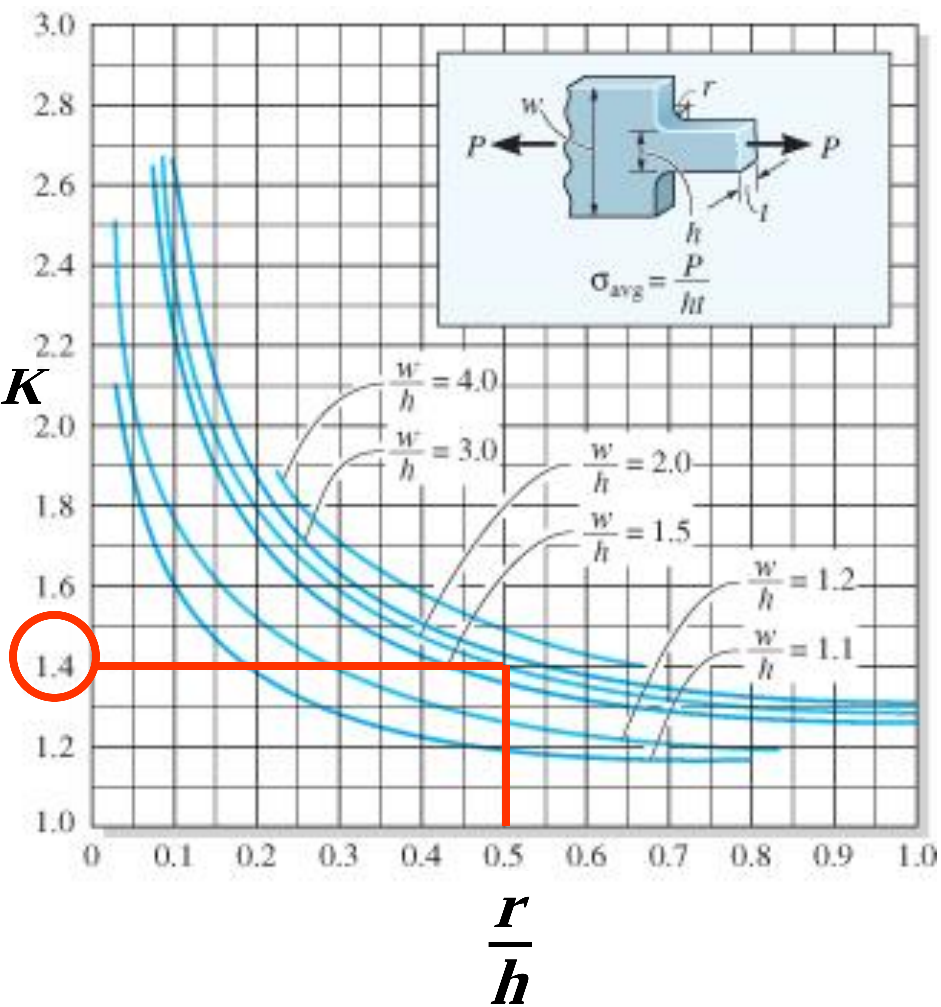
EXAMPLE 4-4





Because there is a shoulder fillet, stress-concentrating factor determined using the graph shown

EXAMPLE 4-4



Geometric parameters:

$$\frac{r}{h} = \frac{10\text{ mm}}{20\text{ mm}} = 0.50$$

$$\frac{w}{h} = \frac{40\text{ mm}}{20\text{ mm}} = 2$$

From the graph, $K = 1.4$

EXAMPLE 4-4

Average normal stress at *smallest* cross-section,

$$\sigma_{avg} = \frac{P}{(20\text{ mm})(10\text{ mm})} = 0.005P\text{ N/mm}^2$$

Applying Eqn $K = \dfrac{\sigma_{max}}{\sigma_{avg}}$

which $\sigma_{allow} = \sigma_{max}$ yields ⇨ $\sigma_{allow} = K\,\sigma_{avg}$

$$115\text{ N/mm}^2 = 1.4(0.005P)$$

$$P = 16.43(10^3)\text{ N} = 16.43\text{ kN}$$

CHAPTER REVIEW

- When load applied on a body, a stress distribution is created within the body that becomes more uniformly distributed at regions farther from point of application. This is the *Saint-Venant's principle.*

- Relative displacement at end of axially loaded member relative to other end is determined from

$$\delta = \int_0^L \frac{P(x)\,dx}{A(x)\,E}$$

CHAPTER REVIEW

- If series of constant external forces are applied and *AE* is constant, then

$$\delta = \sum \frac{PL}{AE}$$

- Make sure to use sign convention for internal load **P** and that material does not yield, but remains linear elastic

- Superposition of load & displacement is possible provided material remains linear elastic and no changes in geometry occur

CHAPTER REVIEW

- Reactions on statically indeterminate bar determined using equilibrium and compatibility conditions that specify displacement at the supports. Use the load-displacement relationship, $\delta = PL/AE$

- Change in temperature can cause member made from homogenous isotropic material to change its length by $\delta = \alpha \Delta T L$. If member is confined, expansion will produce thermal stress in the member

CHAPTER REVIEW

- Holes and sharp transitions at cross-section create stress concentrations. For design, obtain stress concentration factor ***K*** from graph, which is determined empirically. The K value is multiplied by average stress to obtain maximum stress at cross-section, $\sigma_{max} = K\sigma_{avg}$

- If loading in bar causes material to yield, then stress distribution that's produced can be determined from the strain distribution and stress-strain diagram

CHAPTER REVIEW

- For perfectly plastic material, yielding causes stress distribution at cross-section of hole or transition to even out and become uniform
- If member is constrained and external loading causes yielding, then when load is released, it will cause residual stress in the material