Mechanics of materials

Chapter five

Torsion

By

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Mechanics of materials

5.1. Torsional deformation of circular shaft

- Assume the circular shaft shown in the figure(a) with the circular grid shown. If a torque is applied to the member as shown in figure (b) and the deformation is assumed to be **small** then

The circles in the grid will remain circles and the longitudinal lines will deform in helical (spiral) line.

- From that , we can assume that the radius and the length of the shaft will remain constant.

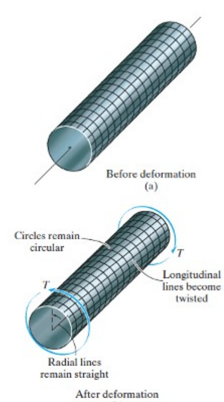

Fig. 5–1

Torsion

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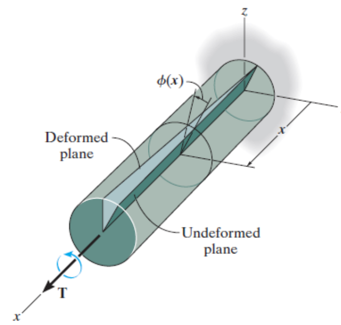
5.1. Torsional deformation of circular shaft

- In torsion, the strain is represented by **twist angle ($\phi(x)$).**
- As we move in the x-axis, a small difference in the twist angle ($\Delta\phi$) will occur. This difference causes the member

to has a **shear strain (γ):**

$$\gamma = \frac{\pi}{2} - \theta'$$



The angle of twist $\phi(x)$ increases as x increases.

Fig. 5–2

Torsion

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TORSION FORMULA

Assumptions:

- Linear and elastic deformation
- Plane section remains plane and undistorted

If we assume that the torsion stress vary linearly from the inside to the outside then $\tau = \frac{\rho}{c}\tau_{max}$

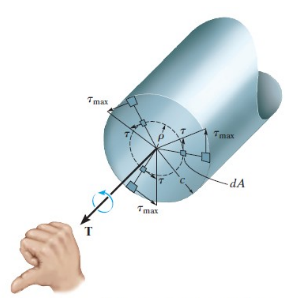
Torsion – shear relationship:

$$T = \int_A \rho(\tau) dA = \int_A \rho\left(\frac{\rho}{c}\right)\tau_{max} dA$$

$$T = \frac{\tau_{max}}{c}\int_A \rho^2 dA$$

$$\tau_{max} = \frac{Tc}{J}$$

Similarly, $\tau = \frac{T\rho}{J}$



Shear stress varies linearly along each radial line of the cross section.

Fig. 5–5

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Torsion

Polar moment of inertia

For solid shaft:

$$J = \int_A \rho^2 dA = \int_0^C \rho^2 (2\pi\rho d\rho) = 2\pi \int_0^C \rho^3 d\rho = 2\pi \left(\frac{1}{4}\right) \rho^4 \Big|_0^C$$

$$J = \frac{\pi}{2} c^4$$

For tubular shaft: $J = \frac{\pi}{2}\left(c_o^4 - c_i^4\right)$

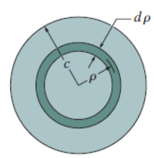
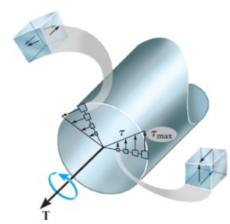
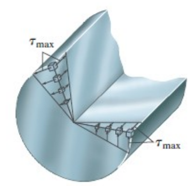


Fig. 5–6



(a)



Shear stress varies linearly along each radial line of the cross section.
(b)

EXAMPLE 5.1

The *solid* shaft of radius c is subjected to a torque **T**, Fig. 5–10*a*. Determine the fraction of T that is resisted by the material contained within the outer region of the shaft, which has an inner radius of $c/2$ and outer radius c.

SOLUTION

The stress in the shaft varies linearly, such that $\tau = (\rho/c)\tau_{max}$, Eq. 5–3. Therefore, the torque dT' on the ring (area) located within the lighter-shaded region, Fig. 5–10*b*, is

$$dT' = \rho(\tau\, dA) = \rho(\rho/c)\tau_{max}(2\pi\rho\, d\rho)$$

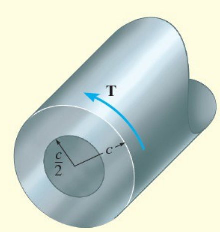
For the entire lighter-shaded area the torque is

$$T' = \frac{2\pi\tau_{max}}{c}\int_{c/2}^{c} \rho^3\, d\rho$$

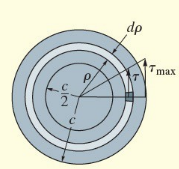
$$= \frac{2\pi\tau_{max}}{c}\frac{1}{4}\rho^4\Big|_{c/2}^{c}$$

So that

$$T' = \frac{15\pi}{32}\tau_{max}c^3 \qquad (1)$$



(a)



(b)

Fig. 5–10

EXAMPLE 5.1 CONTINUED

This torque T' can be expressed in terms of the applied torque T by first using the torsion formula to determine the maximum stress in the shaft. We have

$$\tau_{\max} = \frac{Tc}{J} = \frac{Tc}{(\pi/2)c^4}$$

or

$$\tau_{\max} = \frac{2T}{\pi c^3}$$

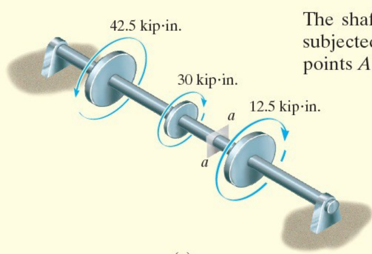
Substituting this into Eq. 1 yields

$$T' = \frac{15}{16}T \qquad \textit{Ans.}$$

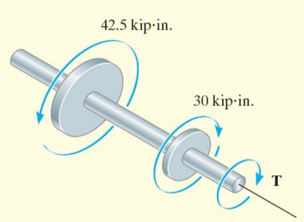
NOTE: Here, approximately 94% of the torque is resisted by the lighter-shaded region, and the remaining 6% (or $\frac{1}{16}$) of T is resisted by the inner "core" of the shaft, $\rho = 0$ to $\rho = c/2$. As a result, the material located at the *outer region* of the shaft is highly effective in resisting torque, which justifies the use of tubular shafts as an efficient means for transmitting torque, and thereby saving material.

EXAMPLE 5.2

The shaft shown in Fig. 5–11*a* is supported by two bearings and is subjected to three torques. Determine the shear stress developed at points A and B, located at section *a–a* of the shaft, Fig. 5–11*c*.



(a)



(b)

EXAMPLE 5.2 CONTINUED

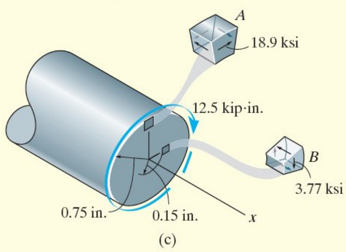


Fig. 5–11

SOLUTION

Internal Torque. The bearing reactions on the shaft are zero, provided the shaft's weight is neglected. Furthermore, the applied torques satisfy moment equilibrium about the shaft's axis.

The internal torque at section *a–a* will be determined from the free-body diagram of the left segment, Fig. 5–11*b*. We have

$$\Sigma M_x = 0; \quad 42.5 \text{ kip} \cdot \text{in.} - 30 \text{ kip} \cdot \text{in.} - T = 0 \qquad T = 12.5 \text{ kip} \cdot \text{in.}$$

Section Property. The polar moment of inertia for the shaft is

$$J = \frac{\pi}{2}(0.75 \text{ in.})^4 = 0.497 \text{ in}^4$$

EXAMPLE 5.2 CONTINUED

Shear Stress. Since point A is at $\rho = c = 0.75$ in.,

$$\tau_A = \frac{Tc}{J} = \frac{(12.5 \text{ kip} \cdot \text{in.})(0.75 \text{ in.})}{(0.497 \text{ in}^4)} = 18.9 \text{ ksi} \qquad \textit{Ans.}$$

Likewise for point B, at $\rho = 0.15$ in., we have

$$\tau_B = \frac{T\rho}{J} = \frac{(12.5 \text{ kip} \cdot \text{in.})(0.15 \text{ in.})}{(0.497 \text{ in}^4)} = 3.77 \text{ ksi} \qquad \textit{Ans.}$$

NOTE: The directions of these stresses on each element at A and B, Fig. 5–11*c*, are established from the direction of the resultant internal torque **T**, shown in Fig. 5–11*b*. Note carefully how the shear stress acts on the planes of each of these elements.

EXAMPLE 5.3

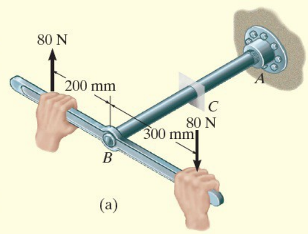
The pipe shown in Fig. 5–12*a* has an inner diameter of 80 mm and an outer diameter of 100 mm. If its end is tightened against the support at *A* using a torque wrench at *B*, determine the shear stress developed in the material at the inner and outer walls along the central portion of the pipe when the 80-N forces are applied to the wrench.

SOLUTION

Internal Torque. A section is taken at an intermediate location *C* along the pipe's axis, Fig. 5–12*b*. The only unknown at the section is the internal torque **T**. We require

$$\Sigma M_y = 0; \quad 80\text{ N}(0.3\text{ m}) + 80\text{ N}(0.2\text{ m}) - T = 0$$

$$T = 40\text{ N}\cdot\text{m}$$



(a)

Section Property. The polar moment of inertia for the pipe's cross-sectional area is

$$J = \frac{\pi}{2}[(0.05\text{ m})^4 - (0.04\text{ m})^4] = 5.796(10^{-6})\text{ m}^4$$

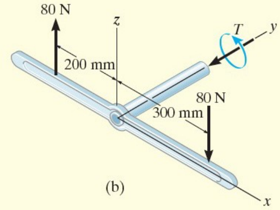
EXAMPLE 5.3 CONTINUED

Shear Stress. For any point lying on the outside surface of the pipe, $\rho = c_o = 0.05$ m, we have

$$\tau_o = \frac{Tc_o}{J} = \frac{40\text{ N}\cdot\text{m}(0.05\text{ m})}{5.796(10^{-6})\text{ m}^4} = 0.345\text{ MPa} \qquad \textit{Ans.}$$

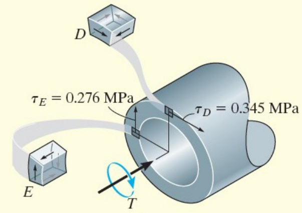
And for any point located on the inside surface, $\rho = c_i = 0.04$ m, so that

$$\tau_i = \frac{Tc_i}{J} = \frac{40\text{ N}\cdot\text{m}(0.04\text{ m})}{5.796(10^{-6})\text{ m}^4} = 0.276\text{ MPa} \qquad \textit{Ans.}$$



(b)

NOTE: To show how these stresses act at representative points *D* and *E* on the cross-section, we will first view the cross section from the front of segment *CA* of the pipe, Fig. 5–12*a*. On this section, Fig. 5–12*c*, the resultant internal torque is equal but opposite to that shown in Fig. 5–12*b*. The shear stresses at *D* and *E* contribute to this torque and therefore act on the shaded faces of the elements in the directions shown. As a consequence, notice how the shear-stress components act on the other three faces. Furthermore, since the top face of *D* and the inner face of *E* are in stress-free regions taken from the pipe's outer and inner walls, no shear stress can exist on these faces or on the other corresponding faces of the elements.



(c)

Fig. 5–12

Mechanics of materials

Torsion

Power transmission

The power (P) transmit by the torque (T) for a shaft that has an angular velocity (ω) is given by

$$P = T.\omega$$

- P is the power in Watt
- T is the torque in N.m
- ω angular velocity in rad/s

If the frequency of the machine (f) is given in Hz, the power become

$$P = T.(2\pi f)$$

Mechanics of materials

Torsion

Power transmission

Shaft design: if the power transmitted and the machine frequency are known, the torque can be found as

$$T = \frac{P}{\omega} = \frac{P}{2\pi f}$$

The value of (T) mustn't exceed the allowable torsion stress (τ_{allow}). The torsion stress formula is substituted in the power equation above, we will have a design criteria for solid shaft that has radius equal (c)

$$\frac{J}{c} = \frac{T}{\tau_{allow}}$$

What you need to find is the minimum c that bear the applied stress. Remember that ***J*** contains the term c

EXAMPLE 5.4

A solid steel shaft AB shown in Fig. 5–13 is to be used to transmit 5 hp from the motor M to which it is attached. If the shaft rotates at $\omega = 175$ rpm and the steel has an allowable shear stress of $\tau_{allow} = 14.5$ ksi, determine the required diameter of the shaft to the nearest $\frac{1}{8}$ in.

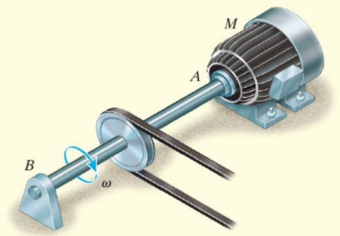


Fig. 5–13

EXAMPLE 5.4 CONTINUED

SOLUTION

The torque on the shaft is determined from Eq. 5–10, that is, $P = T\omega$. Expressing P in foot-pounds per second and ω in radians/second, we have

$$P = 5\text{ hp}\left(\frac{550\text{ ft}\cdot\text{lb/s}}{1\text{ hp}}\right) = 2750\text{ ft}\cdot\text{lb/s}$$

$$\omega = \frac{175\text{ rev}}{\text{min}}\left(\frac{2\pi\text{ rad}}{1\text{ rev}}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right) = 18.33\text{ rad/s}$$

Thus,

$P = T\omega$; $\quad 2750\text{ ft}\cdot\text{lb/s} = T(18.33\text{ rad/s})$

$$T = 150.1\text{ ft}\cdot\text{lb}$$

Applying Eq. 5–12 yields

$$\frac{J}{c} = \frac{\pi}{2}\frac{c^4}{c} = \frac{T}{\tau_{allow}}$$

$$c = \left(\frac{2T}{\pi\tau_{allow}}\right)^{1/3} = \left(\frac{2(150.1\text{ ft}\cdot\text{lb})(12\text{ in./ft})}{\pi(14\,500\text{ lb/in}^2)}\right)^{1/3}$$

$$c = 0.429\text{ in.}$$

Since $2c = 0.858$ in., select a shaft having a diameter of

$$d = \frac{7}{8}\text{ in.} = 0.875\text{ in.}$$ *Ans.*

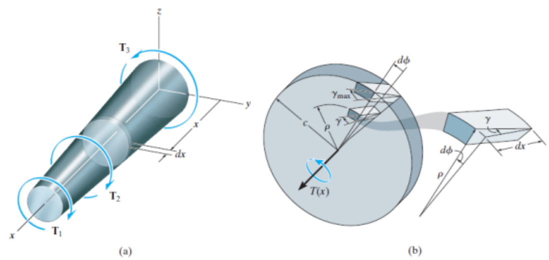
Mechanics of materials

Torsion

ANGLE OF TWIST

$$d\phi = \gamma \frac{dx}{\rho}$$

$$d\phi = \frac{T(x)}{J(x)G} dx$$



For constant torque and cross-sectional area:

$$\phi = \frac{TL}{JG}$$

For multiple torques:

$$\phi = \sum \frac{TL}{JG}$$

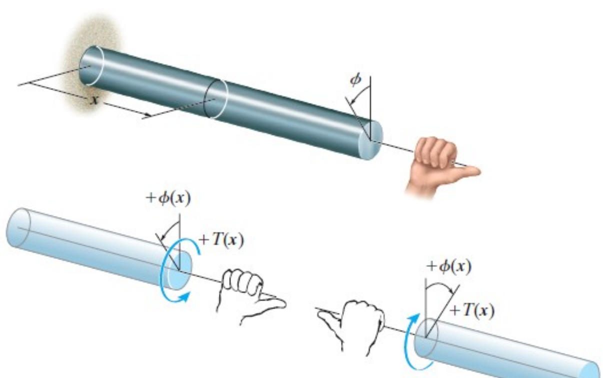
Mechanics of materials

Torsion

ANGLE OF TWIST

Sign convention for both torque and angle of twist

- positive if (right hand) thumb directs outward from the shaft

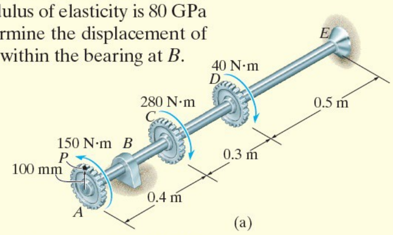


Positive sign convention for T and ϕ.

Fig. 5–17

EXAMPLE 5.5

The gears attached to the fixed-end steel shaft are subjected to the torques shown in Fig. 5–19*a*. If the shear modulus of elasticity is 80 GPa and the shaft has a diameter of 14 mm, determine the displacement of the tooth *P* on gear *A*. The shaft turns freely within the bearing at *B*.



(a)

SOLUTION

Internal Torque. By inspection, the torques in segments *AC*, *CD*, and *DE* are different yet *constant* throughout each segment. Free-body diagrams of appropriate segments of the shaft along with the calculated internal torques are shown in Fig. 5–19*b*. Using the right-hand rule and the established sign convention that positive torque is directed away from the sectioned end of the shaft, we have

$$T_{AC} = +150\ \text{N}\cdot\text{m} \qquad T_{CD} = -130\ \text{N}\cdot\text{m} \qquad T_{DE} = -170\ \text{N}\cdot\text{m}$$

These results are also shown on the torque diagram, Fig. 5–19*c*.

EXAMPLE 5.5 CONTINUED

SOLUTION

Internal Torque. By inspection, the torques in segments *AC*, *CD*, and *DE* are different yet *constant* throughout each segment. Free-body diagrams of appropriate segments of the shaft along with the calculated internal torques are shown in Fig. 5–19*b*. Using the right-hand rule and the established sign convention that positive torque is directed away from the sectioned end of the shaft, we have

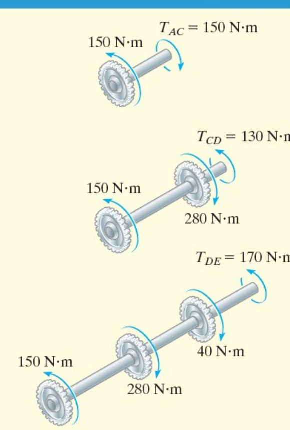
$$T_{AC} = +150\ \text{N}\cdot\text{m} \qquad T_{CD} = -130\ \text{N}\cdot\text{m} \qquad T_{DE} = -170\ \text{N}\cdot\text{m}$$

These results are also shown on the torque diagram, Fig. 5–19*c*.

Angle of Twist. The polar moment of inertia for the shaft is

$$J = \frac{\pi}{2}(0.007\ \text{m})^4 = 3.771(10^{-9})\ \text{m}^4$$

Applying Eq. 5–16 to each segment and adding the results algebraically, we have



(b)

EXAMPLE 5.5 CONTINUED

Angle of Twist. The polar moment of inertia for the shaft is

$$J = \frac{\pi}{2}(0.007\text{ m})^4 = 3.771(10^{-9})\text{ m}^4$$

Applying Eq. 5–16 to each segment and adding the results algebraically, we have

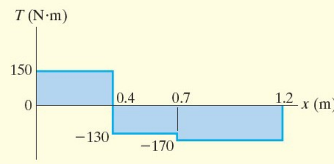
$$\phi_A = \sum\frac{TL}{JG} = \frac{(+150\text{ N}\cdot\text{m})(0.4\text{ m})}{3.771(10^{-9})\text{ m}^4\,[80(10^9)\text{ N/m}^2]}$$
$$+ \frac{(-130\text{ N}\cdot\text{m})(0.3\text{ m})}{3.771(10^{-9})\text{ m}^4\,[80(10^9)\text{ N/m}^2)]}$$
$$+ \frac{(-170\text{ N}\cdot\text{m})(0.5\text{ m})}{3.771(10^{-9})\text{ m}^4\,[80(10^9)\text{ N/m}^2)]} = -0.2121\text{ rad}$$

Since the answer is negative, by the right-hand rule the thumb is directed *toward* the end E of the shaft, and therefore gear A will rotate as shown in Fig. 5–19*d*.

The displacement of tooth P on gear A is

$$s_P = \phi_A r = (0.2121\text{ rad})(100\text{ mm}) = 21.2\text{ mm} \qquad \textit{Ans.}$$

NOTE: Remember that this analysis is valid only if the shear stress does not exceed the proportional limit of the material.



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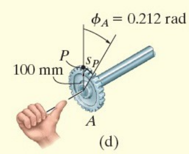
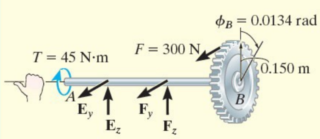


(d)

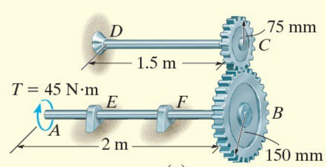
Fig. 5–19

EXAMPLE 5.6

The two solid steel shafts shown in Fig. 5–20*a* are coupled together using the meshed gears. Determine the angle of twist of end A of shaft AB when the torque $T = 45\text{ N}\cdot\text{m}$ is applied. Take $G = 80$ GPa. Shaft AB is free to rotate within bearings E and F, whereas shaft DC is fixed at D. Each shaft has a diameter of 20 mm.

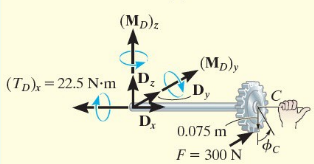


(b)



(a)

Fig. 5–20



(c)

EXAMPLE **5.6** CONTINUED

SOLUTION

Internal Torque. Free-body diagrams for each shaft are shown in Fig. 5–20*b* and 5–20*c*. Summing moments along the x axis of shaft AB yields the tangential reaction between the gears of $F = 45 \text{ N}\cdot\text{m}/0.15 \text{ m} = 300 \text{ N}$. Summing moments about the x axis of shaft DC, this force then creates a torque of $(T_D)_x = 300 \text{ N } (0.075 \text{ m}) = 22.5 \text{ N}\cdot\text{m}$ on shaft DC.

Angle of Twist. To solve the problem, we will first calculate the rotation of gear C due to the torque of $22.5 \text{ N}\cdot\text{m}$ in shaft DC, Fig. 5–20*c*. This angle of twist is

$$\phi_C = \frac{TL_{DC}}{JG} = \frac{(+22.5 \text{ N}\cdot\text{m})(1.5 \text{ m})}{(\pi/2)(0.010 \text{ m})^4[80(10^9) \text{ N/m}^2]} = +0.0269 \text{ rad}$$

Since the gears at the end of the shaft are in mesh, the rotation ϕ_C of gear C causes gear B to rotate ϕ_B, Fig. 5–20*b*, where

$$\phi_B(0.15 \text{ m}) = (0.0269 \text{ rad})(0.075 \text{ m})$$
$$\phi_B = 0.0134 \text{ rad}$$

EXAMPLE **5.6** CONTINUED

We will now determine the angle of twist of end A with respect to end B of shaft AB caused by the $45 \text{ N}\cdot\text{m}$ torque, Fig. 5–20*b*. We have

$$\phi_{A/B} = \frac{T_{AB}L_{AB}}{JG} = \frac{(+45 \text{ N}\cdot\text{m})(2 \text{ m})}{(\pi/2)(0.010 \text{ m})^4[80(10^9) \text{ N/m}^2]} = +0.0716 \text{ rad}$$

The rotation of end A is therefore determined by adding ϕ_B and $\phi_{A/B}$, since both angles are in the *same direction*, Fig. 5–20*b*. We have

$$\phi_A = \phi_B + \phi_{A/B} = 0.0134 \text{ rad} + 0.0716 \text{ rad} = +0.0850 \text{ rad} \quad \textit{Ans.}$$

Torsion

STATICALLY INDETERMINATE TORQUE-LOADED MEMBERS

Procedure for analysis:
use both equilibrium and compatibility equations

Equilibrium

Draw a free-body diagram of the shaft in order to identify all the torques that act on it. Then write the equations of moment equilibrium about the axis of the shaft.

Compatibility

To write the compatibility equation, investigate the way the shaft will twist when subjected to the external loads, and give consideration as to how the supports constrain the shaft when it is twisted.

Torsion

Mechanics of materials

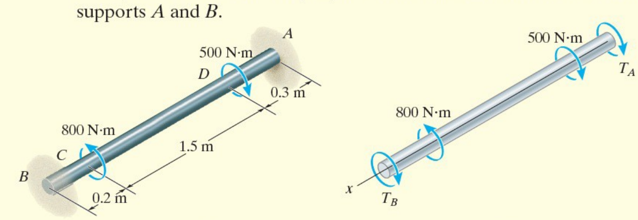
STATICALLY INDETERMINATE TORQUE-LOADED MEMBERS

Express the compatibility condition in terms of the rotational displacements caused by the reactive torques, and then use a torque-displacement relation, such as $\Phi = TL/JG$, to relate the unknown torques to the unknown displacements.

Solve the equilibrium and compatibility equations for the unknown reactive torques. If any of the magnitudes have a negative numerical value, it indicates that this torque acts in the opposite sense of direction to that indicated on the free-body diagram.

EXAMPLE 5.8

The solid steel shaft shown in Fig. 5–23*a* has a diameter of 20 mm. If it is subjected to the two torques, determine the reactions at the fixed supports *A* and *B*.



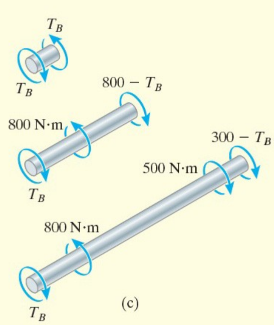
(a) (b)

SOLUTION

Equilibrium. By inspection of the free-body diagram, Fig. 5–23*b*, it is seen that the problem is statically indeterminate since there is only *one* available equation of equilibrium and there are two unknowns. We require

$$\Sigma M_x = 0; \qquad -T_B + 800\ \text{N}\cdot\text{m} - 500\ \text{N}\cdot\text{m} - T_A = 0 \qquad (1)$$

EXAMPLE 5.8 CONTINUED



(c)

Fig. 5–23

Compatibility. Since the ends of the shaft are fixed, the angle of twist of one end of the shaft with respect to the other must be zero. Hence, the compatibility equation becomes

$$\phi_{A/B} = 0$$

This condition can be expressed in terms of the unknown torques by using the load–displacement relationship, $\phi = TL/JG$. Here there are three regions of the shaft where the internal torque is constant. On the free-body diagrams in Fig. 5–23*c* we have shown the internal torques acting on the left segments of the shaft which are sectioned in each of these regions. This way the internal torque is only a function of T_B. Using the sign convention established in Sec. 5.4, we have

$$\frac{-T_B(0.2\ \text{m})}{JG} + \frac{(800 - T_B)(1.5\ \text{m})}{JG} + \frac{(300 - T_B)(0.3\ \text{m})}{JG} = 0$$

so that

$$T_B = 645\ \text{N}\cdot\text{m} \qquad \textit{Ans.}$$

Using Eq. 1,

$$T_A = -345\ \text{N}\cdot\text{m} \qquad \textit{Ans.}$$

The negative sign indicates that $\mathbf{T}_A$ acts in the opposite direction of that shown in Fig. 5–23*b*.

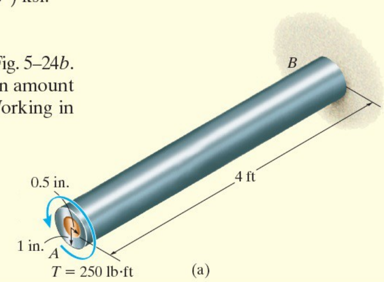
EXAMPLE 5.9

The shaft shown in Fig. 5–24*a* is made from a steel tube, which is bonded to a brass core. If a torque of $T = 250\ \text{lb}\cdot\text{ft}$ is applied at its end, plot the shear-stress distribution along a radial line of its cross-sectional area. Take $G_{st} = 11.4(10^3)$ ksi, $G_{br} = 5.20(10^3)$ ksi.

SOLUTION

Equilibrium. A free-body diagram of the shaft is shown in Fig. 5–24*b*. The reaction at the wall has been represented by the unknown amount of torque resisted by the steel, T_{st}, and by the brass, T_{br}. Working in units of pounds and inches, equilibrium requires

$$-T_{st} - T_{br} + (250\ \text{lb}\cdot\text{ft})(12\ \text{in./ft}) = 0 \qquad (1)$$



(a)

EXAMPLE 5.9 CONTINUED

Compatibility. We require the angle of twist of end A to be the same for both the steel and brass since they are bonded together. Thus,

$$\phi = \phi_{st} = \phi_{br}$$

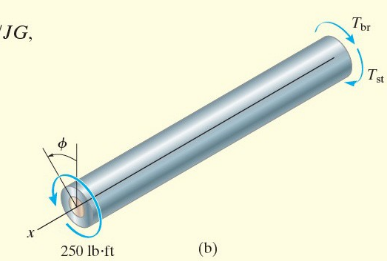
Applying the load–displacement relationship, $\phi = TL/JG$,

$$\frac{T_{st}L}{(\pi/2)[(1\ \text{in.})^4 - (0.5\ \text{in.})^4]11.4(10^3)\ \text{kip/in}^2} = \frac{T_{br}L}{(\pi/2)(0.5\ \text{in.})^4\, 5.20(10^3)\ \text{kip/in}^2}$$

$$T_{st} = 32.88\,T_{br} \qquad (2)$$

Solving Eqs. 1 and 2, we get

$$T_{st} = 2911.5\ \text{lb}\cdot\text{in.} = 242.6\ \text{lb}\cdot\text{ft}$$
$$T_{br} = 88.5\ \text{lb}\cdot\text{in.} = 7.38\ \text{lb}\cdot\text{ft}$$



(b)

EXAMPLE **5.9** CONTINUED

The shear stress in the brass core varies from zero at its center to a maximum at the interface where it contacts the steel tube. Using the torsion formula,

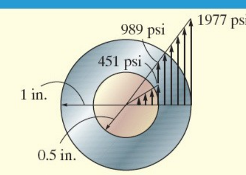
$$(\tau_{br})_{max} = \frac{(88.5 \text{ lb}\cdot\text{in.})(0.5 \text{ in.})}{(\pi/2)(0.5 \text{ in.})^4} = 451 \text{ psi}$$

For the steel, the minimum and maximum shear stresses are

$$(\tau_{st})_{min} = \frac{(2911.5 \text{ lb}\cdot\text{in.})(0.5 \text{ in.})}{(\pi/2)[(1 \text{ in.})^4 - (0.5 \text{ in.})^4]} = 989 \text{ psi}$$

$$(\tau_{st})_{max} = \frac{(2911.5 \text{ lb}\cdot\text{in.})(1 \text{ in.})}{(\pi/2)[(1 \text{ in.})^4 - (0.5 \text{ in.})^4]} = 1977 \text{ psi}$$

The results are plotted in Fig. 5–24*c*. Note the discontinuity of *shear stress* at the brass and steel interface. This is to be expected, since the materials have different moduli of rigidity; i.e., steel is stiffer than brass ($G_{st} > G_{br}$) and thus it carries more shear stress at the interface. Although the shear stress is discontinuous here, the *shear strain* is not. Rather, the shear strain is the *same* for both the brass and the steel.



Shear–stress distribution

(c)

Fig. 5–24