15.   Let λ be an eigenvalue of the $n \times n$ matrix A and $\mathbf{x} \neq \mathbf{0}$ be an associated eigenvector.

    a.   Show that λ is also an eigenvalue of A^t.

    b.   Show that for any integer $k \geq 1$, λ^k is an eigenvalue of A^k with eigenvector $\mathbf{x}$.

    c.   Show that if A^{-1} exists, then $1/\lambda$ is an eigenvalue of A^{-1} with eigenvector $\mathbf{x}$.

    d.   Generalize parts (b) and (c) to $(A^{-1})^k$ for integers $k \geq 2$.

    e.   Given the polynomial $q(x) = q_0 + q_1 x + \cdots + q_k x^k$, define $q(A)$ to be the matrix $q(A) = q_0 I + q_1 A + \cdots + q_k A^k$. Show that $q(\lambda)$ is an eigenvalue of $q(A)$ with eigenvector $\mathbf{x}$.

    f.   Let $\alpha \neq \lambda$ be given. Show that if $A - \alpha I$ is nonsingular, then $1/(\lambda - \alpha)$ is an eigenvalue of $(A - \alpha I)^{-1}$ with eigenvector $\mathbf{x}$.

16.   Show that if A is symmetric, then $||A||_2 = \rho(A)$.

17.   In Exercise 15 of Section 6.3, we assumed that the contribution a female beetle of a certain type made to the future years' beetle population could be expressed in terms of the matrix

$$A = \begin{bmatrix} 0 & 0 & 6 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix},$$

where the entry in the ith row and jth column represents the probabilistic contribution of a beetle of age j onto the next year's female population of age i.

    a.   Does the matrix A have any real eigenvalues? If so, determine them and any associated eigenvectors.

    b.   If a sample of this species was needed for laboratory test purposes that would have a constant proportion in each age group from year to year, what criteria could be imposed on the initial population to ensure that this requirement would be satisfied?

18.   Find matrices A and B for which $\rho(A + B) > \rho(A) + \rho(B)$. (This shows that $\rho(A)$ cannot be a matrix norm.)

19.   Show that if $|| \cdot ||$ is any natural norm, then $(||A^{-1}||)^{-1} \leq |\lambda| \leq ||A||$ for any eigenvalue λ of the nonsingular matrix A.

7.3   The Jacobi and Gauss-Siedel Iterative Techniques

In this section we describe the Jacobi and the Gauss-Seidel iterative methods, classic methods that date to the late eighteenth century. Iterative techniques are seldom used for solving linear systems of small dimension since the time required for sufficient accuracy exceeds that required for direct techniques such as Gaussian elimination. For large systems with a high percentage of 0 entries, however, these techniques are efficient in terms of both computer storage and computation. Systems of this type arise frequently in circuit analysis and in the numerical solution of boundary-value problems and partial-differential equations.

An iterative technique to solve the $n \times n$ linear system $A\mathbf{x} = \mathbf{b}$ starts with an initial approximation $\mathbf{x}^{(0)}$ to the solution $\mathbf{x}$ and generates a sequence of vectors $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ that converges to $\mathbf{x}$.

Jacobi's Method

The **Jacobi iterative method** is obtained by solving the ith equation in $A\mathbf{x} = \mathbf{b}$ for x_i to obtain (provided $a_{ii} \neq 0$)

$$x_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(-\frac{a_{ij} x_j}{a_{ii}} \right) + \frac{b_i}{a_{ii}}, \qquad \text{for } i = 1, 2, \ldots, n.$$

For each $k \geq 1$, generate the components $x_i^{(k)}$ of $\mathbf{x}^{(k)}$ from the components of $\mathbf{x}^{(k-1)}$ by

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[\sum_{\substack{j=1 \\ j \neq i}}^{n} \left(-a_{ij} x_j^{(k-1)} \right) + b_i \right], \qquad \text{for } i = 1, 2, \ldots, n. \qquad (7.5)$$

Example 1 The linear system $A\mathbf{x} = \mathbf{b}$ given by

Carl Gustav Jacob Jacobi
(1804–1851) was initially
recognized for his work in the
area of number theory and elliptic
functions, but his mathematical
interests and abilities were very
broad. He had a strong
personality that was influential in
establishing a research-oriented
attitude that became the nucleus
of a revival of mathematics at
German universities in the 19th
century.

$$
\begin{aligned}
E_1: \quad & 10x_1 - x_2 + 2x_3 && = 6, \\
E_2: \quad & -x_1 + 11x_2 - x_3 + 3x_4 && = 25, \\
E_3: \quad & 2x_1 - x_2 + 10x_3 - x_4 && = -11, \\
E_4: \quad & 3x_2 - x_3 + 8x_4 && = 15
\end{aligned}
$$

has the unique solution $\mathbf{x} = (1, 2, -1, 1)^t$. Use Jacobi's iterative technique to find approximations $\mathbf{x}^{(k)}$ to $\mathbf{x}$ starting with $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$ until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < 10^{-3}.$$

Solution We first solve equation E_i for x_i, for each $i = 1, 2, 3, 4$, to obtain

$$
\begin{aligned}
x_1 &= \frac{1}{10}x_2 - \frac{1}{5}x_3 + \frac{3}{5}, \\
x_2 &= \frac{1}{11}x_1 + \frac{1}{11}x_3 - \frac{3}{11}x_4 + \frac{25}{11}, \\
x_3 &= -\frac{1}{5}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_4 - \frac{11}{10}, \\
x_4 &= -\frac{3}{8}x_2 + \frac{1}{8}x_3 + \frac{15}{8}.
\end{aligned}
$$

From the initial approximation $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$ we have $\mathbf{x}^{(1)}$ given by

$$
\begin{aligned}
x_1^{(1)} &= \frac{1}{10}x_2^{(0)} - \frac{1}{5}x_3^{(0)} + \frac{3}{5} = 0.6000, \\
x_2^{(1)} &= \frac{1}{11}x_1^{(0)} + \frac{1}{11}x_3^{(0)} - \frac{3}{11}x_4^{(0)} + \frac{25}{11} = 2.2727, \\
x_3^{(1)} &= -\frac{1}{5}x_1^{(0)} + \frac{1}{10}x_2^{(0)} + \frac{1}{10}x_4^{(0)} - \frac{11}{10} = -1.1000, \\
x_4^{(1)} &= -\frac{3}{8}x_2^{(0)} + \frac{1}{8}x_3^{(0)} + \frac{15}{8} = 1.8750.
\end{aligned}
$$

Additional iterates, $\mathbf{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)})^t$, are generated in a similar manner and are presented in Table 7.1.

Table 7.1

k	0	1	2	3	4	5	6	7	8	9	10
$x_1^{(k)}$	0.0000	0.6000	1.0473	0.9326	1.0152	0.9890	1.0032	0.9981	1.0006	0.9997	1.0001
$x_2^{(k)}$	0.0000	2.2727	1.7159	2.053	1.9537	2.0114	1.9922	2.0023	1.9987	2.0004	1.9998
$x_3^{(k)}$	0.0000	−1.1000	−0.8052	−1.0493	−0.9681	−1.0103	−0.9945	−1.0020	−0.9990	−1.0004	−0.9998
$x_4^{(k)}$	0.0000	1.8750	0.8852	1.1309	0.9739	1.0214	0.9944	1.0036	0.9989	1.0006	0.9998

We stopped after ten iterations because

$$\frac{\|\mathbf{x}^{(10)} - \mathbf{x}^{(9)}\|_\infty}{\|\mathbf{x}^{(10)}\|_\infty} = \frac{8.0 \times 10^{-4}}{1.9998} < 10^{-3}.$$

In fact, $\|\mathbf{x}^{(10)} - \mathbf{x}\|_\infty = 0.0002$. ∎

In general, iterative techniques for solving linear systems involve a process that converts the system $A\mathbf{x} = \mathbf{b}$ into an equivalent system of the form $\mathbf{x} = T\mathbf{x} + \mathbf{c}$ for some fixed matrix T and vector $\mathbf{c}$. After the initial vector $\mathbf{x}^{(0)}$ is selected, the sequence of approximate solution vectors is generated by computing

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c},$$

for each $k = 1, 2, 3, \ldots$. This should be reminiscent of the fixed-point iteration studied in Chapter 2.

The Jacobi method can be written in the form $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$ by splitting A into its diagonal and off-diagonal parts. To see this, let D be the diagonal matrix whose diagonal entries are those of A, $-L$ be the strictly lower-triangular part of A, and $-U$ be the strictly upper-triangular part of A. With this notation,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

is split into

$$A = \begin{bmatrix} a_{11} & 0 & \cdots\cdots & 0 \\ 0 & a_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots\cdots & 0 & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & \cdots\cdots\cdots & 0 \\ -a_{21} & & & \vdots \\ \vdots & & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ \vdots & & \ddots & \vdots \\ & & & -a_{n-1,n} \\ 0 & \cdots\cdots\cdots & 0 \end{bmatrix}$$

$$= D - L - U.$$

The equation $A\mathbf{x} = \mathbf{b}$, or $(D - L - U)\mathbf{x} = \mathbf{b}$, is then transformed into

$$D\mathbf{x} = (L + U)\mathbf{x} + \mathbf{b},$$

and, if D^{-1} exists, that is, if $a_{ii} \neq 0$ for each i, then

$$\mathbf{x} = D^{-1}(L + U)\mathbf{x} + D^{-1}\mathbf{b}.$$

This results in the matrix form of the Jacobi iterative technique:

$$\mathbf{x}^{(k)} = D^{-1}(L + U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b}, \quad k = 1, 2, \ldots. \tag{7.6}$$

Introducing the notation $T_j = D^{-1}(L + U)$ and $\mathbf{c}_j = D^{-1}\mathbf{b}$ gives the Jacobi technique the form

$$\mathbf{x}^{(k)} = T_j\mathbf{x}^{(k-1)} + \mathbf{c}_j. \tag{7.7}$$

In practice, Eq. (7.5) is used in computation and Eq. (7.7) for theoretical purposes.

Example 2   Express the Jacobi iteration method for the linear system $A\mathbf{x} = \mathbf{b}$ given by

$$
\begin{aligned}
E_1 : \quad & 10x_1 - \quad x_2 + \ 2x_3 \qquad\quad = 6, \\
E_2 : \quad & -x_1 + 11x_2 - \quad x_3 + 3x_4 = 25, \\
E_3 : \quad & 2x_1 - \quad x_2 + 10x_3 - \quad x_4 = -11, \\
E_4 : \quad & \qquad\qquad 3x_2 - \quad x_3 + 8x_4 = 15
\end{aligned}
$$

in the form $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$.

Solution We saw in Example 1 that the Jacobi method for this system has the form

$$
\begin{aligned}
x_1 = & \qquad\qquad \frac{1}{10}x_2 - \frac{1}{5}x_3 \qquad\qquad + \frac{3}{5}, \\
x_2 = & \ \frac{1}{11}x_1 \qquad\qquad + \frac{1}{11}x_3 - \frac{3}{11}x_4 + \frac{25}{11}, \\
x_3 = & -\frac{1}{5}x_1 + \frac{1}{10}x_2 \qquad\qquad + \frac{1}{10}x_4 - \frac{11}{10}, \\
x_4 = & \qquad\qquad -\frac{3}{8}x_2 + \frac{1}{8}x_3 \qquad\qquad + \frac{15}{8}.
\end{aligned}
$$

Hence we have

$$
T = \begin{bmatrix} 0 & \frac{1}{10} & -\frac{1}{5} & 0 \\ \frac{1}{11} & 0 & \frac{1}{11} & -\frac{3}{11} \\ -\frac{1}{5} & \frac{1}{10} & 0 & \frac{1}{10} \\ 0 & -\frac{3}{8} & \frac{1}{8} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} \frac{3}{5} \\ \frac{25}{11} \\ -\frac{11}{10} \\ \frac{15}{8} \end{bmatrix}. \qquad \blacksquare
$$

Algorithm 7.1 implements the Jacobi iterative technique.

**ALGORITHM
7.1**

Jacobi Iterative

To solve $A\mathbf{x} = \mathbf{b}$ given an initial approximation $\mathbf{x}^{(0)}$:

INPUT   the number of equations and unknowns n; the entries a_{ij}, $1 \le i, j \le n$ of the matrix A; the entries b_i, $1 \le i \le n$ of $\mathbf{b}$; the entries XO_i, $1 \le i \le n$ of $\mathbf{XO} = \mathbf{x}^{(0)}$; tolerance TOL; maximum number of iterations N.

OUTPUT   the approximate solution $x_1, \dots, x_n$ or a message that the number of iterations was exceeded.

Step 1   Set $k = 1$.

Step 2   While $(k \le N)$ do Steps 3–6.

  Step 3   For $i = 1, \dots, n$

$$
\text{set } x_i = \frac{1}{a_{ii}} \left[-\sum_{\substack{j=1 \\ j \ne i}}^{n} (a_{ij} XO_j) + b_i \right].
$$

  Step 4   If $||\mathbf{x} - \mathbf{XO}|| < TOL$ then OUTPUT $(x_1, \dots, x_n)$;
                         (*The procedure was successful.*)
                         STOP.

  Step 5   Set $k = k + 1$.

> *Step 6* For $i = 1, \ldots, n$ set $XO_i = x_i$.

Step 7 OUTPUT ('Maximum number of iterations exceeded');
(*The procedure was successful*.)
STOP. ■

Step 3 of the algorithm requires that $a_{ii} \neq 0$, for each $i = 1, 2, \ldots, n$. If one of the a_{ii} entries is 0 and the system is nonsingular, a reordering of the equations can be performed so that no $a_{ii} = 0$. To speed convergence, the equations should be arranged so that a_{ii} is as large as possible. This subject is discussed in more detail later in this chapter.

Another possible stopping criterion in Step 4 is to iterate until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|}{\|\mathbf{x}^{(k)}\|}$$

is smaller than some prescribed tolerance. For this purpose, any convenient norm can be used, the usual being the l_∞ norm.

The *NumericalAnalysis* subpackage of the Maple *Student* package implements the Jacobi iterative method. To illustrate this with our example we first enter both *NumericalAnalysis* and *LinearAlgebra*.

with(*Student*[*NumericalAnalysis*]): *with*(*LinearAlgebra*):

Colons are used at the end of the commands to suppress output for both packages. Enter the matrix with

$A := Matrix([[10, -1, 2, 0, 6], [-1, 11, -1, 3, 25], [2, -1, 10, -1, -11], [0, 3, -1, 8, 15]])$

The following command gives a collection of output that is in agreement with the results in Table 7.1.

IterativeApproximate(A, *initialapprox* = *Vector*([0., 0., 0., 0.]), *tolerance* = 10^{-3}, *maxiterations* = 20, *stoppingcriterion* = *relative*(*infinity*), *method* = *jacobi*, *output* = *approximates*)

If the option *output = approximates* is omitted, then only the final approximation result is output. Notice that the initial approximations was specified by [0., 0., 0., 0.], with decimal points placed after the entries. This was done so that Maple will give the results as 10-digit decimals. If the specification had simply been [0, 0, 0, 0], the output would have been given in fractional form.

The Gauss-Seidel Method

Phillip Ludwig Seidel (1821–1896) worked as an assistant to Jacobi solving problems on systems of linear equations that resulted from Gauss's work on least squares. These equations generally had off-diagonal elements that were much smaller than those on the diagonal, so the iterative methods were particularly effective. The iterative techniques now known as Jacobi and Gauss-Seidel were both known to Gauss before being applied in this situation, but Gauss's results were not often widely communicated.

A possible improvement in Algorithm 7.1 can be seen by reconsidering Eq. (7.5). The components of $\mathbf{x}^{(k-1)}$ are used to compute all the components $x_i^{(k)}$ of $\mathbf{x}^{(k)}$. But, for $i > 1$, the components $x_1^{(k)}, \ldots, x_{i-1}^{(k)}$ of $\mathbf{x}^{(k)}$ have already been computed and are expected to be better approximations to the actual solutions $x_1, \ldots, x_{i-1}$ than are $x_1^{(k-1)}, \ldots, x_{i-1}^{(k-1)}$. It seems reasonable, then, to compute $x_i^{(k)}$ using these most recently calculated values. That is, to use

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[-\sum_{j=1}^{i-1}(a_{ij}x_j^{(k)}) - \sum_{j=i+1}^{n}(a_{ij}x_j^{(k-1)}) + b_i \right], \tag{7.8}$$

for each $i = 1, 2, \ldots, n$, instead of Eq. (7.5). This modification is called the **Gauss-Seidel iterative technique** and is illustrated in the following example.

Example 3   Use the Gauss-Seidel iterative technique to find approximate solutions to

$$
\begin{aligned}
10x_1 - x_2 + 2x_3 \qquad &= 6, \\
-x_1 + 11x_2 - x_3 + 3x_4 &= 25, \\
2x_1 - x_2 + 10x_3 - x_4 &= -11, \\
3x_2 - x_3 + 8x_4 &= 15
\end{aligned}
$$

starting with $\mathbf{x} = (0, 0, 0, 0)^t$ and iterating until

$$
\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < 10^{-3}.
$$

Solution   The solution $\mathbf{x} = (1, 2, -1, 1)^t$ was approximated by Jacobi's method in Example 1. For the Gauss-Seidel method we write the system, for each $k = 1, 2, \ldots$ as

$$
\begin{aligned}
x_1^{(k)} &= \qquad\qquad \frac{1}{10}x_2^{(k-1)} - \frac{1}{5}x_3^{(k-1)} \qquad\qquad + \frac{3}{5}, \\
x_2^{(k)} &= \frac{1}{11}x_1^{(k)} \qquad\qquad + \frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)} + \frac{25}{11}, \\
x_3^{(k)} &= -\frac{1}{5}x_1^{(k)} + \frac{1}{10}x_2^{(k)} \qquad\qquad + \frac{1}{10}x_4^{(k-1)} - \frac{11}{10}, \\
x_4^{(k)} &= \qquad\quad -\frac{3}{8}x_2^{(k)} + \frac{1}{8}x_3^{(k)} \qquad\qquad + \frac{15}{8}.
\end{aligned}
$$

When $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$, we have $\mathbf{x}^{(1)} = (0.6000, 2.3272, -0.9873, 0.8789)^t$. Subsequent iterations give the values in Table 7.2.

Table 7.2

k	0	1	2	3	4	5
$x_1^{(k)}$	0.0000	0.6000	1.030	1.0065	1.0009	1.0001
$x_2^{(k)}$	0.0000	2.3272	2.037	2.0036	2.0003	2.0000
$x_3^{(k)}$	0.0000	-0.9873	-1.014	-1.0025	-1.0003	-1.0000
$x_4^{(k)}$	0.0000	0.8789	0.9844	0.9983	0.9999	1.0000

Because

$$
\frac{\|\mathbf{x}^{(5)} - \mathbf{x}^{(4)}\|_\infty}{\|\mathbf{x}^{(5)}\|_\infty} = \frac{0.0008}{2.000} = 4 \times 10^{-4},
$$

$\mathbf{x}^{(5)}$ is accepted as a reasonable approximation to the solution. Note that Jacobi's method in Example 1 required twice as many iterations for the same accuracy.   ∎

To write the Gauss-Seidel method in matrix form, multiply both sides of Eq. (7.8) by a_{ii} and collect all kth iterate terms, to give

$$
a_{i1}x_1^{(k)} + a_{i2}x_2^{(k)} + \cdots + a_{ii}x_i^{(k)} = -a_{i,i+1}x_{i+1}^{(k-1)} - \cdots - a_{in}x_n^{(k-1)} + b_i,
$$

for each $i = 1, 2, \ldots, n$. Writing all n equations gives

$$
\begin{aligned}
a_{11}x_1^{(k)} \qquad\qquad\qquad &= -a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)} - \cdots - a_{1n}x_n^{(k-1)} + b_1, \\
a_{21}x_1^{(k)} + a_{22}x_2^{(k)} \qquad\qquad &= \qquad\qquad -a_{23}x_3^{(k-1)} - \cdots - a_{2n}x_n^{(k-1)} + b_2, \\
\vdots \qquad\qquad\qquad & \\
a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \cdots + a_{nn}x_n^{(k)} &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad b_n;
\end{aligned}
$$

with the definitions of D, L, and U given previously, we have the Gauss-Seidel method represented by

$$(D - L)\mathbf{x}^{(k)} = U\mathbf{x}^{(k-1)} + \mathbf{b}$$

and

$$\mathbf{x}^{(k)} = (D - L)^{-1}U\mathbf{x}^{(k-1)} + (D - L)^{-1}\mathbf{b}, \quad \text{for each } k = 1, 2, \ldots. \tag{7.9}$$

Letting $T_g = (D - L)^{-1}U$ and $\mathbf{c}_g = (D - L)^{-1}\mathbf{b}$, gives the Gauss-Seidel technique the form

$$\mathbf{x}^{(k)} = T_g\mathbf{x}^{(k-1)} + \mathbf{c}_g. \tag{7.10}$$

For the lower-triangular matrix $D - L$ to be nonsingular, it is necessary and sufficient that $a_{ii} \neq 0$, for each $i = 1, 2, \ldots, n$.

Algorithm 7.2 implements the Gauss-Seidel method.

ALGORITHM
7.2

Gauss-Seidel Iterative

To solve $A\mathbf{x} = \mathbf{b}$ given an initial approximation $\mathbf{x}^{(0)}$:

INPUT the number of equations and unknowns n; the entries a_{ij}, $1 \leq i, j \leq n$ of the matrix A; the entries b_i, $1 \leq i \leq n$ of $\mathbf{b}$; the entries XO_i, $1 \leq i \leq n$ of $\mathbf{XO} = \mathbf{x}^{(0)}$; tolerance *TOL*; maximum number of iterations N.

OUTPUT the approximate solution $x_1, \ldots, x_n$ or a message that the number of iterations was exceeded.

Step 1 Set $k = 1$.

Step 2 While ($k \leq N$) do Steps 3–6.

   Step 3 For $i = 1, \ldots, n$

$$\text{set } x_i = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1} a_{ij}x_j - \sum_{j=i+1}^{n} a_{ij}XO_j + b_i \right].$$

   Step 4 If $||\mathbf{x} - \mathbf{XO}|| < TOL$ then OUTPUT $(x_1, \ldots, x_n)$;
                                               (The procedure was successful.)
                                              STOP.

   Step 5 Set $k = k + 1$.

   Step 6 For $i = 1, \ldots, n$ set $XO_i = x_i$.

Step 7 OUTPUT ('Maximum number of iterations exceeded');
       (The procedure was successful.)
       STOP.

     The comments following Algorithm 7.1 regarding reordering and stopping criteria also apply to the Gauss-Seidel Algorithm 7.2.

     The results of Examples 1 and 2 appear to imply that the Gauss-Seidel method is superior to the Jacobi method. This is almost always true, but there are linear systems for which the Jacobi method converges and the Gauss-Seidel method does not (see Exercises 9 and 10).

The *NumericalAnalysis* subpackage of the Maple *Student* package implements the Gauss-Siedel method in a manner similar to that of the Jacobi iterative method. The results in Table 7.2 are obtained by loading both *NumericalAnalysis* and *LinearAlgebra*, the matrix *A*, and then using the command

IterativeApproximate(A, initialapprox = Vector([0., 0., 0., 0.]), tolerance = 10^{-3}, maxiterations = 20, stoppingcriterion = relative(infinity), method = gaussseidel, output = approximates)

If we change the final option to *output* $=$ [*approximates, distances*], the output also includes the l_∞ distances between the approximations and the actual solution.

General Iteration Methods

To study the convergence of general iteration techniques, we need to analyze the formula

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}, \quad \text{for each } k = 1, 2, \ldots,$$

where $\mathbf{x}^{(0)}$ is arbitrary. The next lemma and Theorem 7.17 on page 449 provide the key for this study.

Lemma 7.18 If the spectral radius satisfies $\rho(T) < 1$, then $(I - T)^{-1}$ exists, and

$$(I - T)^{-1} = I + T + T^2 + \cdots = \sum_{j=0}^{\infty} T^j. \qquad \blacksquare$$

Proof Because $T\mathbf{x} = \lambda\mathbf{x}$ is true precisely when $(I - T)\mathbf{x} = (1 - \lambda)\mathbf{x}$, we have λ as an eigenvalue of T precisely when $1 - \lambda$ is an eigenvalue of $I - T$. But $|\lambda| \leq \rho(T) < 1$, so $\lambda = 1$ is not an eigenvalue of T, and 0 cannot be an eigenvalue of $I - T$. Hence, $(I - T)^{-1}$ exists.

Let $S_m = I + T + T^2 + \cdots + T^m$. Then

$$(I - T)S_m = (1 + T + T^2 + \cdots + T^m) - (T + T^2 + \cdots + T^{m+1}) = I - T^{m+1},$$

and, since T is convergent, Theorem 7.17 implies that

$$\lim_{m \to \infty} (I - T)S_m = \lim_{m \to \infty} (I - T^{m+1}) = I.$$

Thus, $(I - T)^{-1} = \lim_{m \to \infty} S_m = I + T + T^2 + \cdots = \sum_{j=0}^{\infty} T^j. \qquad \blacksquare\ \blacksquare\ \blacksquare$

Theorem 7.19 For any $\mathbf{x}^{(0)} \in \mathbb{R}^n$, the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}, \quad \text{for each } k \geq 1, \tag{7.11}$$

converges to the unique solution of $\mathbf{x} = T\mathbf{x} + \mathbf{c}$ if and only if $\rho(T) < 1$. $\qquad \blacksquare$

Proof First assume that $\rho(T) < 1$. Then,

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$$
$$= T(T\mathbf{x}^{(k-2)} + \mathbf{c}) + \mathbf{c}$$
$$= T^2\mathbf{x}^{(k-2)} + (T + I)\mathbf{c}$$
$$\vdots$$
$$= T^k\mathbf{x}^{(0)} + (T^{k-1} + \cdots + T + I)\mathbf{c}.$$

Because $\rho(T) < 1$, Theorem 7.17 implies that T is convergent, and

$$\lim_{k \to \infty} T^k \mathbf{x}^{(0)} = \mathbf{0}.$$

Lemma 7.18 implies that

$$\lim_{k \to \infty} \mathbf{x}^{(k)} = \lim_{k \to \infty} T^k \mathbf{x}^{(0)} + \left(\sum_{j=0}^{\infty} T^j \right) \mathbf{c} = \mathbf{0} + (I - T)^{-1} \mathbf{c} = (I - T)^{-1} \mathbf{c}.$$

Hence, the sequence $\{\mathbf{x}^{(k)}\}$ converges to the vector $\mathbf{x} \equiv (I - T)^{-1} \mathbf{c}$ and $\mathbf{x} = T\mathbf{x} + \mathbf{c}$.

To prove the converse, we will show that for any $\mathbf{z} \in \mathbb{R}^n$, we have $\lim_{k \to \infty} T^k \mathbf{z} = \mathbf{0}$. By Theorem 7.17, this is equivalent to $\rho(T) < 1$.

Let $\mathbf{z}$ be an arbitrary vector, and $\mathbf{x}$ be the unique solution to $\mathbf{x} = T\mathbf{x} + \mathbf{c}$. Define $\mathbf{x}^{(0)} = \mathbf{x} - \mathbf{z}$, and, for $k \geq 1$, $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$. Then $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}$. Also,

$$\mathbf{x} - \mathbf{x}^{(k)} = (T\mathbf{x} + \mathbf{c}) - \left(T\mathbf{x}^{(k-1)} + \mathbf{c} \right) = T \left(\mathbf{x} - \mathbf{x}^{(k-1)} \right),$$

so

$$\mathbf{x} - \mathbf{x}^{(k)} = T \left(\mathbf{x} - \mathbf{x}^{(k-1)} \right) = T^2 \left(\mathbf{x} - \mathbf{x}^{(k-2)} \right) = \cdots = T^k \left(\mathbf{x} - \mathbf{x}^{(0)} \right) = T^k \mathbf{z}.$$

Hence $\lim_{k \to \infty} T^k \mathbf{z} = \lim_{k \to \infty} T^k \left(\mathbf{x} - \mathbf{x}^{(0)} \right) = \lim_{k \to \infty} \left(\mathbf{x} - \mathbf{x}^{(k)} \right) = \mathbf{0}$.

But $\mathbf{z} \in \mathbb{R}^n$ was arbitrary, so by Theorem 7.17, T is convergent and $\rho(T) < 1$. ▪ ▪ ▪

The proof of the following corollary is similar to the proofs in Corollary 2.5 on page 62. It is considered in Exercise 13.

Corollary 7.20   If $\|T\| < 1$ for any natural matrix norm and $\mathbf{c}$ is a given vector, then the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$ converges, for any $\mathbf{x}^{(0)} \in \mathbb{R}^n$, to a vector $\mathbf{x} \in \mathbb{R}^n$, with $\mathbf{x} = T\mathbf{x} + \mathbf{c}$, and the following error bounds hold:

(i) $\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \|T\|^k \|\mathbf{x}^{(0)} - \mathbf{x}\|$;   (ii) $\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \frac{\|T\|^k}{1 - \|T\|} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|$.   ▪

We have seen that the Jacobi and Gauss-Seidel iterative techniques can be written

$$\mathbf{x}^{(k)} = T_j \mathbf{x}^{(k-1)} + \mathbf{c}_j \quad \text{and} \quad \mathbf{x}^{(k)} = T_g \mathbf{x}^{(k-1)} + \mathbf{c}_g,$$

using the matrices

$$T_j = D^{-1}(L + U) \quad \text{and} \quad T_g = (D - L)^{-1} U.$$

If $\rho(T_j)$ or $\rho(T_g)$ is less than 1, then the corresponding sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ will converge to the solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$. For example, the Jacobi scheme has

$$\mathbf{x}^{(k)} = D^{-1}(L + U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b},$$

and, if $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ converges to $\mathbf{x}$, then

$$\mathbf{x} = D^{-1}(L + U)\mathbf{x} + D^{-1}\mathbf{b}.$$

This implies that

$$D\mathbf{x} = (L + U)\mathbf{x} + \mathbf{b} \quad \text{and} \quad (D - L - U)\mathbf{x} = \mathbf{b}.$$

Since $D - L - U = A$, the solution $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{b}$.

We can now give easily verified sufficiency conditions for convergence of the Jacobi and Gauss-Seidel methods. (To prove convergence for the Jacobi scheme see Exercise 14, and for the Gauss-Seidel scheme see [Or2], p. 120.)

Theorem 7.21   If A is strictly diagonally dominant, then for any choice of $\mathbf{x}^{(0)}$, both the Jacobi and Gauss-Seidel methods give sequences $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ that converge to the unique solution of $A\mathbf{x} = \mathbf{b}$. ∎

The relationship of the rapidity of convergence to the spectral radius of the iteration matrix T can be seen from Corollary 7.20. The inequalities hold for any natural matrix norm, so it follows from the statement after Theorem 7.15 on page 446 that

$$\|\mathbf{x}^{(k)} - \mathbf{x}\| \approx \rho(T)^k \|\mathbf{x}^{(0)} - \mathbf{x}\|. \tag{7.12}$$

Thus we would like to select the iterative technique with minimal $\rho(T) < 1$ for a particular system $A\mathbf{x} = \mathbf{b}$. No general results exist to tell which of the two techniques, Jacobi or Gauss-Seidel, will be most successful for an arbitrary linear system. In special cases, however, the answer is known, as is demonstrated in the following theorem. The proof of this result can be found in [Y], pp. 120–127.

Theorem 7.22   **(Stein-Rosenberg)**

If $a_{ij} \leq 0$, for each $i \neq j$ and $a_{ii} > 0$, for each $i = 1, 2, \ldots, n$, then one and only one of the following statements holds:

    (i) $0 \leq \rho(T_g) < \rho(T_j) < 1$;   **(ii)** $1 < \rho(T_j) < \rho(T_g)$;

    (iii) $\rho(T_j) = \rho(T_g) = 0$;   **(iv)** $\rho(T_j) = \rho(T_g) = 1$. ∎

For the special case described in Theorem 7.22, we see from part (i) that when one method gives convergence, then both give convergence, and the Gauss-Seidel method converges faster than the Jacobi method. Part (ii) indicates that when one method diverges then both diverge, and the divergence is more pronounced for the Gauss-Seidel method.

EXERCISE SET 7.3

1. Find the first two iterations of the Jacobi method for the following linear systems, using $\mathbf{x}^{(0)} = \mathbf{0}$:

  a. $\begin{aligned} 3x_1 - x_2 + x_3 &= 1, \\ 3x_1 + 6x_2 + 2x_3 &= 0, \\ 3x_1 + 3x_2 + 7x_3 &= 4. \end{aligned}$
      b. $\begin{aligned} 10x_1 - x_2 \qquad\quad &= 9, \\ -x_1 + 10x_2 - 2x_3 &= 7, \\ - 2x_2 + 10x_3 &= 6. \end{aligned}$

  c. $\begin{aligned} 10x_1 + 5x_2 \qquad\qquad\quad &= 6, \\ 5x_1 + 10x_2 - 4x_3 \qquad\quad &= 25, \\ - 4x_2 + 8x_3 - x_4 &= -11, \\ - x_3 + 5x_4 &= -11. \end{aligned}$
  d. $\begin{aligned} 4x_1 + x_2 + x_3 + \qquad x_5 &= 6, \\ -x_1 - 3x_2 + x_3 + x_4 \qquad &= 6, \\ 2x_1 + x_2 + 5x_3 - x_4 - x_5 &= 6, \\ -x_1 - x_2 - x_3 + 4x_4 \qquad &= 6, \\ 2x_2 - x_3 + x_4 + 4x_5 &= 6. \end{aligned}$

2. Find the first two iterations of the Jacobi method for the following linear systems, using $\mathbf{x}^{(0)} = \mathbf{0}$:

  a. $\begin{aligned} 4x_1 + x_2 - x_3 &= 5, \\ -x_1 + 3x_2 + x_3 &= -4, \\ 2x_1 + 2x_2 + 5x_3 &= 1. \end{aligned}$
      b. $\begin{aligned} -2x_1 + x_2 + \tfrac{1}{2}x_3 &= 4, \\ x_1 - 2x_2 - \tfrac{1}{2}x_3 &= -4, \\ x_2 + 2x_3 &= 0. \end{aligned}$

  c. $\begin{aligned} 4x_1 + x_2 - x_3 + x_4 &= -2, \\ x_1 + 4x_2 - x_3 - x_4 &= -1, \\ -x_1 - x_2 + 5x_3 + x_4 &= 0, \\ x_1 - x_2 + x_3 + 3x_4 &= 1. \end{aligned}$
  d. $\begin{aligned} 4x_1 - x_2 \qquad - x_4 \qquad\qquad &= 0, \\ -x_1 + 4x_2 - x_3 \qquad - x_5 \qquad &= 5, \\ - x_2 + 4x_3 \qquad\qquad - x_6 &= 0, \\ -x_1 \qquad\qquad + 4x_4 - x_5 \qquad &= 6, \\ - x_2 \qquad - x_4 + 4x_5 - x_6 &= -2, \\ - x_3 \qquad - x_5 + 4x_6 &= 6. \end{aligned}$

3. Repeat Exercise 1 using the Gauss-Seidel method.

4. Repeat Exercise 2 using the Gauss-Seidel method.

5. Use the Jacobi method to solve the linear systems in Exercise 1, with $TOL = 10^{-3}$ in the l_∞ norm.

6. Use the Jacobi method to solve the linear systems in Exercise 2, with $TOL = 10^{-3}$ in the l_∞ norm.

7. Use the Gauss-Seidel method to solve the linear systems in Exercise 1, with $TOL = 10^{-3}$ in the l_∞ norm.

8. Use the Gauss-Seidel method to solve the linear systems in Exercise 2, with $TOL = 10^{-3}$ in the l_∞ norm.

9. The linear system

$$
\begin{aligned}
2x_1 - x_2 + x_3 &= -1, \\
2x_1 + 2x_2 + 2x_3 &= 4, \\
-x_1 - x_2 + 2x_3 &= -5
\end{aligned}
$$

has the solution $(1, 2, -1)^t$.

a. Show that $\rho(T_j) = \frac{\sqrt{5}}{2} > 1$.

b. Show that the Jacobi method with $\mathbf{x}^{(0)} = \mathbf{0}$ fails to give a good approximation after 25 iterations.

c. Show that $\rho(T_g) = \frac{1}{2}$.

d. Use the Gauss-Seidel method with $\mathbf{x}^{(0)} = \mathbf{0}$ to approximate the solution to the linear system to within 10^{-5} in the l_∞ norm.

10. The linear system

$$
\begin{aligned}
x_1 + 2x_2 - 2x_3 &= 7, \\
x_1 + x_2 + x_3 &= 2, \\
2x_1 + 2x_2 + x_3 &= 5
\end{aligned}
$$

has the solution $(1, 2, -1)^t$.

a. Show that $\rho(T_j) = 0$.

b. Use the Jacobi method with $\mathbf{x}^{(0)} = \mathbf{0}$ to approximate the solution to the linear system to within 10^{-5} in the l_∞ norm.

c. Show that $\rho(T_g) = 2$.

d. Show that the Gauss-Seidel method applied as in part (b) fails to give a good approximation in 25 iterations.

11. The linear system

$$
\begin{aligned}
x_1 \quad\quad - x_3 &= 0.2, \\
-\frac{1}{2}x_1 + x_2 - \frac{1}{4}x_3 &= -1.425, \\
x_1 - \frac{1}{2}x_2 + x_3 &= 2.
\end{aligned}
$$

has the solution $(0.9, -0.8, 0.7)^t$.

a. Is the coefficient matrix

$$
A = \begin{bmatrix} 1 & 0 & -1 \\ -\frac{1}{2} & 1 & -\frac{1}{4} \\ 1 & -\frac{1}{2} & 1 \end{bmatrix}
$$

strictly diagonally dominant?

b. Compute the spectral radius of the Gauss-Seidel matrix T_g.

c. Use the Gauss-Seidel iterative method to approximate the solution to the linear system with a tolerance of 10^{-2} and a maximum of 300 iterations.

d. What happens in part (c) when the system is changed to

$$
\begin{aligned}
x_1 \quad\quad - 2x_3 &= 0.2, \\
-\frac{1}{2}x_1 + x_2 - \frac{1}{4}x_3 &= -1.425, \\
x_1 - \frac{1}{2}x_2 + x_3 &= 2.
\end{aligned}
$$