c. $y' = (y^2 + y)/t, \quad 1 \le t \le 3, \quad y(1) = -2,$ with $h = 0.2$

d. $y' = -ty + 4t/y, \quad 0 \le t \le 1, \quad y(0) = 1,$ with $h = 0.1$

7. Repeat Exercise 5 using Taylor's method of order four.

8. Repeat Exercise 6 using Taylor's method of order four.

9. Given the initial-value problem

$$y' = \frac{2}{t}y + t^2e^t, \quad 1 \le t \le 2, \quad y(1) = 0,$$

with exact solution $y(t) = t^2(e^t - e)$:

a. Use Taylor's method of order two with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.

b. Use the answers generated in part (a) and linear interpolation to approximate y at the following values, and compare them to the actual values of y.

i. $y(1.04)$ **ii.** $y(1.55)$ **iii.** $y(1.97)$

c. Use Taylor's method of order four with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.

d. Use the answers generated in part (c) and piecewise cubic Hermite interpolation to approximate y at the following values, and compare them to the actual values of y.

i. $y(1.04)$ **ii.** $y(1.55)$ **iii.** $y(1.97)$

10. Given the initial-value problem

$$y' = \frac{1}{t^2} - \frac{y}{t} - y^2, \quad 1 \le t \le 2, \quad y(1) = -1,$$

with exact solution $y(t) = -1/t$:

a. Use Taylor's method of order two with $h = 0.05$ to approximate the solution, and compare it with the actual values of y.

b. Use the answers generated in part (a) and linear interpolation to approximate the following values of y, and compare them to the actual values.

i. $y(1.052)$ **ii.** $y(1.555)$ **iii.** $y(1.978)$

c. Use Taylor's method of order four with $h = 0.05$ to approximate the solution, and compare it with the actual values of y.

d. Use the answers generated in part (c) and piecewise cubic Hermite interpolation to approximate the following values of y, and compare them to the actual values.

i. $y(1.052)$ **ii.** $y(1.555)$ **iii.** $y(1.978)$

11. A projectile of mass $m = 0.11$ kg shot vertically upward with initial velocity $v(0) = 8$ m/s is slowed due to the force of gravity, $F_g = -mg$, and due to air resistance, $F_r = -kv|v|$, where $g = 9.8$ m/s^2 and $k = 0.002$ kg/m. The differential equation for the velocity v is given by

$$mv' = -mg - kv|v|.$$

a. Find the velocity after $0.1, 0.2, \ldots, 1.0$ s.

b. To the nearest tenth of a second, determine when the projectile reaches its maximum height and begins falling.

12. Use the Taylor method of order two with $h = 0.1$ to approximate the solution to

$$y' = 1 + t\sin(ty), \quad 0 \le t \le 2, \quad y(0) = 0.$$

5.4 Runge-Kutta Methods

The Taylor methods outlined in the previous section have the desirable property of high-order local truncation error, but the disadvantage of requiring the computation and evaluation of the derivatives of $f(t, y)$. This is a complicated and time-consuming procedure for most problems, so the Taylor methods are seldom used in practice.

In the later 1800s, Carl Runge (1856–1927) used methods similar to those in this section to derive numerous formulas for approximating the solution to initial-value problems.

Runge-Kutta methods have the high-order local truncation error of the Taylor methods but eliminate the need to compute and evaluate the derivatives of $f(t, y)$. Before presenting the ideas behind their derivation, we need to consider Taylor's Theorem in two variables. The proof of this result can be found in any standard book on advanced calculus (see, for example, [Fu], p. 331).

Theorem 5.13 Suppose that $f(t, y)$ and all its partial derivatives of order less than or equal to $n + 1$ are continuous on $D = \{(t, y) \mid a \le t \le b, c \le y \le d\}$, and let $(t_0, y_0) \in D$. For every $(t, y) \in D$, there exists ξ between t and t_0 and μ between y and y_0 with

$$f(t, y) = P_n(t, y) + R_n(t, y),$$

In 1901, Martin Wilhelm Kutta (1867–1944) generalized the methods that Runge developed in 1895 to incorporate systems of first-order differential equations. These techniques differ slightly from those we currently call Runge-Kutta methods.

where

$$\begin{aligned}
P_n(t, y) = f(t_0, y_0) &+ \left[(t - t_0)\frac{\partial f}{\partial t}(t_0, y_0) + (y - y_0)\frac{\partial f}{\partial y}(t_0, y_0)\right] \\
&+ \left[\frac{(t - t_0)^2}{2}\frac{\partial^2 f}{\partial t^2}(t_0, y_0) + (t - t_0)(y - y_0)\frac{\partial^2 f}{\partial t \partial y}(t_0, y_0)\right. \\
&+ \left.\frac{(y - y_0)^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0, y_0)\right] + \cdots \\
&+ \left[\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(t - t_0)^{n-j}(y - y_0)^j\frac{\partial^n f}{\partial t^{n-j}\partial y^j}(t_0, y_0)\right]
\end{aligned}$$

and

$$R_n(t, y) = \frac{1}{(n + 1)!}\sum_{j=0}^{n+1}\binom{n + 1}{j}(t - t_0)^{n+1-j}(y - y_0)^j\frac{\partial^{n+1} f}{\partial t^{n+1-j}\partial y^j}(\xi, \mu).$$

The function $P_n(t, y)$ is called the ***n*th Taylor polynomial in two variables** for the function f about (t_0, y_0), and $R_n(t, y)$ is the remainder term associated with $P_n(t, y)$. ■

Example 1 Use Maple to determine $P_2(t, y)$, the second Taylor polynomial about $(2, 3)$ for the function

$$f(t, y) = \exp\left[-\frac{(t - 2)^2}{4} - \frac{(y - 3)^2}{4}\right]\cos(2t + y - 7)$$

Solution To determine $P_2(t, y)$ we need the values of f and its first and second partial derivatives at $(2, 3)$. The evaluation of the function is easy

$$f(2, 3) = e^{\left(-0^2/4 - 0^2/4\right)}\cos(4 + 3 - 7) = 1,$$

but the computations involved with the partial derivatives are quite tedious. However, higher dimensional Taylor polynomials are available in the *MultivariateCalculus* subpackage of the *Student* package, which is accessed with the command

with(*Student*[*MultivariateCalculus*])

The first option of the *TaylorApproximation* command is the function, the second specifies the point (t_0, y_0) where the polynomial is centered, and the third specifies the degree of the polynomial. So we issue the command

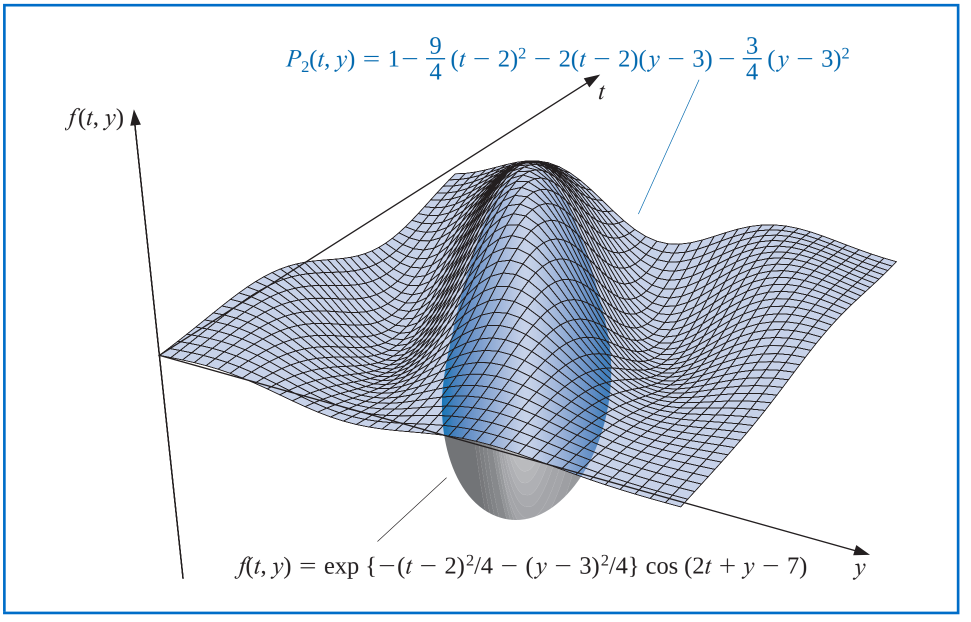
$$TaylorApproximation\left(e^{-\frac{(t-2)^2}{4}-\frac{(y-3)^2}{4}}\cos(2t+y-7),[t,y]=[2,3],2\right)$$

The response from this Maple command is the polynomial

$$1-\frac{9}{4}(t-2)^2-2(t-2)(y-3)-\frac{3}{4}(y-3)^2$$

A plot option is also available by adding a fourth option to the *TaylorApproximation* command in the form *output* = *plot*. The plot in the default form is quite crude, however, because not many points are plotted for the function and the polynomial. A better illustration is seen in Figure 5.5.

Figure 5.5



The final parameter in this command indicates that we want the second multivariate Taylor polynomial, that is, the quadratic polynomial. If this parameter is 2, we get the quadratic polynomial, and if it is 0 or 1, we get the constant polynomial 1, because there are no linear terms. When this parameter is omitted, it defaults to 6 and gives the sixth Taylor polynomial. ■

Runge-Kutta Methods of Order Two

The first step in deriving a Runge-Kutta method is to determine values for a_1, α_1, and β_1 with the property that $a_1 f(t+\alpha_1, y+\beta_1)$ approximates

$$T^{(2)}(t,y)=f(t,y)+\frac{h}{2}f'(t,y),$$

with error no greater than $O(h^2)$, which is same as the order of the local truncation error for the Taylor method of order two. Since

$$f'(t,y)=\frac{df}{dt}(t,y)=\frac{\partial f}{\partial t}(t,y)+\frac{\partial f}{\partial y}(t,y)\cdot y'(t) \quad \text{and} \quad y'(t)=f(t,y),$$

we have

$$T^{(2)}(t, y) = f(t, y) + \frac{h}{2}\frac{\partial f}{\partial t}(t, y) + \frac{h}{2}\frac{\partial f}{\partial y}(t, y) \cdot f(t, y). \tag{5.18}$$

Expanding $f(t + \alpha_1, y + \beta_1)$ in its Taylor polynomial of degree one about (t, y) gives

$$\begin{aligned} a_1 f(t + \alpha_1, y + \beta_1) = a_1 f(t, y) + a_1\alpha_1 \frac{\partial f}{\partial t}(t, y) \\ + a_1\beta_1 \frac{\partial f}{\partial y}(t, y) + a_1 \cdot R_1(t + \alpha_1, y + \beta_1), \end{aligned} \tag{5.19}$$

where

$$R_1(t + \alpha_1, y + \beta_1) = \frac{\alpha_1^2}{2}\frac{\partial^2 f}{\partial t^2}(\xi, \mu) + \alpha_1\beta_1 \frac{\partial^2 f}{\partial t \partial y}(\xi, \mu) + \frac{\beta_1^2}{2}\frac{\partial^2 f}{\partial y^2}(\xi, \mu), \tag{5.20}$$

for some ξ between t and $t + \alpha_1$ and μ between y and $y + \beta_1$.

Matching the coefficients of f and its derivatives in Eqs. (5.18) and (5.19) gives the three equations

$$f(t, y): \; a_1 = 1; \quad \frac{\partial f}{\partial t}(t, y): \; a_1\alpha_1 = \frac{h}{2}; \quad \text{and} \quad \frac{\partial f}{\partial y}(t, y): \; a_1\beta_1 = \frac{h}{2} f(t, y).$$

The parameters a_1, α_1, and β_1 are therefore

$$a_1 = 1, \quad \alpha_1 = \frac{h}{2}, \quad \text{and} \quad \beta_1 = \frac{h}{2} f(t, y),$$

so

$$T^{(2)}(t, y) = f\left(t + \frac{h}{2}, y + \frac{h}{2} f(t, y)\right) - R_1\left(t + \frac{h}{2}, y + \frac{h}{2} f(t, y)\right),$$

and from Eq. (5.20),

$$\begin{aligned} R_1\left(t + \frac{h}{2}, y + \frac{h}{2} f(t, y)\right) = \frac{h^2}{8}\frac{\partial^2 f}{\partial t^2}(\xi, \mu) + \frac{h^2}{4} f(t, y) \frac{\partial^2 f}{\partial t \partial y}(\xi, \mu) \\ + \frac{h^2}{8}(f(t, y))^2 \frac{\partial^2 f}{\partial y^2}(\xi, \mu). \end{aligned}$$

If all the second-order partial derivatives of f are bounded, then

$$R_1\left(t + \frac{h}{2}, y + \frac{h}{2} f(t, y)\right)$$

is $O(h^2)$. As a consequence:

- The order of error for this new method is the same as that of the Taylor method of order two.

The difference-equation method resulting from replacing $T^{(2)}(t, y)$ in Taylor's method of order two by $f(t + (h/2), y + (h/2) f(t, y))$ is a specific Runge-Kutta method known as the *Midpoint method*.

Midpoint Method

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)\right), \quad \text{for } i = 0, 1, \ldots, N-1.$$

Only three parameters are present in $a_1 f(t + \alpha_1, y + \beta_1)$ and all are needed in the match of $T^{(2)}$. So a more complicated form is required to satisfy the conditions for any of the higher-order Taylor methods.

The most appropriate four-parameter form for approximating

$$T^{(3)}(t, y) = f(t, y) + \frac{h}{2}f'(t, y) + \frac{h^2}{6}f''(t, y)$$

is

$$a_1 f(t, y) + a_2 f(t + \alpha_2, y + \delta_2 f(t, y)); \tag{5.21}$$

and even with this, there is insufficient flexibility to match the term

$$\frac{h^2}{6}\left[\frac{\partial f}{\partial y}(t, y)\right]^2 f(t, y),$$

resulting from the expansion of $(h^2/6)f''(t, y)$. Consequently, the best that can be obtained from using (5.21) are methods with $O(h^2)$ local truncation error.

The fact that (5.21) has four parameters, however, gives a flexibility in their choice, so a number of $O(h^2)$ methods can be derived. One of the most important is the *Modified Euler method*, which corresponds to choosing $a_1 = a_2 = \frac{1}{2}$ and $\alpha_2 = \delta_2 = h$. It has the following difference-equation form.

Modified Euler Method

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))], \quad \text{for} \quad i = 0, 1, \ldots, N-1.$$

Example 2 Use the Midpoint method and the Modified Euler method with $N = 10$, $h = 0.2$, $t_i = 0.2i$, and $w_0 = 0.5$ to approximate the solution to our usual example,

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

Solution The difference equations produced from the various formulas are

$$\text{Midpoint method:} \quad w_{i+1} = 1.22w_i - 0.0088i^2 - 0.008i + 0.218;$$

$$\text{Modified Euler method:} \quad w_{i+1} = 1.22w_i - 0.0088i^2 - 0.008i + 0.216,$$

for each $i = 0, 1, \ldots, 9$. The first two steps of these methods give

$$\text{Midpoint method:} \quad w_1 = 1.22(0.5) - 0.0088(0)^2 - 0.008(0) + 0.218 = 0.828;$$

$$\text{Modified Euler method:} \quad w_1 = 1.22(0.5) - 0.0088(0)^2 - 0.008(0) + 0.216 = 0.826,$$

and

$$\text{Midpoint method:} \quad w_2 = 1.22(0.828) - 0.0088(0.2)^2 - 0.008(0.2) + 0.218 = 1.21136;$$

$$\text{Modified Euler method:} \quad w_2 = 1.22(0.826) - 0.0088(0.2)^2 - 0.008(0.2) + 0.216 = 1.20692,$$

Table 5.6 lists all the results of the calculations. For this problem, the Midpoint method is superior to the Modified Euler method. ■

Table 5.6

t_i	$y(t_i)$	Midpoint Method	Error	Modified Euler Method	Error
0.0	0.5000000	0.5000000	0	0.5000000	0
0.2	0.8292986	0.8280000	0.0012986	0.8260000	0.0032986
0.4	1.2140877	1.2113600	0.0027277	1.2069200	0.0071677
0.6	1.6489406	1.6446592	0.0042814	1.6372424	0.0116982
0.8	2.1272295	2.1212842	0.0059453	2.1102357	0.0169938
1.0	2.6408591	2.6331668	0.0076923	2.6176876	0.0231715
1.2	3.1799415	3.1704634	0.0094781	3.1495789	0.0303627
1.4	3.7324000	3.7211654	0.0112346	3.6936862	0.0387138
1.6	4.2834838	4.2706218	0.0128620	4.2350972	0.0483866
1.8	4.8151763	4.8009586	0.0142177	4.7556185	0.0595577
2.0	5.3054720	5.2903695	0.0151025	5.2330546	0.0724173

Runge-Kutta methods are also options within the Maple command *InitialValueProblem*. The form and output for Runge-Kutta methods are the same as available under the Euler's and Taylor's methods, as discussed in Sections 5.1 and 5.2.

Higher-Order Runge-Kutta Methods

The term $T^{(3)}(t, y)$ can be approximated with error $O(h^3)$ by an expression of the form

$$f(t + \alpha_1, y + \delta_1 f(t + \alpha_2, y + \delta_2 f(t, y))),$$

involving four parameters, the algebra involved in the determination of $\alpha_1, \delta_1, \alpha_2$, and δ_2 is quite involved. The most common $O(h^3)$ is Heun's method, given by

Karl Heun (1859–1929) was a professor at the Technical University of Karlsruhe. He introduced this technique in a paper published in 1900. [Heu]

$$w_0 = \alpha$$

$$w_{i+1} = w_i + \frac{h}{4}\left(f(t_i, w_i) + 3f\left(t_i + \frac{2h}{3}, w_i + \frac{2h}{3} f\left(t_i + \frac{h}{3}, w_i + \frac{h}{3} f(t_i, w_i)\right)\right)\right),$$

$$\text{for} \quad i = 0, 1, \ldots, N - 1.$$

Illustration Applying Heun's method with $N = 10$, $h = 0.2$, $t_i = 0.2i$, and $w_0 = 0.5$ to approximate the solution to our usual example,

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

gives the values in Table 5.7. Note the decreased error throughout the range over the Midpoint and Modified Euler approximations. □

Table 5.7

t_i	$y(t_i)$	Heun's Method	Error
0.0	0.5000000	0.5000000	0
0.2	0.8292986	0.8292444	0.0000542
0.4	1.2140877	1.2139750	0.0001127
0.6	1.6489406	1.6487659	0.0001747
0.8	2.1272295	2.1269905	0.0002390
1.0	2.6408591	2.6405555	0.0003035
1.2	3.1799415	3.1795763	0.0003653
1.4	3.7324000	3.7319803	0.0004197
1.6	4.2834838	4.2830230	0.0004608
1.8	4.8151763	4.8146966	0.0004797
2.0	5.3054720	5.3050072	0.0004648

Runge-Kutta methods of order three are not generally used. The most common Runge-Kutta method in use is of order four in difference-equation form, is given by the following.

Runge-Kutta Order Four

$$
\begin{aligned}
w_0 &= \alpha,\\
k_1 &= hf(t_i, w_i),\\
k_2 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right),\\
k_3 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right),\\
k_4 &= hf(t_{i+1}, w_i + k_3),\\
w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),
\end{aligned}
$$

for each $i = 0, 1, \ldots, N-1$. This method has local truncation error $O(h^4)$, provided the solution $y(t)$ has five continuous derivatives. We introduce the notation k_1, k_2, k_3, k_4 into the method is to eliminate the need for successive nesting in the second variable of $f(t, y)$. Exercise 32 shows how complicated this nesting becomes.

Algorithm 5.2 implements the Runge-Kutta method of order four.

ALGORITHM 5.2

Runge-Kutta (Order Four)

To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

at $(N+1)$ equally spaced numbers in the interval $[a, b]$:

INPUT endpoints a, b; integer N; initial condition α.

OUTPUT approximation w to y at the $(N+1)$ values of t.