Chapter 9

When a pressure tank of any shape is designed, it is important to be able to determine its center of gravity, calculate its volume and surface area, and determine the forces of the liquids they contain. All of these topics will be covered in this chapter.

Center of Gravity and Centroid

CHAPTER OBJECTIVES

- To discuss the concept of the center of gravity, center of mass, and the centroid.
- To show how to determine the location of the center of gravity and centroid for a system of discrete particles and a body of arbitrary shape.
- To use the theorems of Pappus and Guldinus for finding the surface area and volume for a body having axial symmetry.
- To present a method for finding the resultant of a general distributed loading and show how it applies to finding the resultant force of a pressure loading caused by a fluid.

9.1 Center of Gravity, Center of Mass, and the Centroid of a Body

In this section we will first show how to locate the center of gravity for a body, and then we will show that the center of mass and the centroid of a body can be developed using this same method.

Center of Gravity. A body is composed of an infinite number of particles of differential size, and so if the body is located within a gravitational field, then each of these particles will have a weight dW, Fig. 9–1*a*. These weights will form an approximately parallel force system, and the resultant of this system is the total weight of the body, which passes through a single point called the *center of gravity*, G, Fig. 9–1*b*.*

*This is true as long as the gravity field is assumed to have the same magnitude and direction everywhere. That assumption is appropriate for most engineering applications, since gravity does not vary appreciably between, for instance, the bottom and the top of a building.

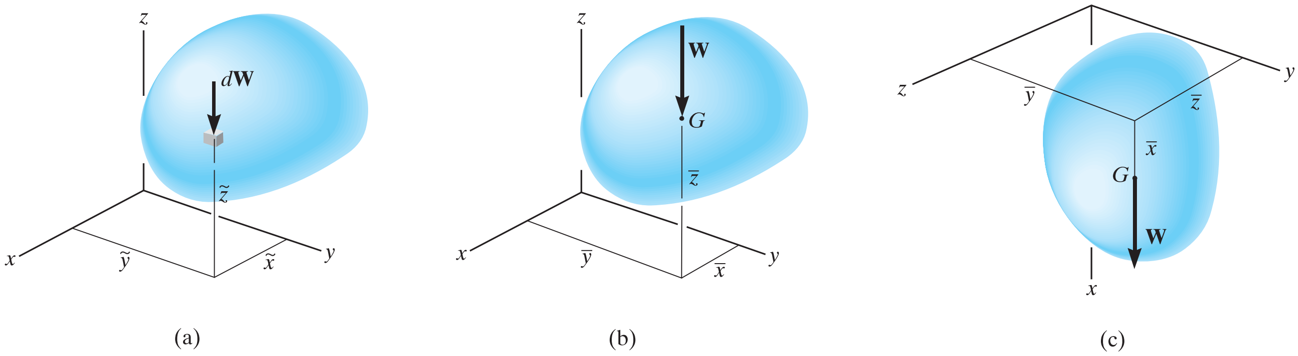


Fig. 9–1

Using the methods outlined in Sec. 4.8, the weight of the body is the sum of the weights of all of its particles, that is

$$+\downarrow F_R = \Sigma F_z; \qquad W = \int dW$$

The location of the center of gravity, measured from the y axis, is determined by equating the moment of W about the y axis, Fig. 9–1b, to the sum of the moments of the weights of the particles about this same axis. If dW is located at point ($\tilde{x}$, $\tilde{y}$, $\tilde{z}$), Fig. 9–1a, then

$$(M_R)_y = \Sigma M_y; \qquad \bar{x}W = \int \tilde{x}dW$$

Similarly, if moments are summed about the x axis,

$$(M_R)_x = \Sigma M_x; \qquad \bar{y}W = \int \tilde{y}dW$$

Finally, imagine that the body is fixed within the coordinate system and this system is rotated 90° about the y axis, Fig. 9–1c. Then the sum of the moments about the y axis gives

$$(M_R)_y = \Sigma M_y; \qquad \bar{z}W = \int \tilde{z}dW$$

Therefore, the location of the center of gravity G with respect to the x, y, z axes becomes

$$\bar{x} = \frac{\int \tilde{x}\, dW}{\int dW} \qquad \bar{y} = \frac{\int \tilde{y}\, dW}{\int dW} \qquad \bar{z} = \frac{\int \tilde{z}\, dW}{\int dW} \tag{9–1}$$

Here

$\bar{x}, \bar{y}, \bar{z}$ are the coordinates of the center of gravity G, Fig. 9–1b.
$\tilde{x}, \tilde{y}, \tilde{z}$ are the coordinates of each particle in the body, Fig. 9–1a.

Center of Mass of a Body. In order to study the *dynamic response* or accelerated motion of a body, it becomes important to locate the body's center of mass C_m, Fig. 9–2. This location can be determined by substituting $dW = g\,dm$ into Eqs. 9–1. Since g is constant, it cancels out, and so

$$\bar{x} = \frac{\int \tilde{x}\,dm}{\int dm} \qquad \bar{y} = \frac{\int \tilde{y}\,dm}{\int dm} \qquad \bar{z} = \frac{\int \tilde{z}\,dm}{\int dm} \tag{9–2}$$

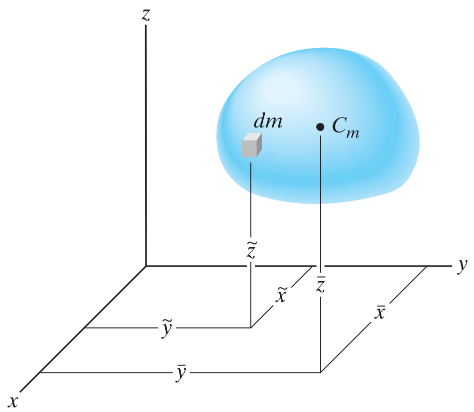


Fig. 9–2

Centroid of a Volume. If the body in Fig. 9–3 is made from a homogeneous material, then its density ρ (rho) will be constant. Therefore, a differential element of volume dV has a mass $dm = \rho\,dV$. Substituting this into Eqs. 9–2 and canceling out ρ, we obtain formulas that locate the *centroid* C or geometric center of the body; namely

$$\bar{x} = \frac{\int_V \tilde{x}\,dV}{\int_V dV} \qquad \bar{y} = \frac{\int_V \tilde{y}\,dV}{\int_V dV} \qquad \bar{z} = \frac{\int_V \tilde{z}\,dV}{\int_V dV} \tag{9–3}$$

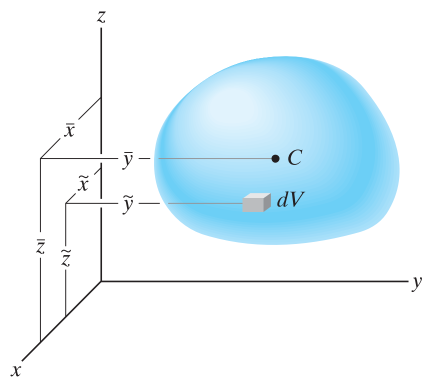


Fig. 9–3

These equations represent a balance of the moments of the volume of the body. Therefore, if the volume possesses two planes of symmetry, then its centroid must lie along the line of intersection of these two planes. For example, the cone in Fig. 9–4 has a centroid that lies on the y axis so that $\bar{x} = \bar{z} = 0$. The location $\bar{y}$ can be found using a single integration by choosing a differential element represented by a *thin disk* having a thickness dy and radius $r = z$. Its volume is $dV = \pi r^2\,dy = \pi z^2\,dy$ and its centroid is at $\tilde{x} = 0$, $\tilde{y} = y$, $\tilde{z} = 0$.

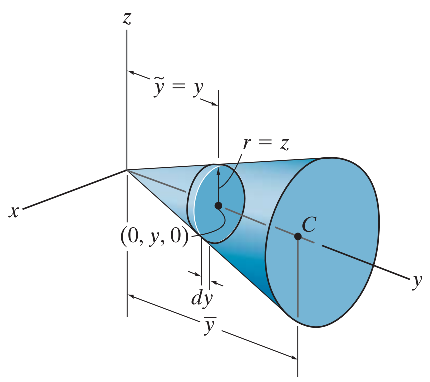


Fig. 9–4

9

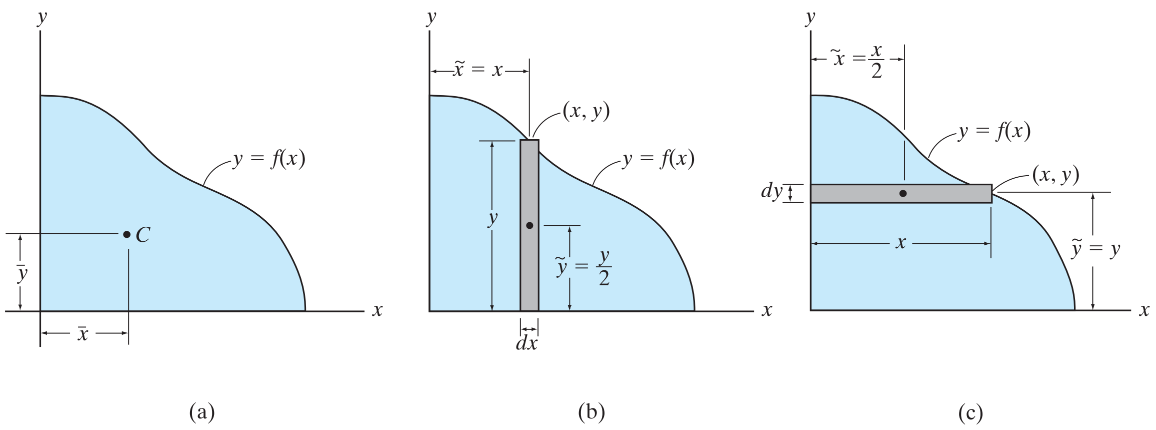
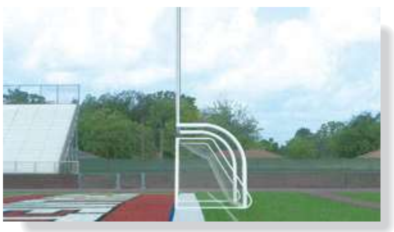


Fig. 9–5



Integration must be used to determine the location of the center of gravity of this goal post due to the curvature of the supporting member.

Centroid of an Area.

If an area lies in the x–y plane and is bounded by the curve $y = f(x)$, as shown in Fig. 9–5*a*, then its centroid will be in this plane and can be determined from integrals similar to Eqs. 9–3, namely,

$$\bar{x} = \frac{\int_A \tilde{x}\, dA}{\int_A dA} \qquad \bar{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA} \tag{9–4}$$

These integrals can be evaluated by performing a *single integration* if we use a *rectangular strip* for the differential area element. For example, if a vertical strip is used, Fig. 9–5*b*, the area of the element is $dA = y\,dx$, and its centroid is located at $\tilde{x} = x$ and $\tilde{y} = y/2$. If we consider a horizontal strip, Fig. 9–5*c*, then $dA = x\,dy$, and its centroid is located at $\tilde{x} = x/2$ and $\tilde{y} = y$.

Centroid of a Line.

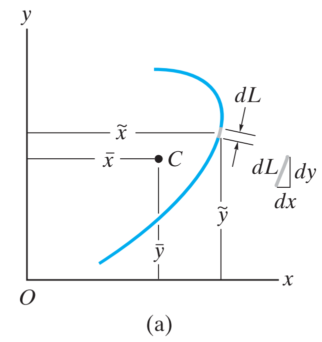
If a line segment (or rod) lies within the x–y plane and it can be described by a thin curve $y = f(x)$, Fig. 9–6*a*, then its centroid is determined from

$$\bar{x} = \frac{\int_L \tilde{x}\, dL}{\int_L dL} \qquad \bar{y} = \frac{\int_L \tilde{y}\, dL}{\int_L dL} \tag{9–5}$$

Here, the length of the differential element is given by the Pythagorean theorem, $dL = \sqrt{(dx)^2 + (dy)^2}$, which can also be written in the form

$$dL = \sqrt{\left(\frac{dx}{dx}\right)^2 dx^2 + \left(\frac{dy}{dx}\right)^2 dx^2}$$

$$= \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right) dx$$



or

$$dL = \sqrt{\left(\frac{dx}{dy}\right)^2 dy^2 + \left(\frac{dy}{dy}\right)^2 dy^2}$$

$$= \left(\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\right) dy$$

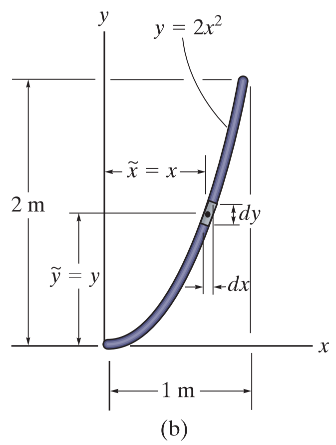


Fig. 9–6

Either one of these expressions can be used; however, for application, the one that will result in a simpler integration should be selected. For example, consider the rod in Fig. 9–6*b*, defined by $y = 2x^2$. The length of the element is $dL = \sqrt{1 + (dy/dx)^2}\,dx$, and since $dy/dx = 4x$, then $dL = \sqrt{1 + (4x)^2}\,dx$. The centroid for this element is located at $\tilde{x} = x$ and $\tilde{y} = y$.

Important Points

- The centroid represents the geometric center of a body. This point coincides with the center of mass or the center of gravity only if the material composing the body is uniform or homogeneous.
- Formulas used to locate the center of gravity or the centroid simply represent a balance between the sum of moments of all the parts of the system and the moment of the "resultant" for the system.
- In some cases the centroid is located at a point that is not on the object, as in the case of a ring, where the centroid is at its center. Also, this point will lie on any axis of symmetry for the body, Fig. 9–7.

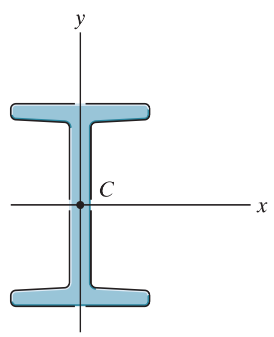


Fig. 9–7

9

Procedure for Analysis

The center of gravity or centroid of an object or shape can be determined by single integrations using the following procedure.

Differential Element.

- Select an appropriate coordinate system, specify the coordinate axes, and then choose a differential element for integration.
- For lines the element is represented by a differential line segment of length dL.
- For areas the element is generally a rectangle of area dA, having a finite length and differential width.
- For volumes the element can be a circular disk of volume dV, having a finite radius and differential thickness.
- Locate the element so that it touches the arbitrary point (x, y, z) on the curve that defines the boundary of the shape.

Size and Moment Arms.

- Express the length dL, area dA, or volume dV of the element in terms of the coordinates describing the curve.
- Express the moment arms $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ for the centroid or center of gravity of the element in terms of the coordinates describing the curve.

Integrations.

- Substitute the formulations for $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ and dL, dA, or dV into the appropriate equations (Eqs. 9–1 through 9–5).
- Express the function in the integrand in terms of the *same variable as the differential thickness of the element*.
- The limits of the integral are defined from the two extreme locations of the element's differential thickness, so that when the elements are "summed" or the integration performed, the entire region is covered.

EXAMPLE 9.1

Locate the centroid of the rod bent into the shape of a parabolic arc as shown in Fig. 9–8.

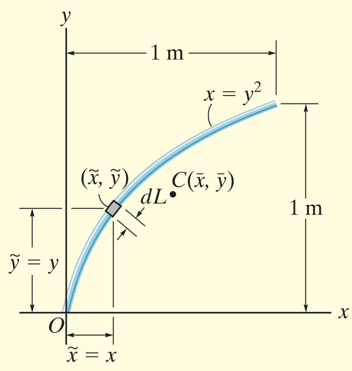


Fig. 9–8

SOLUTION

Differential Element. The differential element is shown in Fig. 9–8. It is located on the curve at the *arbitrary point* (x, y).

Area and Moment Arms. The differential element of length dL can be expressed in terms of the differentials dx and dy using the Pythagorean theorem.

$$dL = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$$

Since $x = y^2$, then $dx/dy = 2y$. Therefore, expressing dL in terms of y and dy, we have

$$dL = \sqrt{(2y)^2 + 1}\, dy$$

As shown in Fig. 9–8, the centroid of the element is located at $\tilde{x} = x$, $\tilde{y} = y$.

Integrations. Applying Eq. 9–5 and using the integration formula to evaluate the integrals, we get

$$\bar{x} = \frac{\int_L \tilde{x}\, dL}{\int_L dL} = \frac{\int_0^{1\text{ m}} x\sqrt{4y^2 + 1}\, dy}{\int_0^{1\text{ m}} \sqrt{4y^2 + 1}\, dy} = \frac{\int_0^{1\text{ m}} y^2\sqrt{4y^2 + 1}\, dy}{\int_0^{1\text{ m}} \sqrt{4y^2 + 1}\, dy}$$

$$= \frac{0.6063}{1.479} = 0.410\text{ m} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\int_L \tilde{y}\, dL}{\int_L dL} = \frac{\int_0^{1\text{ m}} y\sqrt{4y^2 + 1}\, dy}{\int_0^{1\text{ m}} \sqrt{4y^2 + 1}\, dy} = \frac{0.8484}{1.479} = 0.574\text{ m} \qquad \textit{Ans.}$$

NOTE: These results for C seem reasonable when they are plotted on Fig. 9–8.

9

EXAMPLE 9.2

Locate the centroid of the circular wire segment shown in Fig. 9–9.

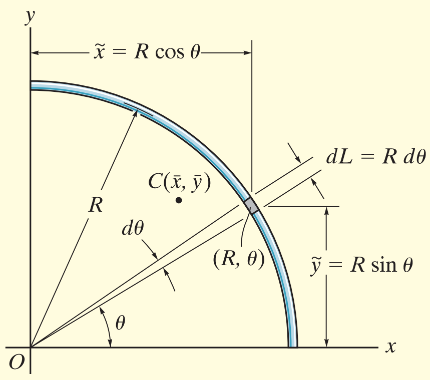


Fig. 9–9

SOLUTION

Polar coordinates will be used to solve this problem since the arc is circular.

Differential Element. A differential circular arc is selected as shown in the figure. This element lies on the curve at (R, θ).

Length and Moment Arm. The length of the differential element is $dL = R\,d\theta$, and its centroid is located at $\tilde{x} = R\cos\theta$ and $\tilde{y} = R\sin\theta$.

Integrations. Applying Eqs. 9–5 and integrating with respect to θ, we obtain

$$\bar{x} = \frac{\int_L \tilde{x}\,dL}{\int_L dL} = \frac{\int_0^{\pi/2} (R\cos\theta)R\,d\theta}{\int_0^{\pi/2} R\,d\theta} = \frac{R^2\int_0^{\pi/2} \cos\theta\,d\theta}{R\int_0^{\pi/2} d\theta} = \frac{2R}{\pi} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\int_L \tilde{y}\,dL}{\int_L dL} = \frac{\int_0^{\pi/2} (R\sin\theta)R\,d\theta}{\int_0^{\pi/2} R\,d\theta} = \frac{R^2\int_0^{\pi/2} \sin\theta\,d\theta}{R\int_0^{\pi/2} d\theta} = \frac{2R}{\pi} \qquad \textit{Ans.}$$

NOTE: As expected, the two coordinates are numerically the same due to the symmetry of the wire.

9

EXAMPLE 9.3

Determine the distance $\bar{y}$ measured from the x axis to the centroid of the area of the triangle shown in Fig. 9–10.

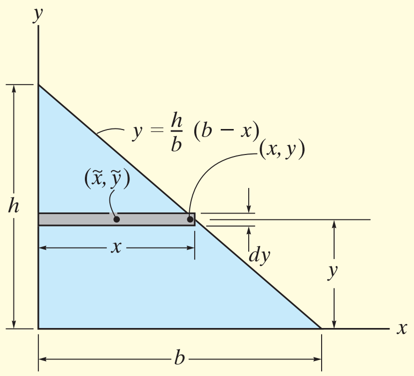


Fig. 9–10

SOLUTION

Differential Element. Consider a rectangular element having a thickness dy, and located in an arbitrary position so that it intersects the boundary at (x, y), Fig. 9–10.

Area and Moment Arms. The area of the element is $dA = x\,dy = \frac{b}{h}(h - y)\,dy$, and its centroid is located a distance $\tilde{y} = y$ from the x axis.

Integration. Applying the second of Eqs. 9–4 and integrating with respect to y yields

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^h y\left[\frac{b}{h}(h - y)\,dy\right]}{\int_0^h \frac{b}{h}(h - y)\,dy} = \frac{\frac{1}{6}bh^2}{\frac{1}{2}bh}$$

$$= \frac{h}{3} \qquad \textit{Ans.}$$

NOTE: This result is valid for any shape of triangle. It states that the centroid is located at one-third the height, measured from the base of the triangle.

9

EXAMPLE 9.4

Locate the centroid for the area of a quarter circle shown in Fig. 9–11.

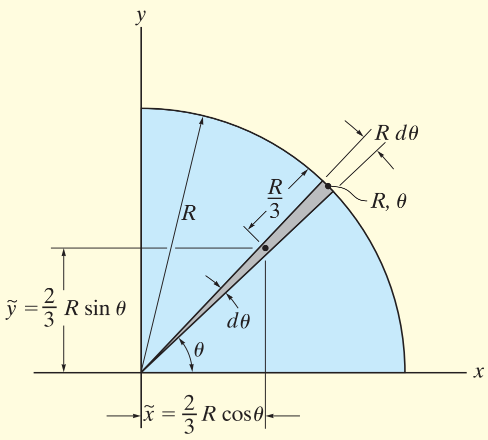


Fig. 9–11

SOLUTION

Differential Element. Polar coordinates will be used, since the boundary is circular. We choose the element in the shape of a *triangle*, Fig. 9–11. (Actually the shape is a circular sector; however, neglecting higher-order differentials, the element becomes triangular.) The element intersects the curve at point (R, θ).

Area and Moment Arms. The area of the element is

$$dA = \tfrac{1}{2}(R)(R\,d\theta) = \frac{R^2}{2}d\theta$$

and using the results of Example 9.3, the centroid of the (triangular) element is located at $\tilde{x} = \tfrac{2}{3}R\cos\theta$, $\tilde{y} = \tfrac{2}{3}R\sin\theta$.

Integrations. Applying Eqs. 9–4 and integrating with respect to θ, we obtain

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA} = \frac{\int_0^{\pi/2}\left(\frac{2}{3}R\cos\theta\right)\frac{R^2}{2}d\theta}{\int_0^{\pi/2}\frac{R^2}{2}d\theta} = \frac{\left(\frac{2}{3}R\right)\int_0^{\pi/2}\cos\theta\,d\theta}{\int_0^{\pi/2}d\theta} = \frac{4R}{3\pi} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{\pi/2}\left(\frac{2}{3}R\sin\theta\right)\frac{R^2}{2}d\theta}{\int_0^{\pi/2}\frac{R^2}{2}d\theta} = \frac{\left(\frac{2}{3}R\right)\int_0^{\pi/2}\sin\theta\,d\theta}{\int_0^{\pi/2}d\theta} = \frac{4R}{3\pi} \qquad \textit{Ans.}$$

9

EXAMPLE 9.5

Locate the centroid of the area shown in Fig. 9–12*a*.

SOLUTION I

Differential Element. A differential element of thickness dx is shown in Fig. 9–12*a*. The element intersects the curve at the *arbitrary point* (x, y), and so it has a height y.

Area and Moment Arms. The area of the element is $dA = y\,dx$, and its centroid is located at $\tilde{x} = x$, $\tilde{y} = y/2$.

Integrations. Applying Eqs. 9–4 and integrating with respect to x yields

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} xy\,dx}{\int_0^{1\text{ m}} y\,dx} = \frac{\int_0^{1\text{ m}} x^3\,dx}{\int_0^{1\text{ m}} x^2\,dx} = \frac{0.250}{0.333} = 0.75\text{ m} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} (y/2)y\,dx}{\int_0^{1\text{ m}} y\,dx} = \frac{\int_0^{1\text{ m}} (x^2/2)x^2\,dx}{\int_0^{1\text{ m}} x^2\,dx} = \frac{0.100}{0.333} = 0.3\text{ m} \qquad \textit{Ans.}$$

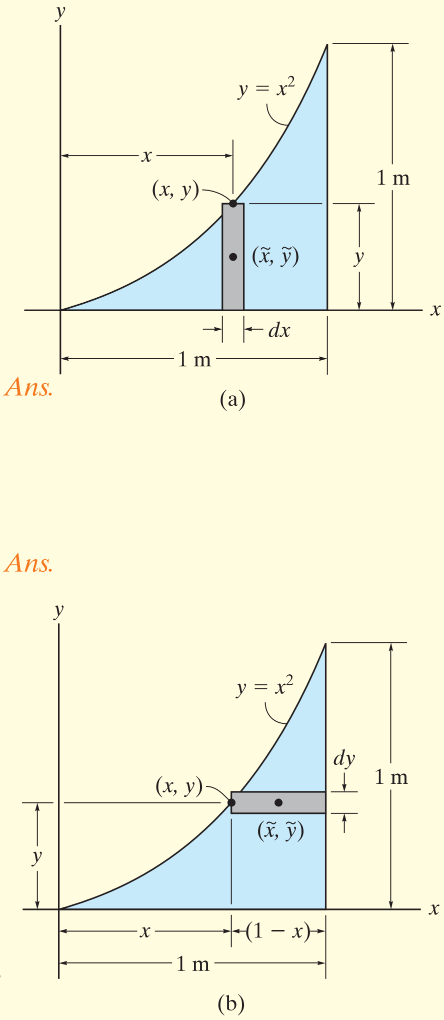


Fig. 9–12

SOLUTION II

Differential Element. The differential element of thickness dy is shown in Fig. 9–12*b*. The element intersects the curve at the *arbitrary point* (x, y), and so it has a length $(1 - x)$.

Area and Moment Arms. The area of the element is $dA = (1 - x)\,dy$, and its centroid is located at

$$\tilde{x} = x + \left(\frac{1 - x}{2}\right) = \frac{1 + x}{2}, \quad \tilde{y} = y$$

Integrations. Applying Eqs. 9–4 and integrating with respect to y, we obtain

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} [(1 + x)/2](1 - x)\,dy}{\int_0^{1\text{ m}} (1 - x)\,dy} = \frac{\frac{1}{2}\int_0^{1\text{ m}} (1 - y)\,dy}{\int_0^{1\text{ m}} (1 - \sqrt{y})\,dy} = \frac{0.250}{0.333} = 0.75\text{ m} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} y(1 - x)\,dy}{\int_0^{1\text{ m}} (1 - x)\,dy} = \frac{\int_0^{1\text{ m}} (y - y^{3/2})\,dy}{\int_0^{1\text{ m}} (1 - \sqrt{y})\,dy} = \frac{0.100}{0.333} = 0.3\text{ m} \qquad \textit{Ans.}$$

NOTE: Plot these results and notice that they seem reasonable. Also, for this problem, elements of thickness dx offer a simpler solution.

EXAMPLE 9.6

Locate the centroid of the semi-elliptical area shown in Fig. 9–13*a*.

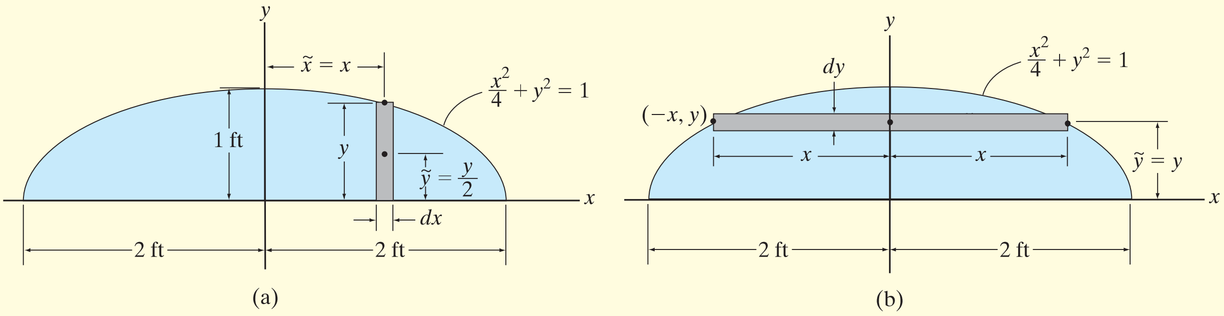


Fig. 9–13

SOLUTION I

Differential Element. The rectangular differential element parallel to the y axis shown shaded in Fig. 9–13*a* will be considered. This element has a thickness of dx and a height of y.

Area and Moment Arms. Thus, the area is $dA = y\,dx$, and its centroid is located at $\tilde{x} = x$ and $\tilde{y} = y/2$.

Integration. Since the area is symmetrical about the y axis,

$$\bar{x} = 0 \qquad \textit{Ans.}$$

Applying the second of Eqs. 9–4 with $y = \sqrt{1 - \dfrac{x^2}{4}}$, we have

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_{-2\text{ ft}}^{2\text{ ft}} \frac{y}{2}(y\,dx)}{\int_{-2\text{ ft}}^{2\text{ ft}} y\,dx} = \frac{\frac{1}{2}\int_{-2\text{ ft}}^{2\text{ ft}}\left(1 - \frac{x^2}{4}\right)dx}{\int_{-2\text{ ft}}^{2\text{ ft}}\sqrt{1 - \frac{x^2}{4}}\,dx} = \frac{4/3}{\pi} = 0.424\text{ ft} \qquad \textit{Ans.}$$

SOLUTION II

Differential Element. The shaded rectangular differential element of thickness dy and width $2x$, parallel to the x axis, will be considered, Fig. 9–13*b*.

Area and Moment Arms. The area is $dA = 2x\,dy$, and its centroid is at $\tilde{x} = 0$ and $\tilde{y} = y$.

Integration. Applying the second of Eqs. 9–4, with $x = 2\sqrt{1 - y^2}$, we have

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ ft}} y(2x\,dy)}{\int_0^{1\text{ ft}} 2x\,dy} = \frac{\int_0^{1\text{ ft}} 4y\sqrt{1 - y^2}\,dy}{\int_0^{1\text{ ft}} 4\sqrt{1 - y^2}\,dy} = \frac{4/3}{\pi}\text{ ft} = 0.424\text{ ft} \qquad \textit{Ans.}$$

EXAMPLE 9.7

Locate the $\bar{y}$ centroid for the paraboloid of revolution, shown in Fig. 9–14.

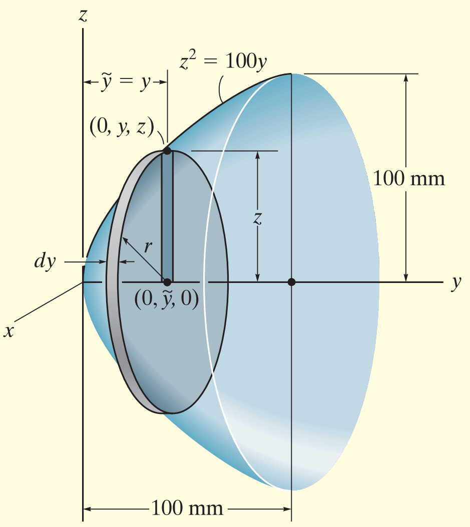


Fig. 9–14

SOLUTION

Differential Element. An element having the shape of a *thin disk* is chosen. This element has a thickness dy, it intersects the generating curve at the *arbitrary point* $(0, y, z)$, and so its radius is $r = z$.

Volume and Moment Arm. The volume of the element is $dV = (\pi z^2)\,dy$, and its centroid is located at $\tilde{y} = y$.

Integration. Applying the second of Eqs. 9–3 and integrating with respect to y yields.

$$\bar{y} = \frac{\int_V \tilde{y}\,dV}{\int_V dV} = \frac{\int_0^{100\text{ mm}} y(\pi z^2)\,dy}{\int_0^{100\text{ mm}} (\pi z^2)\,dy} = \frac{100\pi \int_0^{100\text{ mm}} y^2\,dy}{100\pi \int_0^{100\text{ mm}} y\,dy} = 66.7\text{ mm} \qquad \textit{Ans.}$$

9

EXAMPLE 9.8

Determine the location of the center of mass of the cylinder shown in Fig. 9–15 if its density varies directly with the distance from its base, i.e., $\rho = 200z \text{ kg/m}^3$.

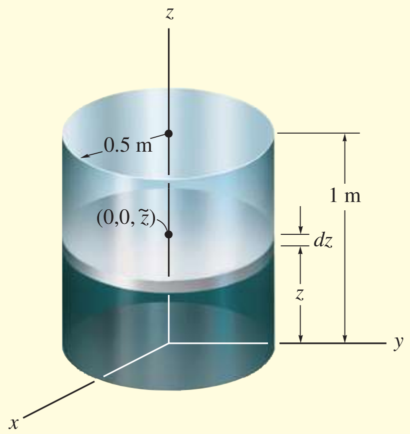


Fig. 9–15

SOLUTION

For reasons of material symmetry,

$$\bar{x} = \bar{y} = 0 \qquad \textit{Ans.}$$

Differential Element. A disk element of radius 0.5 m and thickness dz is chosen for integration, Fig. 9–15, since the *density of the entire element is constant* for a given value of z. The element is located along the z axis at the *arbitrary point* $(0, 0, z)$.

Volume and Moment Arm. The volume of the element is $dV = \pi(0.5)^2\,dz$, and its centroid is located at $\tilde{z} = z$.

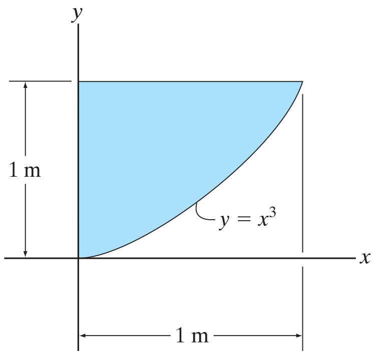
Integrations. Using the third of Eqs. 9–2 with $dm = \rho\,dV$ and integrating with respect to z, noting that $\rho = 200z$, we have

$$\bar{z} = \frac{\int_V \tilde{z}\rho\,dV}{\int_V \rho\,dV} = \frac{\int_0^{1\text{ m}} z(200z)\left[\pi(0.5)^2\,dz\right]}{\int_0^{1\text{ m}} (200z)\pi(0.5)^2\,dz}$$

$$= \frac{\int_0^{1\text{ m}} z^2\,dz}{\int_0^{1\text{ m}} z\,dz} = 0.667 \text{ m} \qquad \textit{Ans.}$$

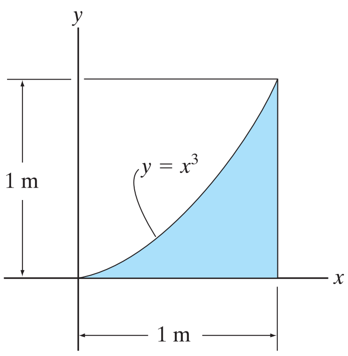
FUNDAMENTAL PROBLEMS

F9–1. Determine the centroid $(\bar{x}, \bar{y})$ of the shaded area.



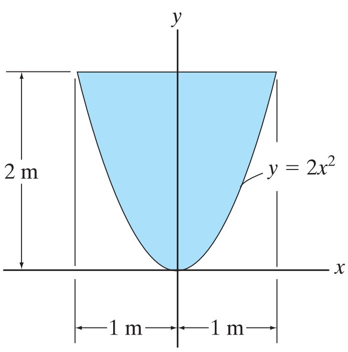
F9–1

F9–2. Determine the centroid $(\bar{x}, \bar{y})$ of the shaded area.



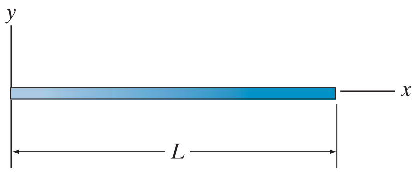
F9–2

F9–3. Determine the centroid $\bar{y}$ of the shaded area.



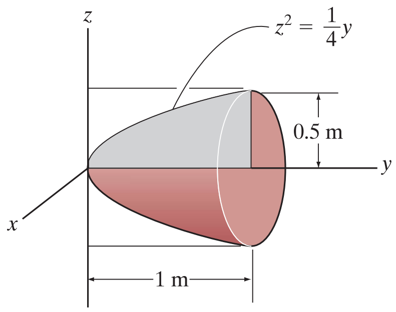
F9–3

F9–4. Locate the center mass $\bar{x}$ of the straight rod if its mass per unit length is given by $m = m_0(1 + x^2/L^2)$.



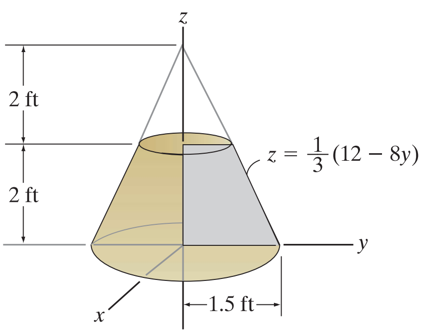
F9–4

F9–5. Locate the centroid $\bar{y}$ of the homogeneous solid formed by revolving the shaded area about the y axis.



F9–5

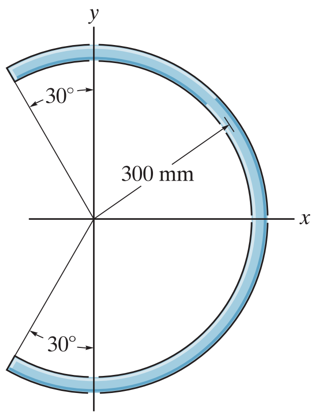
F9–6. Locate the centroid $\bar{z}$ of the homogeneous solid formed by revolving the shaded area about the z axis.



F9–6

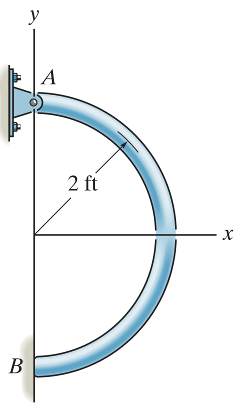
PROBLEMS

9–1. Locate the center of mass of the homogeneous rod bent into the shape of a circular arc.



Prob. 9–1

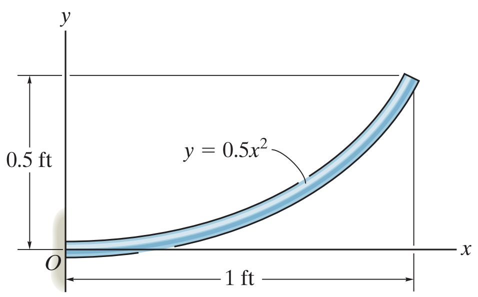
9–2. Locate the center of gravity $\bar{x}$ of the homogeneous rod bent in the form of a semicircular arc. The rod has a weight per unit length of 0.5 lb/ft. Also, determine the horizontal reaction at the smooth support B and the x and y components of reaction at the pin A.



Prob. 9–2

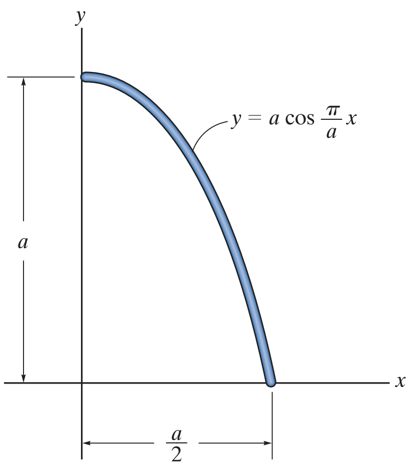
9–3. Locate the distance $\bar{x}$ to the center of gravity of the homogeneous rod bent into the parabolic shape. If the rod has a weight per unit length of 0.5 lb/ft, determine the reactions at the fixed support O.

***9–4.** Locate the distance $\bar{y}$ to the center of gravity of the homogeneous rod bent into the parabolic shape.



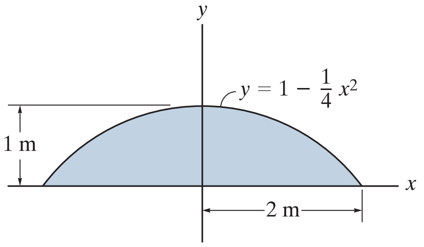
Probs. 9–3/4

9–5. Locate the centroid $(\bar{x}, \bar{y})$ of the uniform rod. Evaluate the integrals using a numerical method.



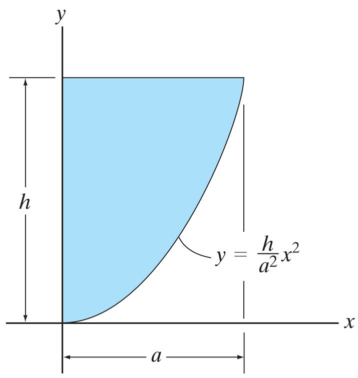
Prob. 9–5

9–6. Locate the centroid $\bar{y}$ of the area.



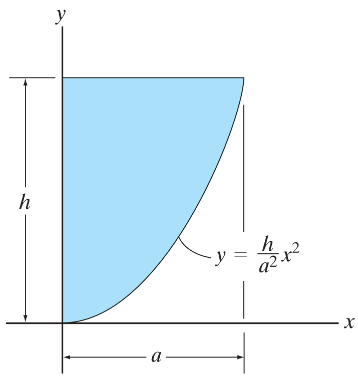
Prob. 9–6

9–7. Locate the centroid $\bar{x}$ of the parabolic area.



Prob. 9–7

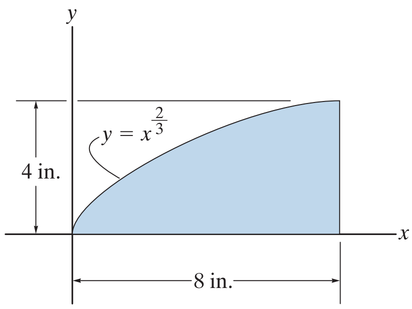
***9–8.** Locate the centroid $\bar{y}$ of the parabolic area.



Prob. 9–8

9–9. Locate the centroid $\bar{x}$ of the area.

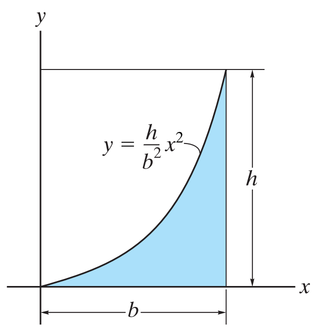
9–10. Locate the centroid $\bar{y}$ of the area.



Probs. 9–9/10

9–11. Locate the centroid $\bar{x}$ of the area.

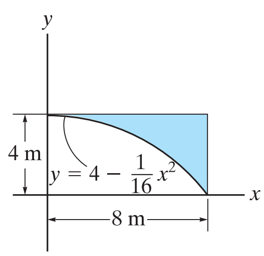
***9–12.** Locate the centroid $\bar{y}$ of the area.



Probs. 9–11/12

9–13. Locate the centroid $\bar{x}$ of the area.

9–14. Locate the centroid $\bar{y}$ of the area.

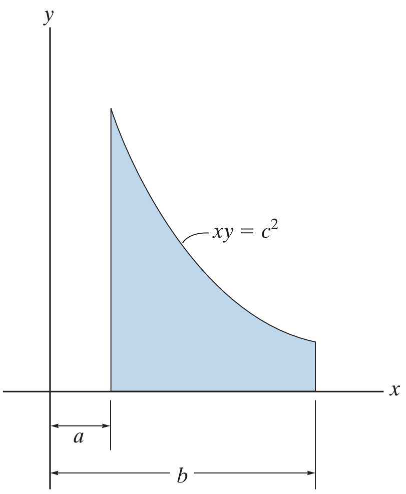


Probs. 9–13/14

9

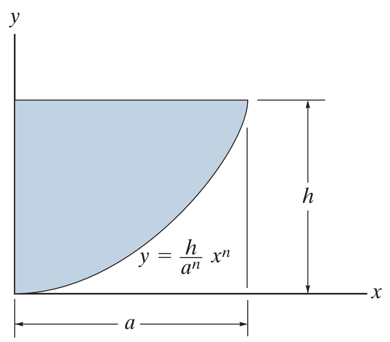
9–15. Locate the centroid $\bar{x}$ of the area.

***9–16.** Locate the centroid $\bar{y}$ of the area.



Probs. 9–15/16

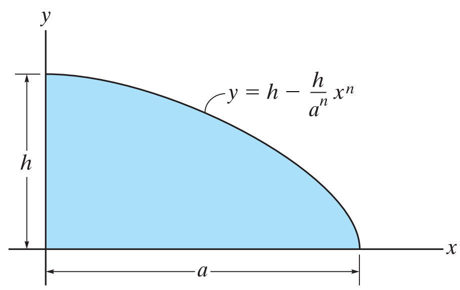
9–17. Locate the centroid $\bar{x}$ of the shaded area.



Prob. 9–17

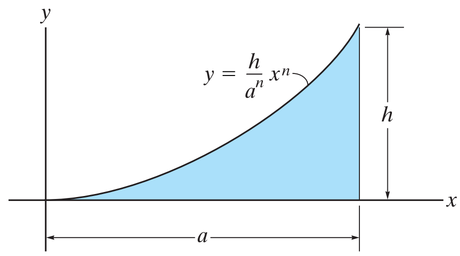
9–18. Locate the centroid $\bar{x}$ of the area.

9–19. Locate the centroid $\bar{y}$ of the area.



Probs. 9–18/19

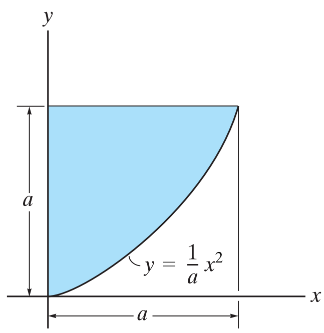
***9–20.** Locate the centroid $\bar{y}$ of the shaded area.



Prob. 9–20

9–21. Locate the centroid $\bar{x}$ of the area.

9–22. Locate the centroid $\bar{y}$ of the area.

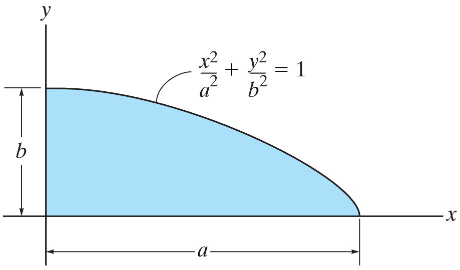


Probs. 9–21/22

9

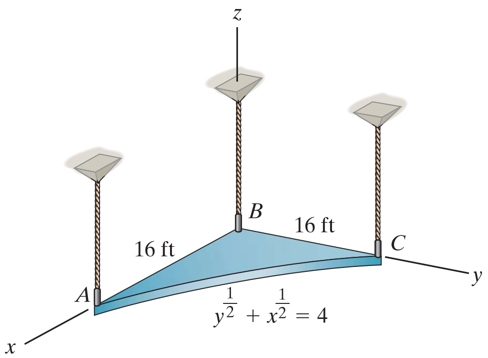
9–23. Locate the centroid $\bar{x}$ of the quarter elliptical area.

***9–24.** Locate the centroid $\bar{y}$ of the quarter elliptical area.



Probs. 9–23/24

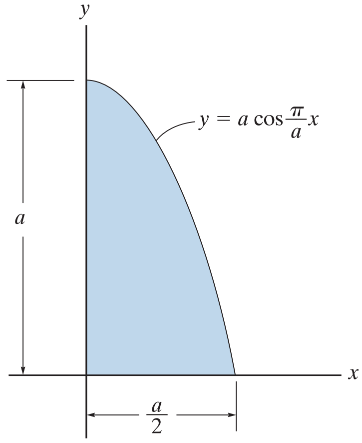
9–25. The plate has a thickness of 0.25 ft and a specific weight of $\gamma = 180 \text{ lb/ft}^3$. Determine the location of its center of gravity. Also, find the tension in each of the cords used to support it.



Prob. 9–25

9–26. Locate the centroid $\bar{x}$ of the area.

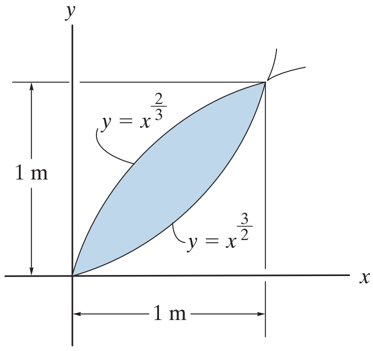
9–27. Locate the centroid $\bar{y}$ of the area.



Probs. 9–26/27

***9–28.** Locate the centroid $\bar{x}$ of the area.

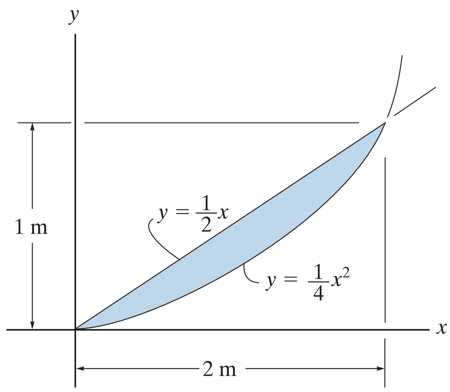
9–29. Locate the centroid $\bar{y}$ of the area.



Probs. 9–28/29

9–30. Locate the centroid $\bar{x}$ of the area.

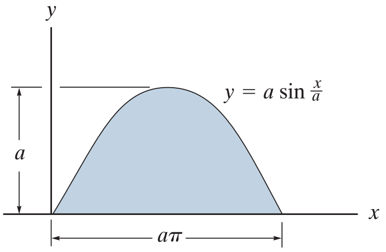
9–31. Locate the centroid $\bar{y}$ of the area.



Probs. 9–30/31

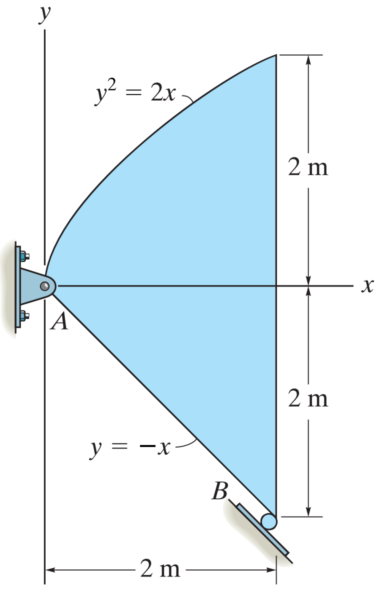
***9–32.** Locate the centroid $\bar{x}$ of the area.

9–33. Locate the centroid $\bar{y}$ of the area.



Probs. 9–32/33

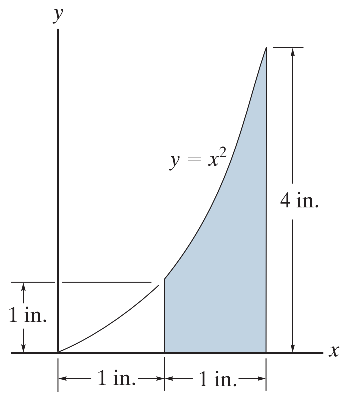
9–34. The steel plate is 0.3 m thick and has a density of $7850\ \text{kg/m}^3$. Determine the location of its center of mass. Also find the reactions at the pin and roller support.



Prob. 9–34

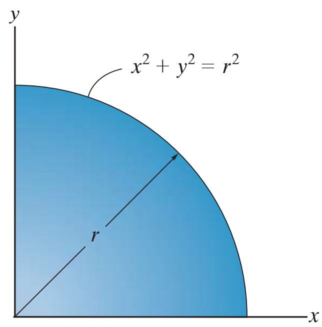
9–35. Locate the centroid $\bar{x}$ of the shaded area.

***9–36.** Locate the centroid $\bar{y}$ of the shaded area.



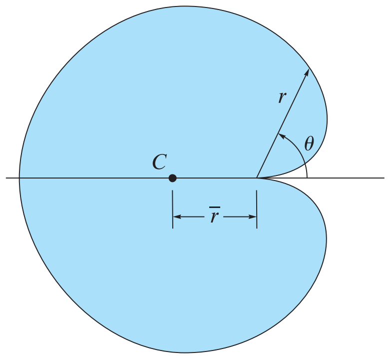
Probs. 9–35/36

9–37. If the density at any point in the quarter circular plate is defined by $\rho = \rho_0 xy$, where ρ_0 is a constant, determine the mass and locate the center of mass $(\bar{x}, \bar{y})$ of the plate. The plate has a thickness t.



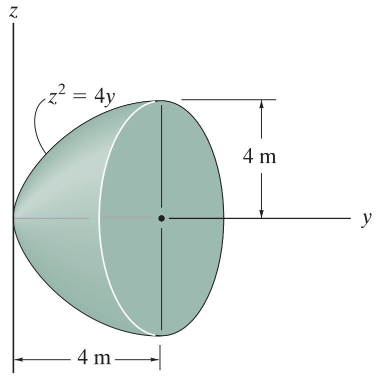
Prob. 9–37

9–38. Determine the location $\bar{r}$ of the centroid C of the cardioid, $r = a(1 - \cos\theta)$.



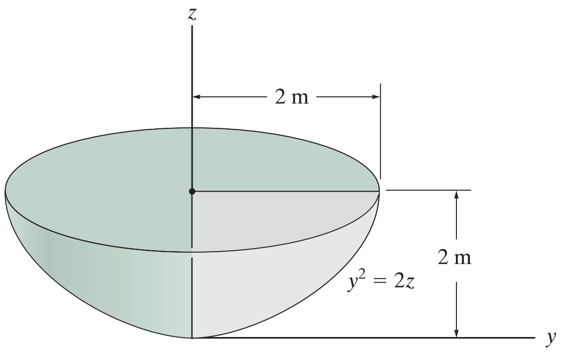
Prob. 9–38

9–39. Locate the centroid $\bar{y}$ of the paraboloid.



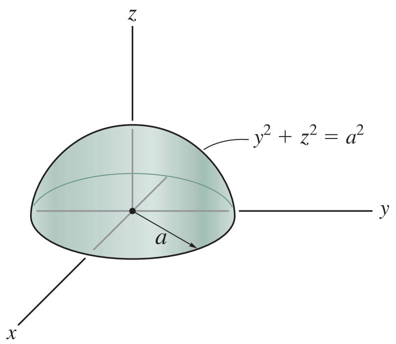
Prob. 9–39

***9–40.** Locate the center of gravity of the volume. The material is homogeneous.



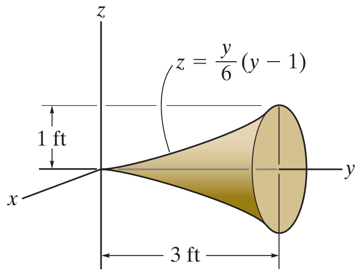
Prob. 9–40

9–41. Locate the centroid $\bar{z}$ of the hemisphere.



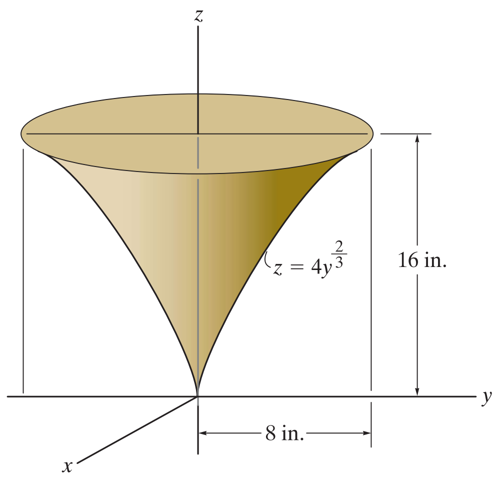
Prob. 9–41

9–42. Determine the centroid $\bar{y}$ of the solid.



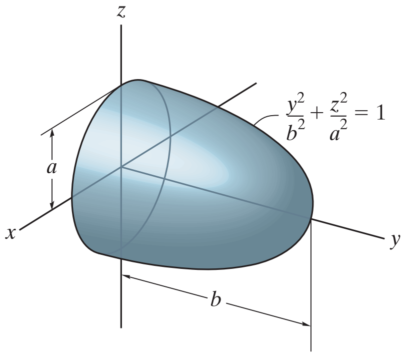
Prob. 9–42

9–43. Locate the center of gravity $\bar{z}$ of the solid.



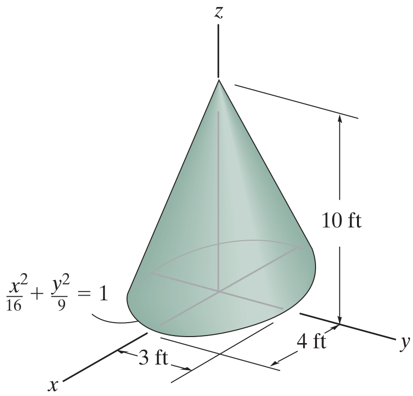
Prob. 9–43

***9–44.** Locate the centroid of the ellipsoid of revolution.



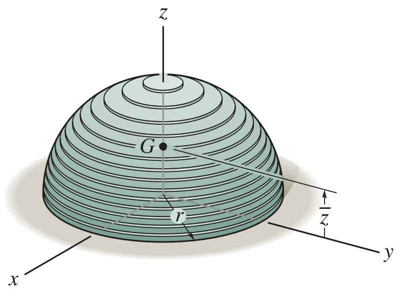
Prob. 9–44

9–45. Locate the centroid $\bar{z}$ of the right-elliptical cone.



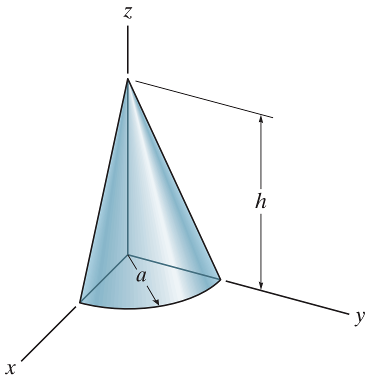
Prob. 9–45

9–46. The hemisphere of radius r is made from a stack of very thin plates such that the density varies with height, $\rho = kz$, where k is a constant. Determine its mass and the distance $\bar{z}$ to the center of mass G.



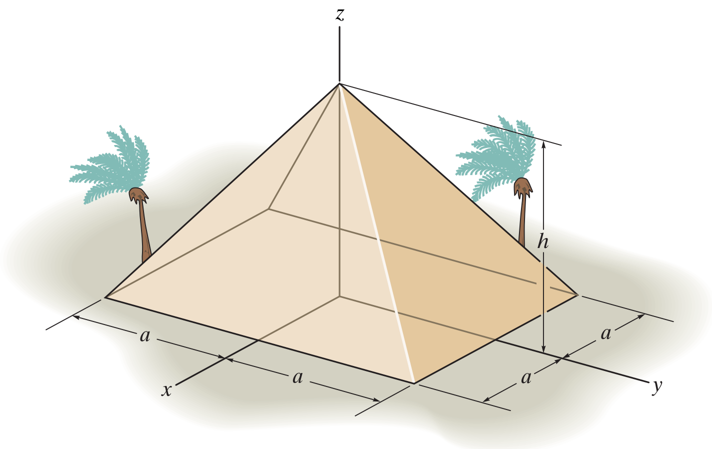
Prob. 9–46

9–47. Locate the centroid of the quarter-cone.



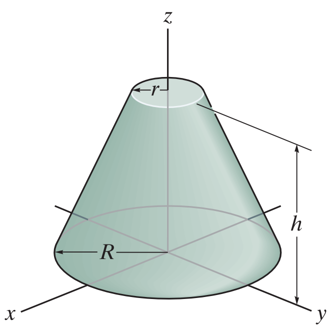
Prob. 9–47

9–49. The king's chamber of the Great Pyramid of Giza is located at its centroid. Assuming the pyramid to be a solid, prove that this point is at $\bar{z} = \frac{1}{4}h$, *Suggestion:* Use a rectangular differential plate element having a thickness dz and area $(2x)(2y)$.



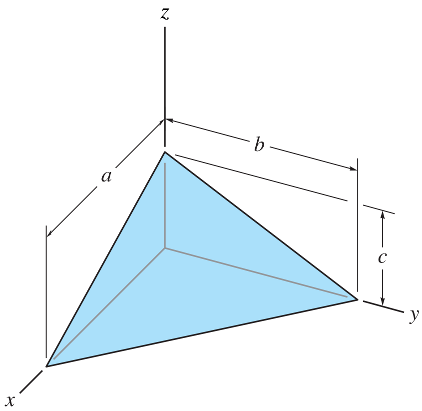
Prob. 9–49

***9–48.** Locate the centroid $\bar{z}$ of the frustum of the right-circular cone.



Prob. 9–48

9–50. Determine the location $\bar{z}$ of the centroid for the tetrahedron. *Suggestion:* Use a triangular "plate" element parallel to the x–y plane and of thickness dz.



Prob. 9–50

9.2 Composite Bodies

A *composite body* consists of a series of connected "simpler" shaped bodies, which may be rectangular, triangular, semicircular, etc. Such a body can often be sectioned or divided into its composite parts and, provided the *weight* and location of the center of gravity of each of these parts are known, we can then eliminate the need for integration to determine the center of gravity for the entire body. The method for doing this follows the same procedure outlined in Sec. 9.1. Formulas analogous to Eqs. 9–1 result; however, rather than account for an infinite number of differential weights, we have instead a finite number of weights. Therefore,

$$\bar{x} = \frac{\Sigma \tilde{x} W}{\Sigma W} \qquad \bar{y} = \frac{\Sigma \tilde{y} W}{\Sigma W} \qquad \bar{z} = \frac{\Sigma \tilde{z} W}{\Sigma W} \tag{9–6}$$

Here

$\bar{x}, \bar{y}, \bar{z}$ represent the coordinates of the center of gravity G of the composite body.

$\tilde{x}, \tilde{y}, \tilde{z}$ represent the coordinates of the center of gravity of each composite part of the body.

ΣW is the sum of the weights of all the composite parts of the body, or simply the total weight of the body.

When the body has a *constant density or specific weight*, the center of gravity *coincides* with the centroid of the body. The centroid for composite lines, areas, and volumes can be found using relations analogous to Eqs. 9–6; however, the W's are replaced by L's, A's, and V's, respectively. Centroids for common shapes of lines, areas, shells, and volumes that often make up a composite body are given in the table on the inside back cover.

In order to determine the force required to tip over this concrete barrier it is first necessary to determine the location of its center of gravity G. Due to symmetry, G will lie on the vertical axis of symmetry.

Procedure for Analysis

The location of the center of gravity of a body or the centroid of a composite geometrical object represented by a line, area, or volume can be determined using the following procedure.

Composite Parts.

- Using a sketch, divide the body or object into a finite number of composite parts that have simpler shapes.
- If a composite body has a *hole*, or a geometric region having no material, then consider the composite body without the hole and consider the hole as an *additional* composite part having *negative* weight or size.

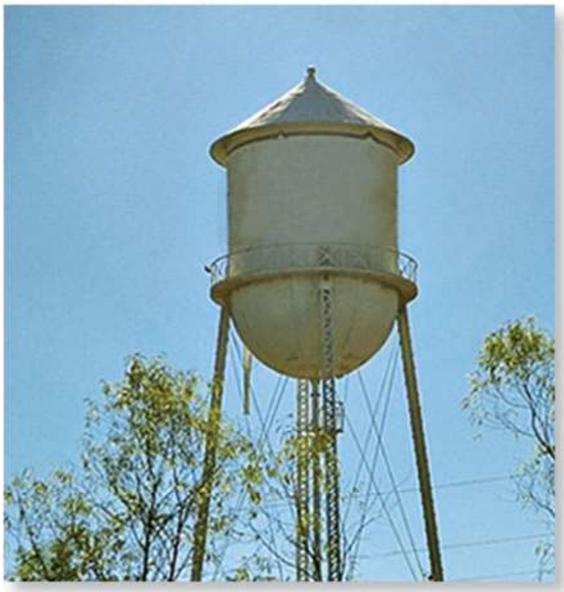
Moment Arms.

- Establish the coordinate axes on the sketch and determine the coordinates $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ of the center of gravity or centroid of each part.

Summations.

- Determine $\bar{x}$, $\bar{y}$, $\bar{z}$ by applying the center of gravity equations, Eqs. 9–6, or the analogous centroid equations.
- If an object is *symmetrical* about an axis, the centroid of the object lies on this axis.

If desired, the calculations can be arranged in tabular form, as indicated in the following three examples.



The center of gravity of this water tank can be determined by dividing it into composite parts and applying Eqs. 9–6.

EXAMPLE 9.9

Locate the centroid of the wire shown in Fig. 9–16*a*.

SOLUTION

Composite Parts. The wire is divided into three segments as shown in Fig. 9–16*b*.

Moment Arms. The location of the centroid for each segment is determined and indicated in the figure. In particular, the centroid of segment (1) is determined either by integration or by using the table on the inside back cover.

Summations. For convenience, the calculations can be tabulated as follows:

Segment	L (mm)	$\tilde{x}$ (mm)	$\tilde{y}$ (mm)	$\tilde{z}$ (mm)	$\tilde{x}L$ (mm^2)	$\tilde{y}L$ (mm^2)	$\tilde{z}L$ (mm^2)
1	$\pi(60) = 188.5$	60	−38.2	0	11 310	−7200	0
2	40	0	20	0	0	800	0
3	20	0	40	−10	0	800	−200
	$\Sigma L = 248.5$				$\Sigma \tilde{x}L = 11\,310$	$\Sigma \tilde{y}L = -5600$	$\Sigma \tilde{z}L = -200$

Thus,

$$\bar{x} = \frac{\Sigma \tilde{x}L}{\Sigma L} = \frac{11\,310}{248.5} = 45.5 \text{ mm} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\Sigma \tilde{y}L}{\Sigma L} = \frac{-5600}{248.5} = -22.5 \text{ mm} \qquad \textit{Ans.}$$

$$\bar{z} = \frac{\Sigma \tilde{z}L}{\Sigma L} = \frac{-200}{248.5} = -0.805 \text{ mm} \qquad \textit{Ans.}$$

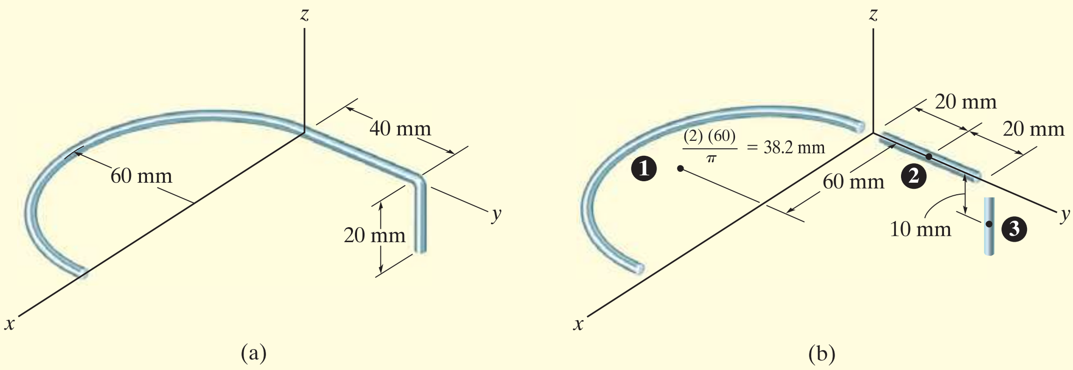
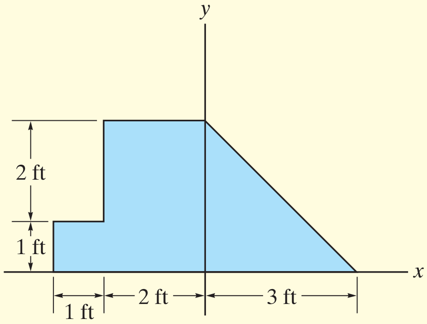


Fig. 9–16

EXAMPLE 9.10

Locate the centroid of the plate area shown in Fig. 9–17*a*.



(a)

Fig. 9–17

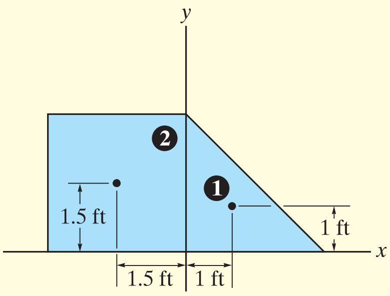
SOLUTION

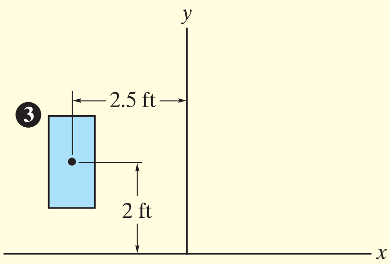
Composite Parts. The plate is divided into three segments as shown in Fig. 9–17*b*. Here the area of the small rectangle (3) is considered "negative" since it must be subtracted from the larger one (2).

Moment Arms. The centroid of each segment is located as indicated in the figure. Note that the $\tilde{x}$ coordinates of (2) and (3) are *negative*.

Summations. Taking the data from Fig. 9–17*b*, the calculations are tabulated as follows:

Segment	A (ft^2)	$\tilde{x}$ (ft)	$\tilde{y}$ (ft)	$\tilde{x}A$ (ft^3)	$\tilde{y}A$ (ft^3)
1	$\frac{1}{2}(3)(3) = 4.5$	1	1	4.5	4.5
2	$(3)(3) = 9$	−1.5	1.5	−13.5	13.5
3	$-(2)(1) = -2$	−2.5	2	5	−4
	$\Sigma A = 11.5$			$\Sigma\tilde{x}A = -4$	$\Sigma\tilde{y}A = 14$





(b)

Thus,

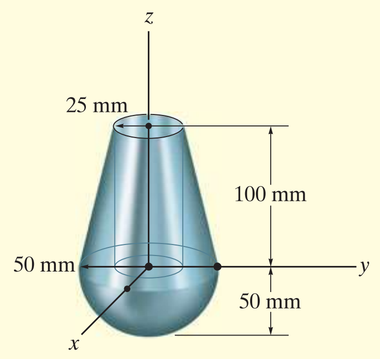
$$\bar{x} = \frac{\Sigma\tilde{x}A}{\Sigma A} = \frac{-4}{11.5} = -0.348 \text{ ft} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\Sigma\tilde{y}A}{\Sigma A} = \frac{14}{11.5} = 1.22 \text{ ft} \qquad \textit{Ans.}$$

NOTE: If these results are plotted in Fig. 9–17*a*, the location of point *C* seems reasonable.

9

EXAMPLE 9.11



(a)

Fig. 9–18

Locate the center of mass of the assembly shown in Fig. 9–18*a*. The conical frustum has a density of $\rho_c = 8\ \text{Mg/m}^3$, and the hemisphere has a density of $\rho_h = 4\ \text{Mg/m}^3$. There is a 25-mm-radius cylindrical hole in the center of the frustum.

SOLUTION

Composite Parts. The assembly can be thought of as consisting of four segments as shown in Fig. 9–18*b*. For the calculations, (3) and (4) must be considered as "negative" segments in order that the four segments, when added together, yield the total composite shape shown in Fig. 9–18*a*.

Moment Arm. Using the table on the inside back cover, the computations for the centroid $\tilde{z}$ of each piece are shown in the figure.

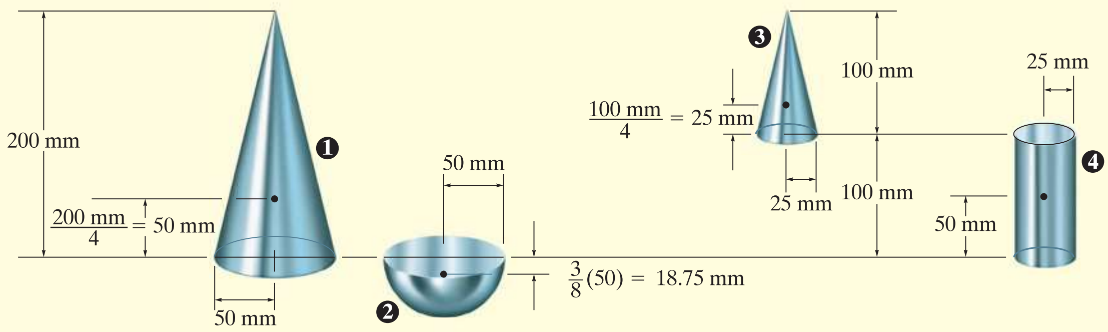
Summations. Because of *symmetry*, note that

$$\bar{x} = \bar{y} = 0 \qquad \textit{Ans.}$$

Since $W = mg$, and g is constant, the third of Eqs. 9–6 becomes $\bar{z} = \Sigma\tilde{z}m/\Sigma m$. The mass of each piece can be computed from $m = \rho V$ and used for the calculations. Also, $1\ \text{Mg/m}^3 = 10^{-6}\ \text{kg/mm}^3$, so that

Segment	m (kg)	$\tilde{z}$ (mm)	$\tilde{z}m$ (kg · mm)
1	$8(10^{-6})\left(\frac{1}{3}\right)\pi(50)^2(200) = 4.189$	50	209.440
2	$4(10^{-6})\left(\frac{2}{3}\right)\pi(50)^3 = 1.047$	-18.75	-19.635
3	$-8(10^{-6})\left(\frac{1}{3}\right)\pi(25)^2(100) = -0.524$	$100 + 25 = 125$	-65.450
4	$-8(10^{-6})\pi(25)^2(100) = -1.571$	50	-78.540
	$\Sigma m = 3.142$		$\Sigma\tilde{z}m = 45.815$

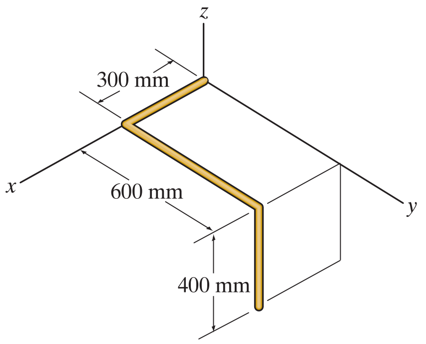
Thus, $$\tilde{z} = \frac{\Sigma\tilde{z}m}{\Sigma m} = \frac{45.815}{3.142} = 14.6\ \text{mm} \qquad \textit{Ans.}$$



(b)

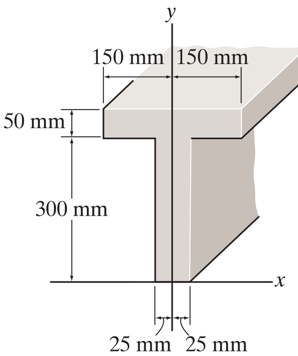
FUNDAMENTAL PROBLEMS

F9–7. Locate the centroid $(\bar{x}, \bar{y}, \bar{z})$ of the wire bent in the shape shown.



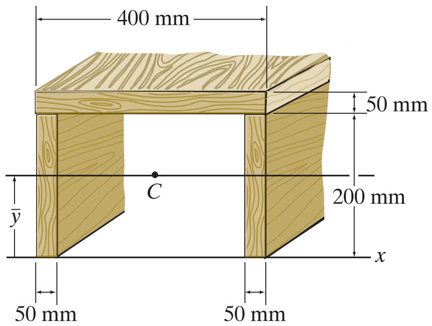
F9–7

F9–8. Locate the centroid $\bar{y}$ of the beam's cross-sectional area.



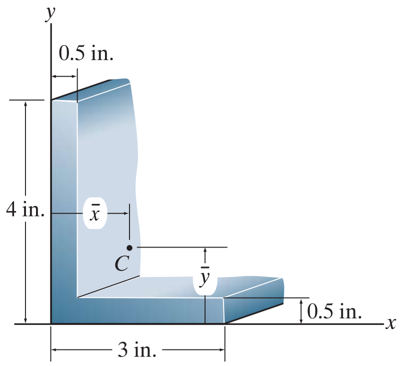
F9–8

F9–9. Locate the centroid $\bar{y}$ of the beam's cross-sectional area.



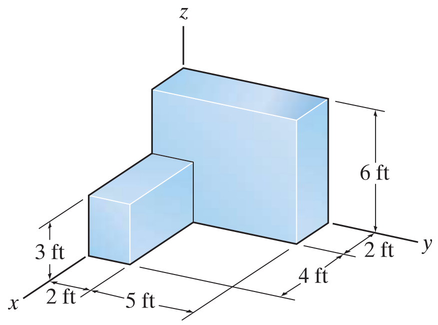
F9–9

F9–10. Locate the centroid $(\bar{x}, \bar{y})$ of the cross-sectional area.



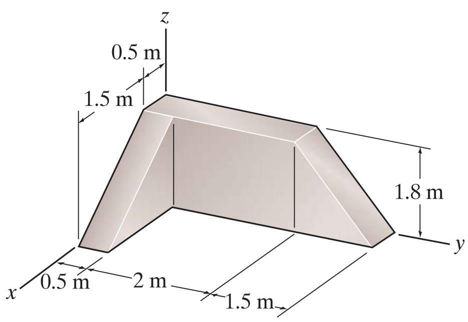
F9–10

F9–11. Locate the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of the homogeneous solid block.



F9–11

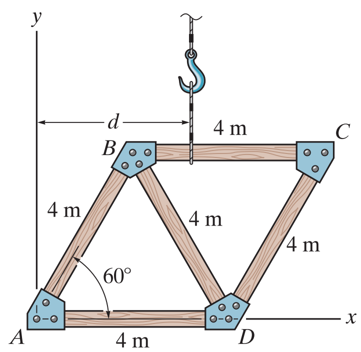
F9–12. Determine the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of the homogeneous solid block.



F9–12

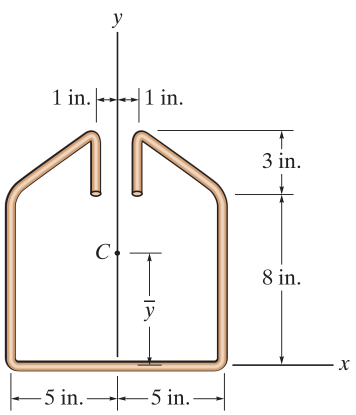
PROBLEMS

9–51. The truss is made from five members, each having a length of 4 m and a mass of 7 kg/m. If the mass of the gusset plates at the joints and the thickness of the members can be neglected, determine the distance d to where the hoisting cable must be attached, so that the truss does not tip (rotate) when it is lifted.



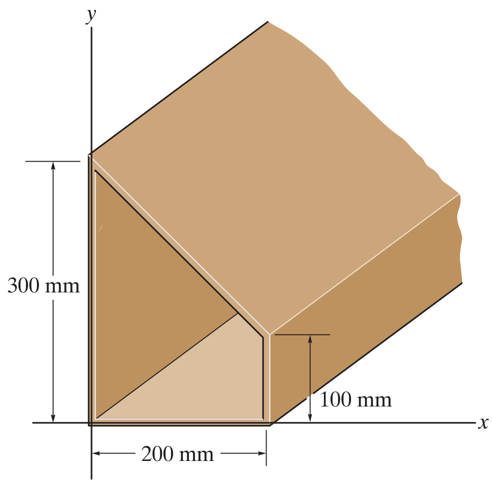
Prob. 9–51

***9–52.** Locate the centroid for the wire. Neglect the thickness of the material and slight bends at the corners.



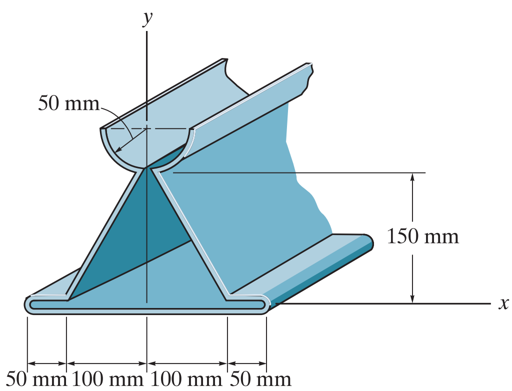
Prob. 9–52

9–53. Locate the centroid $(\bar{x}, \bar{y})$ of the cross section. All the dimensions are measured to the centerline thickness of each thin segment.



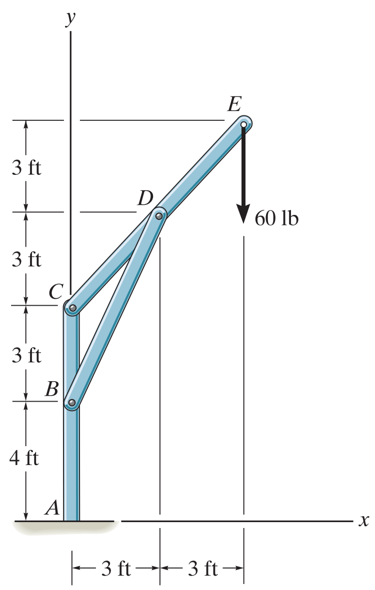
Prob. 9–53

9–54. Locate the centroid $(\bar{x}, \bar{y})$ of the metal cross section. Neglect the thickness of the material and slight bends at the corners.



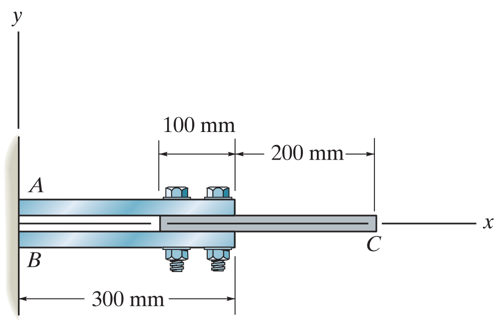
Prob. 9–54

9–55. The three members of the frame each have a weight per unit length of 4 lb/ft. Locate the position $(\bar{x}, \bar{y})$ of the center of gravity. Neglect the size of the pins at the joints and the thickness of the members. Also, calculate the reactions at the fixed support A.



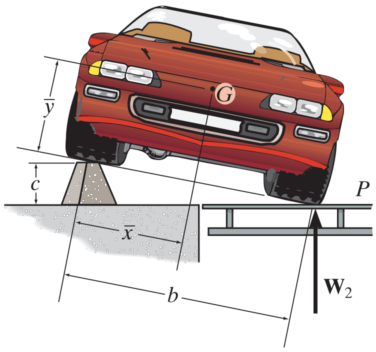
Prob. 9–55

***9–56.** The steel and aluminum plate assembly is bolted together and fastened to the wall. Each plate has a constant width in the z direction of 200 mm and thickness of 20 mm. If the density of A and B is $\rho_s = 7.85\ \text{Mg/m}^3$, and for C, $\rho_{al} = 2.71\ \text{Mg/m}^3$, determine the location $\bar{x}$ of the center of mass. Neglect the size of the bolts.



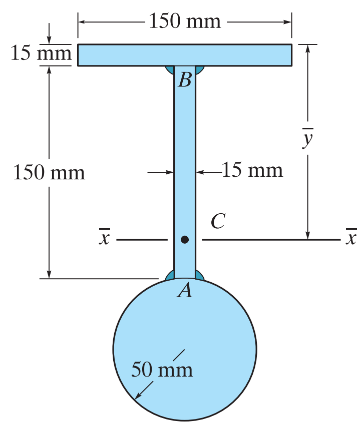
Prob. 9–56

9–57. To determine the location of the center of gravity of the automobile it is first placed in a *level position*, with the two wheels on one side resting on the scale platform P. In this position the scale records a reading of W_1. Then, one side is elevated to a convenient height c as shown. The new reading on the scale is W_2. If the automobile has a total weight of W, determine the location of its center of gravity $G(\bar{x}, \bar{y})$.



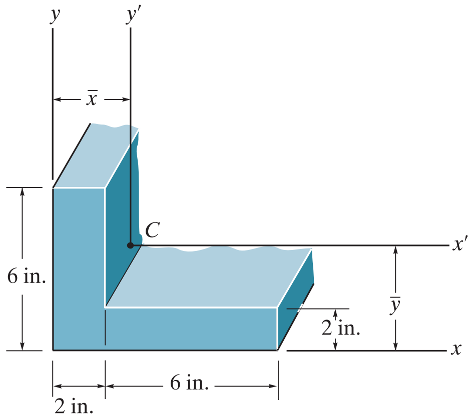
Prob. 9–57

9–58. Determine the location $\bar{y}$ of the centroidal axis $\bar{x}-\bar{x}$ of the beam's cross-sectional area. Neglect the size of the corner welds at A and B for the calculation.



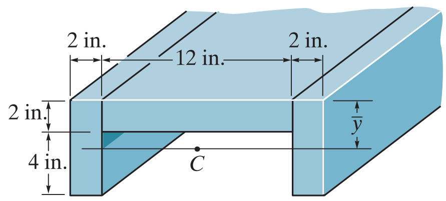
Prob. 9–58

9–59. Locate the centroid $(\bar{x}, \bar{y})$ for the angle's cross-sectional area.



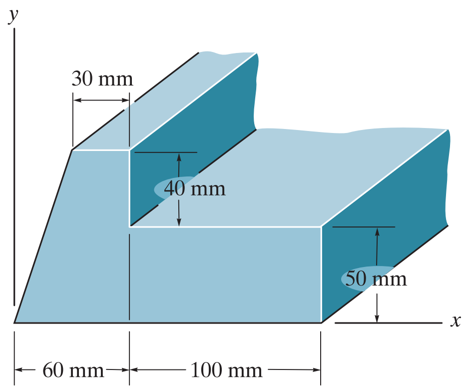
Prob. 9–59

***9–60.** Locate the centroid $\bar{y}$ of the channel's cross-sectional area.



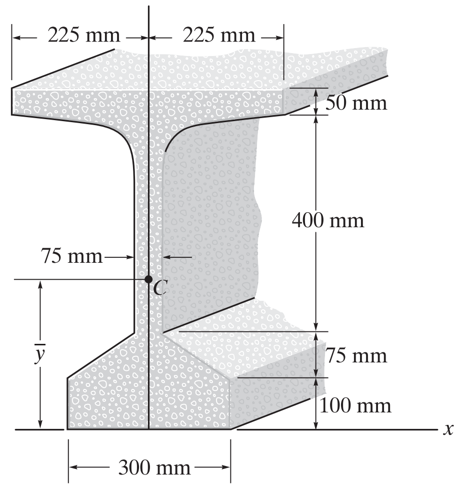
Prob. 9–60

9–61. Locate the centroid $(\bar{x}, \bar{y})$ of the member's cross-sectional area.



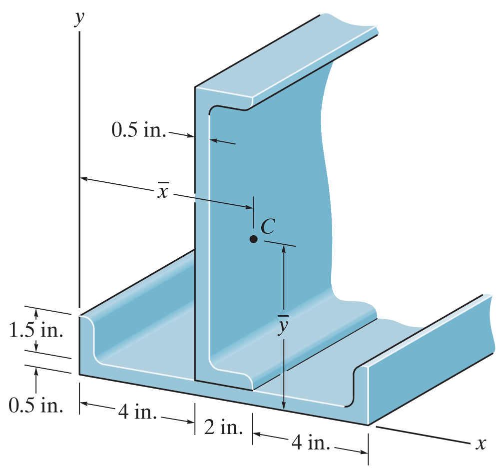
Prob. 9–61

9–62. Locate the centroid $\bar{y}$ of the bulb-tee cross section.



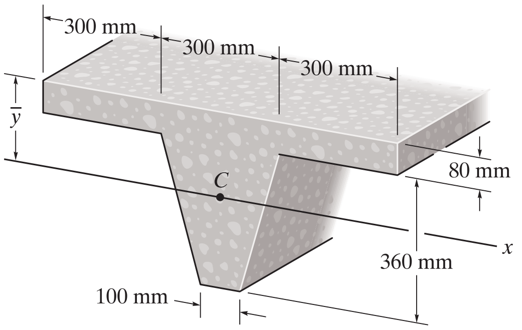
Prob. 9–62

9–63. Determine the location $(\bar{x}, \bar{y})$ of the centroid C of the cross-sectional area for the structural member constructed from two equal-sized channels welded together as shown. Assume all corners are square. Neglect the size of the welds.



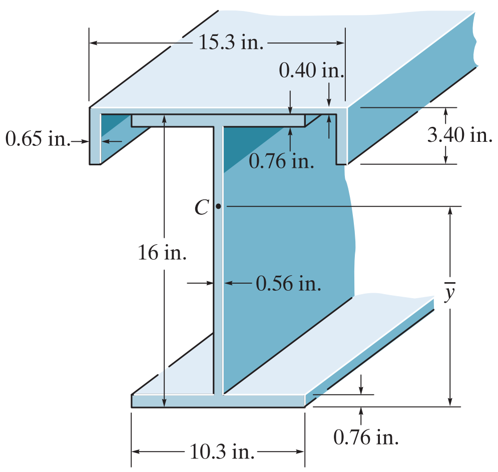
Prob. 9–63

***9–64.** Locate the centroid $\bar{y}$ of the concrete beam having the tapered cross section shown.



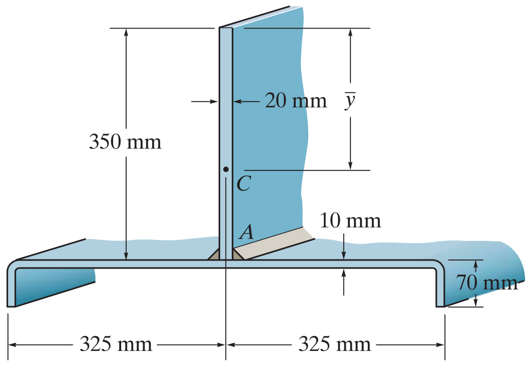
Prob. 9–64

9–65. Locate the centroid $\bar{y}$ of the beam's cross-section built up from a channel and a wide-flange beam.



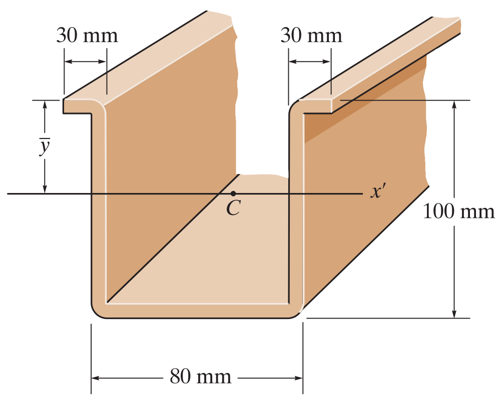
Prob. 9–65

9–66. Locate the centroid $\bar{y}$ of the cross-sectional area of the beam constructed from a channel and a plate. Assume all corners are square and neglect the size of the weld at A.



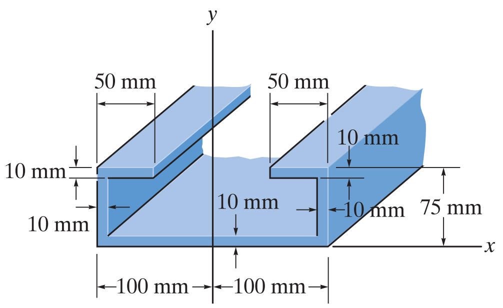
Prob. 9–66

9–67. An aluminum strut has a cross section referred to as a deep hat. Locate the centroid $\bar{y}$ of its area. Each segment has a thickness of 10 mm.



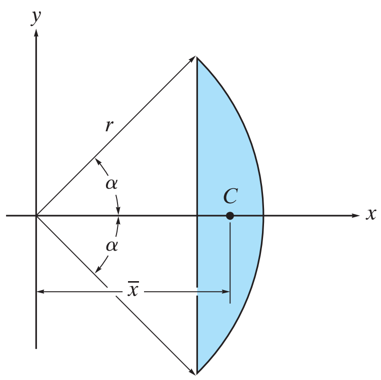
Prob. 9–67

***9–68.** Locate the centroid $\bar{y}$ of the beam's cross-sectional area.



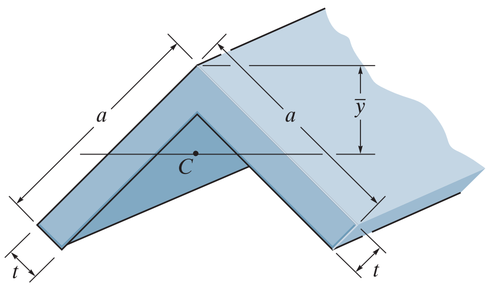
Prob. 9–68

9–69. Determine the location $\bar{x}$ of the centroid C of the shaded area that is part of a circle having a radius r.



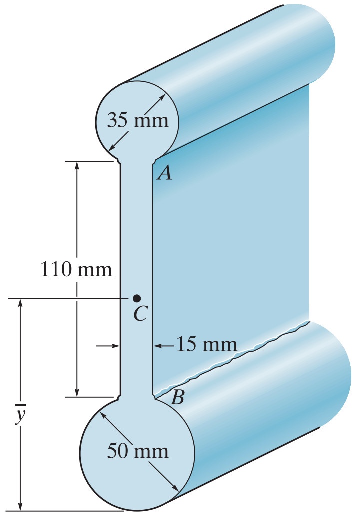
Prob. 9–69

9–70. Locate the centroid $\bar{y}$ for the cross-sectional area of the angle.



Prob. 9–70

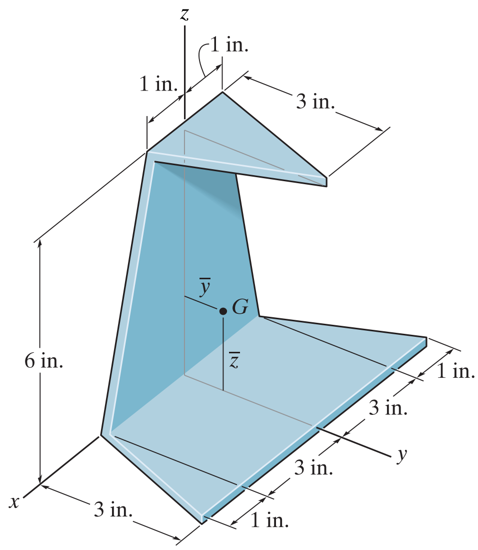
9–71. Determine the location $\bar{y}$ of the centroid of the beam's cross-sectional area. Neglect the size of the corner welds at A and B for the calculation.



Prob. 9–71

***9–72.** A triangular plate made of homogeneous material has a constant thickness that is very small. If it is folded over as shown, determine the location $\bar{y}$ of the plate's center of gravity G.

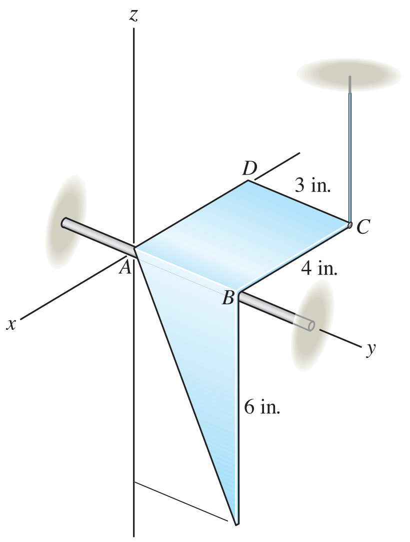
9–73. A triangular plate made of homogeneous material has a constant thickness that is very small. If it is folded over as shown, determine the location $\bar{z}$ of the plate's center of gravity G.



Probs. 9–72/73

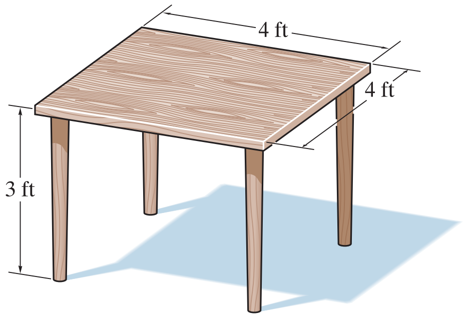
9–74. The sheet metal part has the dimensions shown. Determine the location $(\bar{x}, \bar{y}, \bar{z})$ of its centroid.

9–75. The sheet metal part has a weight per unit area of $2\ \text{lb/ft}^2$ and is supported by the smooth rod and the cord at C. If the cord is cut, the part will rotate about the y axis until it reaches equilibrium. Determine the equilibrium angle of tilt, measured downward from the negative x axis, that AD makes with the $-x$ axis.



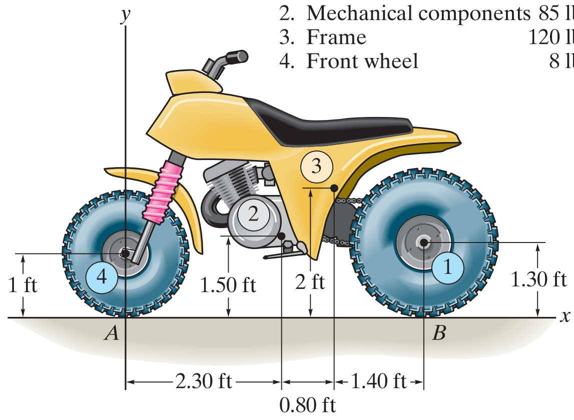
Probs. 9–74/75

***9–76.** The wooden table is made from a square board having a weight of 15 lb. Each of the legs weighs 2 lb and is 3 ft long. Determine how high its center of gravity is from the floor. Also, what is the angle, measured from the horizontal, through which its top surface can be tilted on two of its legs before it begins to overturn? Neglect the thickness of each leg.



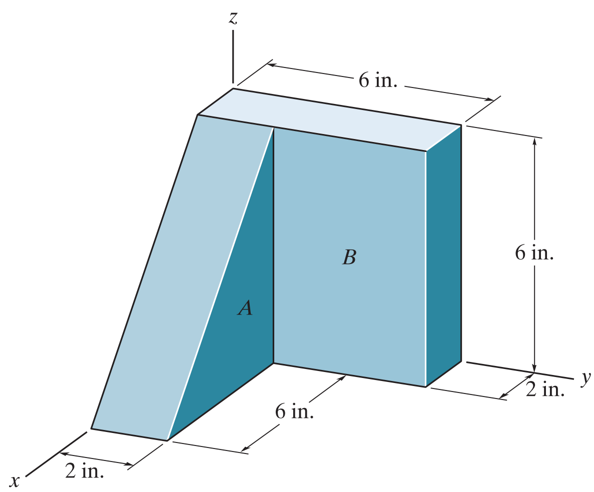
Prob. 9–76

9–77. Determine the location $(\bar{x}, \bar{y})$ of the center of gravity of the three-wheeler. The location of the center of gravity of each component and its weight are tabulated in the figure. If the three-wheeler is symmetrical with respect to the x–y plane, determine the normal reaction each of its wheels exerts on the ground.



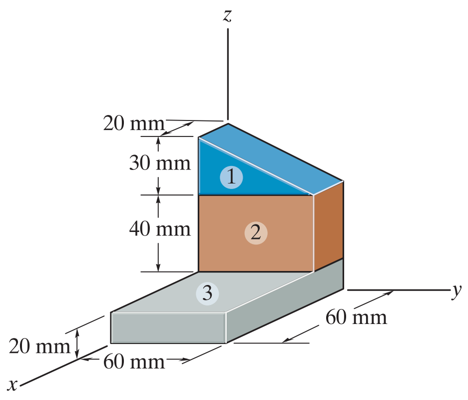
Prob. 9–77

9–78. Locate the center of gravity of the two-block assembly. The specific weights of the materials A and B are $\gamma_A = 150\ \text{lb/ft}^3$ and $\gamma_B = 400\ \text{lb/ft}^3$, respectively.



Prob. 9–78

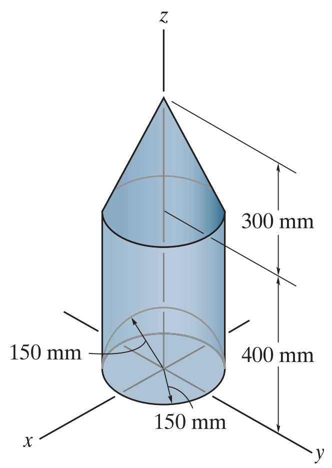
9–79. Locate the center of mass of the block. Materials 1, 2, and 3 have densities of 2.70 Mg/m^3, 5.70 Mg/m^3, and 7.80 Mg/m^3, respectively.



Prob. 9–79

***9–80.** Locate the centroid $\bar{z}$ of the homogenous solid formed by boring a hemispherical hole into the cylinder that is capped with a cone.

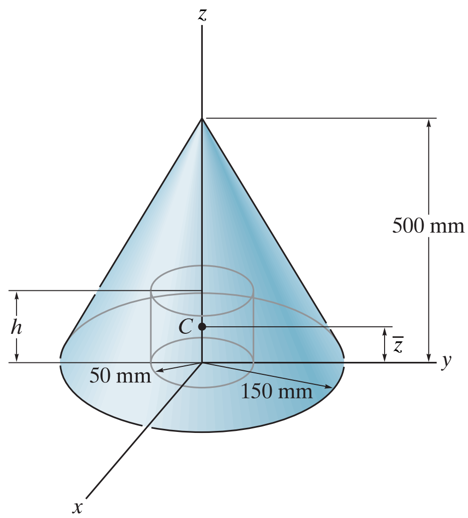
9–81. Locate the center of mass $\bar{z}$ of the solid formed by boring a hemispherical hole into a cylinder that is capped with a cone. The cone and cylinder are made of materials having densities of 7.80 Mg/m^3 and 2.70 Mg/m^3, respectively.



Probs. 9–80/81

9–82. Determine the distance h to which a 100-mm-diameter hole must be bored into the base of the cone so that the center of mass of the resulting shape is located at $\bar{z} = 115$ mm. The material has a density of 8 Mg/m^3.

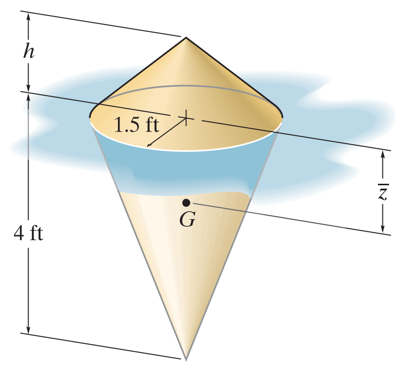
9–83. Determine the distance $\bar{z}$ to the centroid of the shape that consists of a cone with a hole of height $h = 50$ mm bored into its base.



Probs. 9–82/83

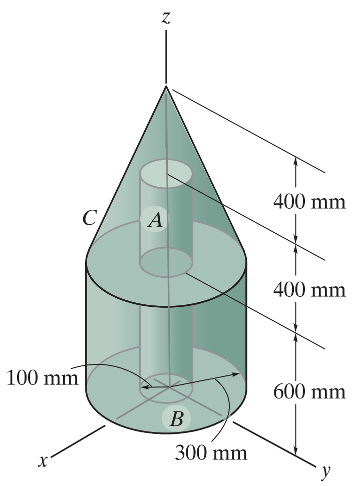
***9–84.** The buoy is made from two homogeneous cones each having a radius of 1.5 ft. If $h = 1.2$ ft, find the distance $\bar{z}$ to the buoy's center of gravity G.

9–85. The buoy is made from two homogeneous cones each having a radius of 1.5 ft. If it is required that the buoy's center of gravity G be located at $\bar{z} = 0.5$ ft, determine the height h of the top cone.



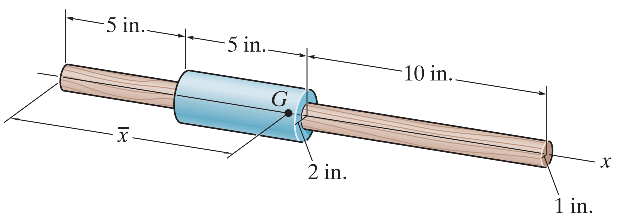
Probs. 9–84/85

9–86. Locate the center of mass $\bar{z}$ of the assembly. The assembly consists of a cylindrical center core, *A*, having a density of 7.90 Mg/m^3, and a cylindrical outer part, *B*, and a cone cap, *C*, each having a density of 2.70 Mg/m^3.



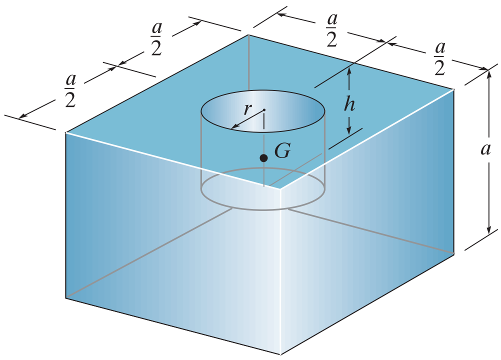
Prob. 9–86

9–87. The assembly consists of a 20-in. wooden dowel rod and a tight-fitting steel collar. Determine the distance $\bar{x}$ to its center of gravity if the specific weights of the materials are $\gamma_w = 150\ \text{lb/ft}^3$ and $\gamma_{st} = 490\ \text{lb/ft}^3$. The radii of the dowel and collar are shown.



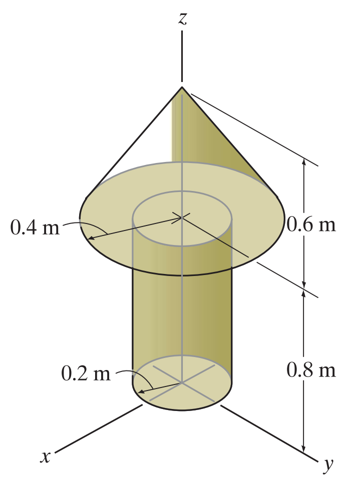
Prob. 9–87

***9–88.** A hole having a radius *r* is to be drilled in the center of the homogeneous block. Determine the depth *h* of the hole so that the center of gravity *G* is as low as possible.



Prob. 9–88

9–89. Locate the center of mass $\bar{z}$ of the assembly. The cylinder and the cone are made from materials having densities of 5 Mg/m^3 and 9 Mg/m^3, respectively.



Prob. 9–89

*9.3 Theorems of Pappus and Guldinus

The two *theorems of Pappus and Guldinus* are used to find the surface area and volume of any body of revolution. They were first developed by Pappus of Alexandria during the fourth century A.D. and then restated at a later time by the Swiss mathematician Paul Guldin or Guldinus (1577–1643).

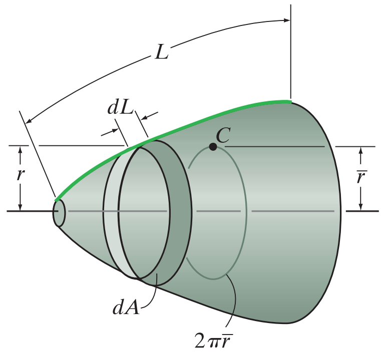


Fig. 9–19

The amount of roofing material used on this storage building can be estimated by using the first theorem of Pappus and Guldinus to determine its surface area.

Surface Area. If we revolve a *plane curve* about an axis that does not intersect the curve we will generate a *surface area of revolution*. For example, the surface area in Fig. 9–19 is formed by revolving the curve of length L about the horizontal axis. To determine this surface area, we will first consider the differential line element of length dL. If this element is revolved 2π radians about the axis, a ring having a surface area of $dA = 2\pi r\, dL$ will be generated. Thus, the surface area of the entire body is $A = 2\pi \int r\, dL$. Since $\int r\, dL = \bar{r}L$ (Eq. 9–5), then $A = 2\pi\bar{r}L$. If the curve is revolved only through an angle θ (radians), then

$$A = \theta\bar{r}L \tag{9–7}$$

where

A = surface area of revolution

θ = angle of revolution measured in radians, $\theta \leq 2\pi$

$\bar{r}$ = perpendicular distance from the axis of revolution to the centroid of the generating curve

L = length of the generating curve

Therefore the first theorem of Pappus and Guldinus states that *the area of a surface of revolution equals the product of the length of the generating curve and the distance traveled by the centroid of the curve in generating the surface area.*

9

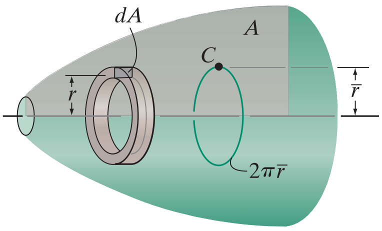


Fig. 9–20

Volume. A *volume* can be generated by revolving a *plane area* about an axis that does not intersect the area. For example, if we revolve the shaded area A in Fig. 9–20 about the horizontal axis, it generates the volume shown. This volume can be determined by first revolving the differential element of area dA 2π radians about the axis, so that a ring having the volume $dV = 2\pi r\, dA$ is generated. The entire volume is then $V = 2\pi \int r\, dA$. However, $\int r\, dA = \bar{r} A$, Eq. 9–4, so that $V = 2\pi\bar{r}A$. If the area is only revolved through an angle θ (radians), then

$$V = \theta \bar{r} A \qquad (9\text{–}8)$$

where

V = volume of revolution

θ = angle of revolution measured in radians, $\theta \leq 2\pi$

$\bar{r}$ = perpendicular distance from the axis of revolution to the centroid of the generating area

A = generating area

Therefore the second theorem of Pappus and Guldinus states that *the volume of a body of revolution equals the product of the generating area and the distance traveled by the centroid of the area in generating the volume.*

Composite Shapes. We may also apply the above two theorems to lines or areas that are composed of a series of composite parts. In this case the total surface area or volume generated is the addition of the surface areas or volumes generated by each of the composite parts. If the perpendicular distance from the axis of revolution to the centroid of each composite part is $\tilde{r}$, then

$$A = \theta \Sigma(\tilde{r} L) \qquad (9\text{–}9)$$

and

$$V = \theta \Sigma(\tilde{r} A) \qquad (9\text{–}10)$$

Application of the above theorems is illustrated numerically in the following examples.

The volume of fertilizer contained within this silo can be determined using the second theorem of Pappus and Guldinus.

9

EXAMPLE 9.12

Show that the surface area of a sphere is $A = 4\pi R^2$ and its volume is $V = \frac{4}{3}\pi R^3$.

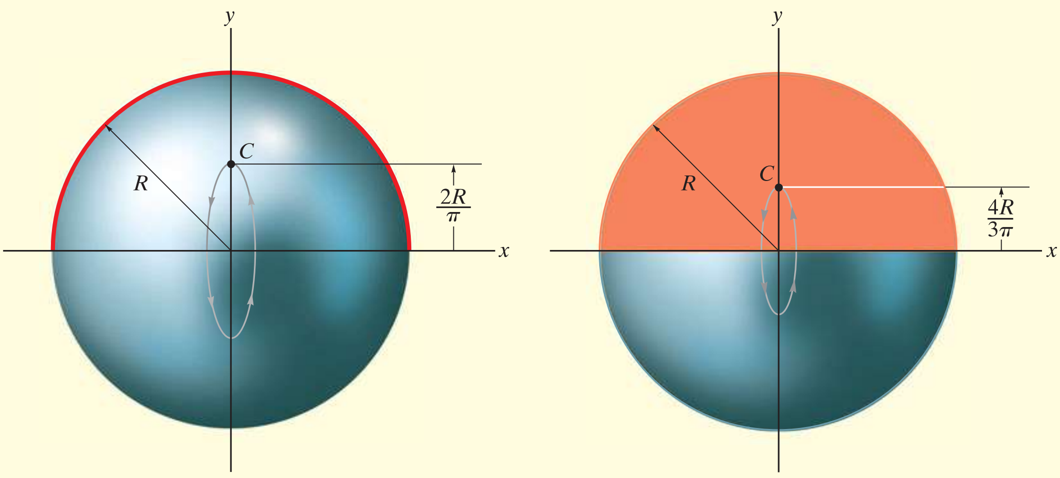


Fig. 9–21

SOLUTION

Surface Area. The surface area of the sphere in Fig. 9–21*a* is generated by revolving a semicircular *arc* about the *x* axis. Using the table on the inside back cover, it is seen that the centroid of this arc is located at a distance $\bar{r} = 2R/\pi$ from the axis of revolution (x axis). Since the centroid moves through an angle of $\theta = 2\pi$ rad to generate the sphere, then applying Eq. 9–7 we have

$$A = \theta\bar{r}L; \qquad A = 2\pi\left(\frac{2R}{\pi}\right)\pi R = 4\pi R^2 \qquad \textit{Ans.}$$

Volume. The volume of the sphere is generated by revolving the semicircular *area* in Fig. 9–21*b* about the *x* axis. Using the table on the inside back cover to locate the centroid of the area, i.e., $\bar{r} = 4R/3\pi$, and applying Eq. 9–8, we have

$$V = \theta\bar{r}A; \qquad V = 2\pi\left(\frac{4R}{3\pi}\right)\left(\frac{1}{2}\pi R^2\right) = \frac{4}{3}\pi R^3 \qquad \textit{Ans.}$$

9

EXAMPLE 9.13

Determine the surface area and volume of the full solid in Fig. 9–22*a*.

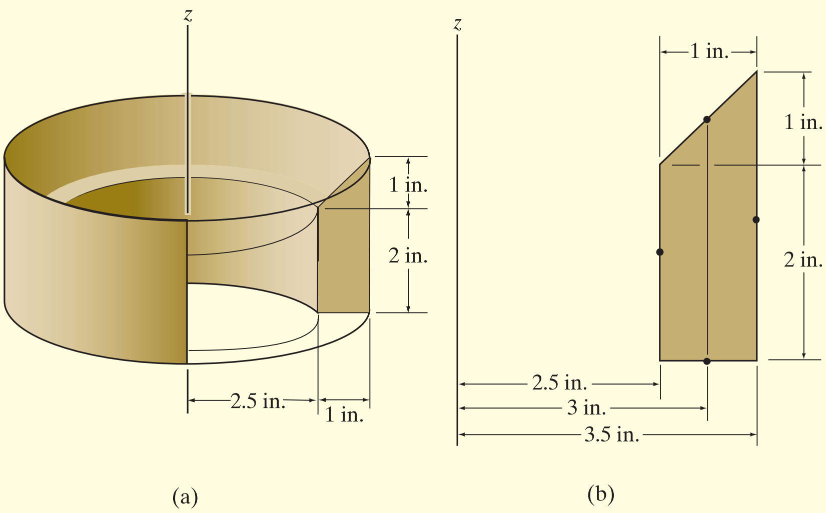


Fig. 9–22

SOLUTION

Surface Area. The surface area is generated by revolving the four line segments shown in Fig. 9–22*b* 2π radians about the z axis. The distances from the centroid of each segment to the z axis are also shown in the figure. Applying Eq. 9–7 yields

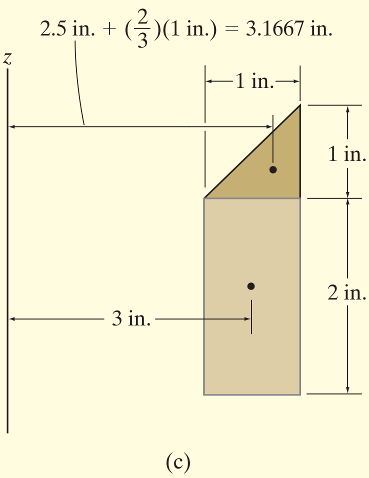
$$A = 2\pi\Sigma\tilde{r}L = 2\pi\left[(2.5\text{ in.})(2\text{ in.}) + (3\text{ in.})\left(\sqrt{(1\text{ in.})^2 + (1\text{ in.})^2}\right) + (3.5\text{ in.})(3\text{ in.}) + (3\text{ in.})(1\text{ in.})\right]$$

$$= 143\text{ in}^2 \qquad \textit{Ans.}$$

Volume. The volume of the solid is generated by revolving the two area segments shown in Fig. 9–22*c* 2π radians about the z axis. The distances from the centroid of each segment to the z axis are also shown in the figure. Applying Eq. 9–10, we have

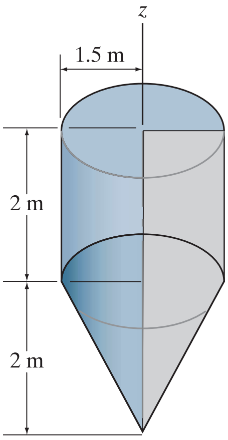
$$V = 2\pi\Sigma\tilde{r}A = 2\pi\left\{(3.1667\text{ in.})\left[\frac{1}{2}(1\text{ in.})(1\text{ in.})\right] + (3\text{ in.})[(2\text{ in.})(1\text{ in.})]\right\}$$

$$= 47.6\text{ in}^3 \qquad \textit{Ans.}$$



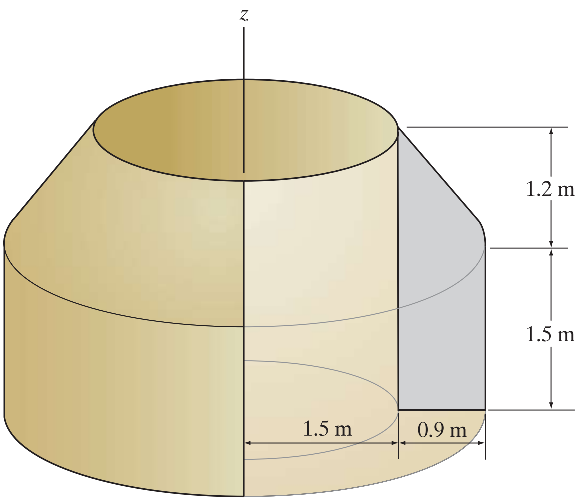
FUNDAMENTAL PROBLEMS

F9–13. Determine the surface area and volume of the solid formed by revolving the shaded area 360° about the z axis.



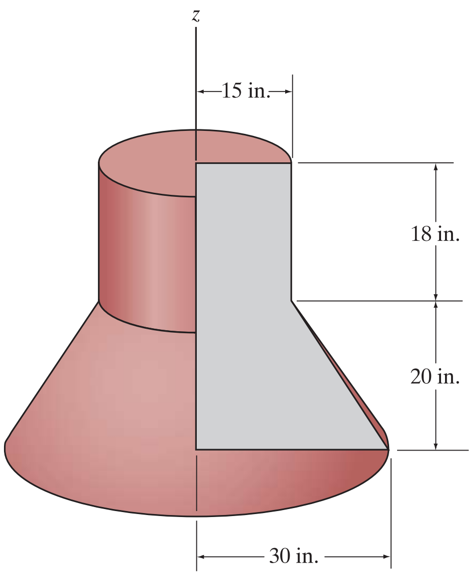
F9–13

F9–14. Determine the surface area and volume of the solid formed by revolving the shaded area 360° about the z axis.



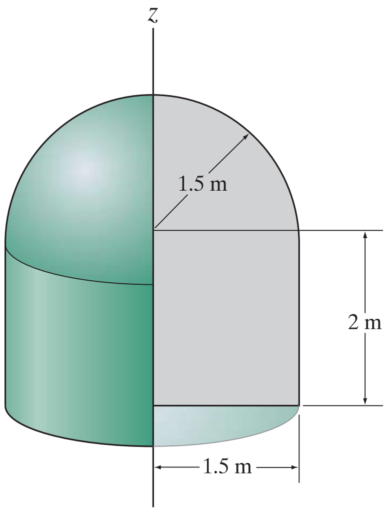
F9–14

F9–15. Determine the surface area and volume of the solid formed by revolving the shaded area 360° about the z axis.



F9–15

F9–16. Determine the surface area and volume of the solid formed by revolving the shaded area 360° about the z axis.

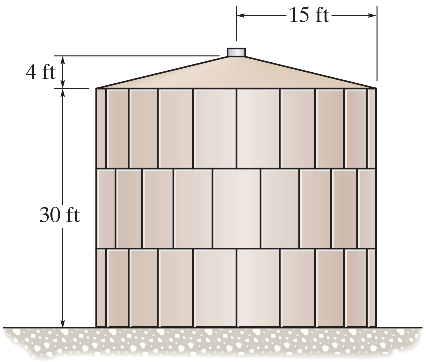


F9–16

PROBLEMS

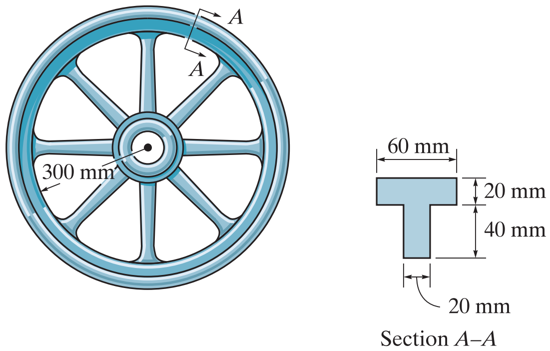
9–90. Determine the outside surface area of the storage tank.

9–91. Determine the volume of the storage tank.



Probs. 9–90/91

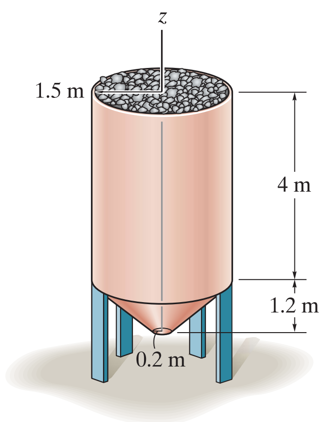
9–94. The *rim* of a flywheel has the cross section *A–A* shown. Determine the volume of material needed for its construction.



Prob. 9–94

***9–92.** Determine the outside surface area of the hopper.

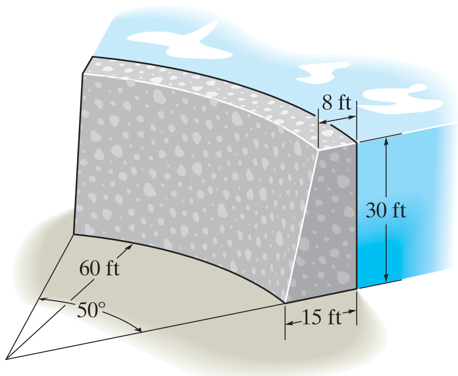
9–93. The hopper is filled to its top with coal. Determine the volume of coal if the voids (air space) are 35 percent of the volume of the hopper.



Probs. 9–92/93

9–95. Determine the surface area of the concrete sea wall, excluding its bottom.

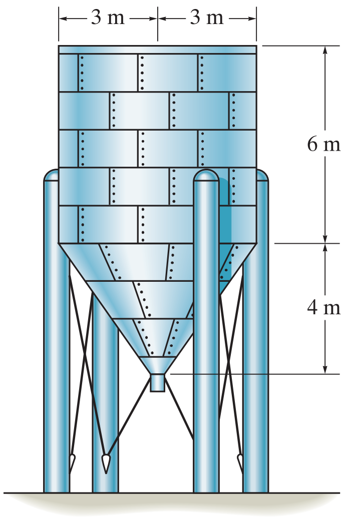
***9–96.** A circular sea wall is made of concrete. Determine the total weight of the wall if the concrete has a specific weight of $\gamma_c = 150\ \text{lb/ft}^3$.



Probs. 9–95/96

9–97. The process tank is used to store liquids during manufacturing. Estimate the outside surface area of the tank. The tank has a flat top and the plates from which the tank is made have negligible thickness.

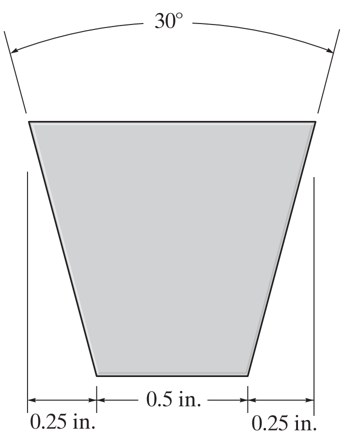
9–98. The process tank is used to store liquids during manufacturing. Estimate the volume of the tank. The tank has a flat top and the plates from which the tank is made have negligible thickness.



Probs. 9–97/98

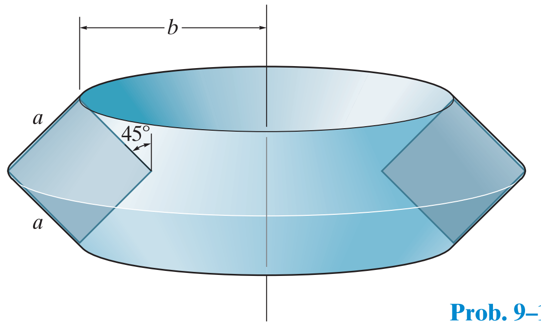
9–99. The V-belt has an inner radius of 6 in. and a cross-sectional area as shown. Determine the outside surface area of the belt.

***9–100.** A V-belt has an inner radius of 6 in., and a cross-sectional area as shown. Determine the volume of material used in making the V-belt.



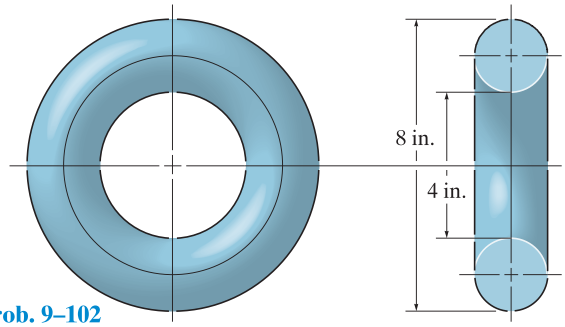
Probs. 9–99/100

9–101. Determine the surface area and the volume of the ring formed by rotating the square about the vertical axis.



Prob. 9–101

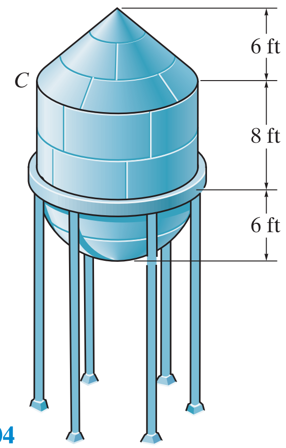
9–102. Determine the surface area of the ring. The cross section is circular as shown.



Prob. 9–102

9–103. The water-supply tank has a hemispherical bottom and cylindrical sides. Determine the weight of water in the tank when it is filled to the top at C. Take $\gamma_w = 62.4 \text{ lb/ft}^3$.

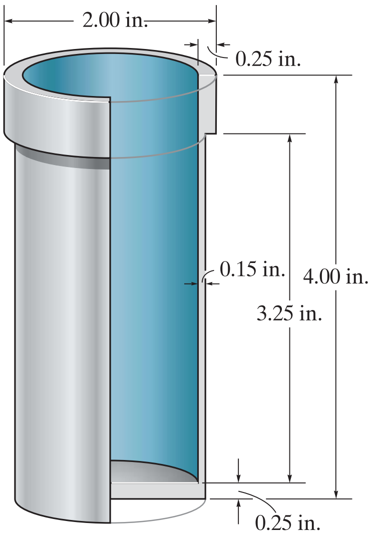
***9–104.** Determine the number of gallons of paint needed to paint the outside surface of the water-supply tank, which consists of a hemispherical bottom, cylindrical sides, and conical top. Each gallon of paint can cover 250 ft^2.



Probs. 9–103/104

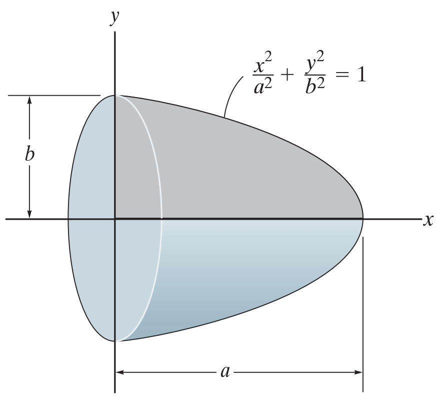
9

9–105. The full circular aluminum housing is used in an automotive brake system. The cross section is shown in the figure. Determine its weight if aluminum has a specific weight of 169 lb/ft^3.



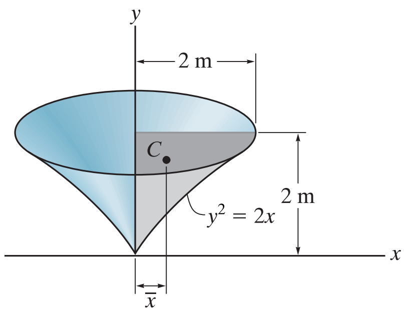
Prob. 9–105

9–106. Determine the volume of an ellipsoid formed by revolving the shaded area about the x axis using the second theorem of Pappus–Guldinus. The area and centroid y of the shaded area should first be obtained by using integration.



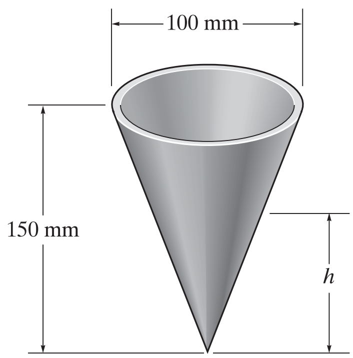
Prob. 9–106

9–107. Using integration, determine both the area and the centroidal distance $\bar{x}$ of the shaded area. Then, using the second theorem of Pappus–Guldinus, determine the volume of the solid generated by revolving the area about the y axis.



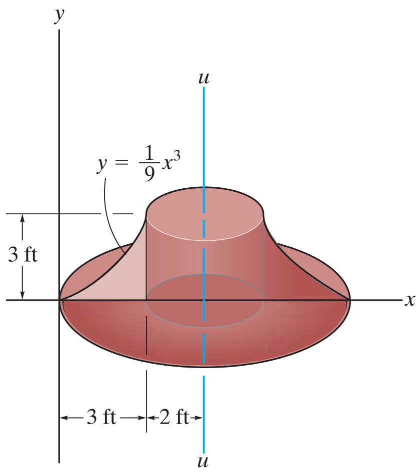
Prob. 9–107

***9–108.** Determine the height h to which liquid should be poured into the conical cup so that it contacts half the surface area on the inside of the cup.



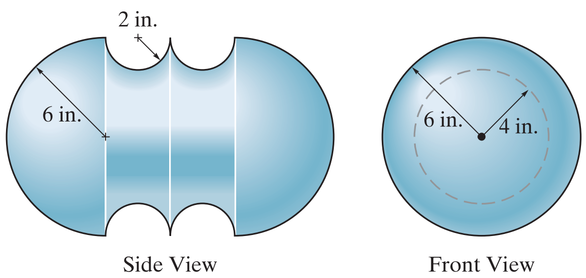
Prob. 9–108

9–109. Determine the volume of the solid formed by revolving the shaded area about the u–u axis using the second theorem of Pappus–Guldinus. The area and centroid of the area should first be obtained by using integration.



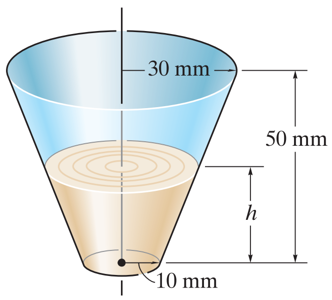
Prob. 9–109

9–110. Determine the volume of material needed to make the casting.



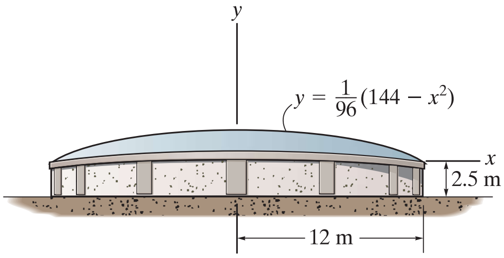
Prob. 9–110

9–111. Determine the height h to which liquid should be poured into the cup so that it contacts half the surface area on the inside of the cup. Neglect the cup's thickness for the calculation.



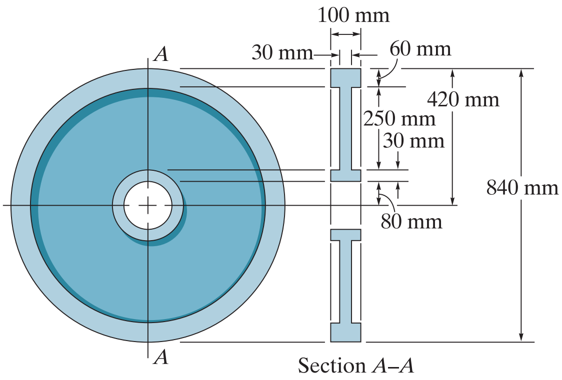
Prob. 9–111

***9–112.** The water tank has a paraboloid-shaped roof. If one liter of paint can cover 3 m^2 of the tank, determine the number of liters required to coat the roof.



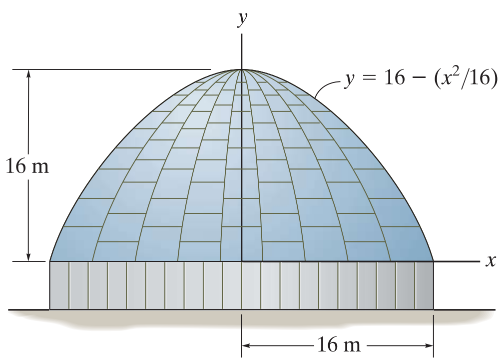
Prob. 9–112

9–113. A steel wheel has a diameter of 840 mm and a cross section as shown in the figure. Determine the total mass of the wheel if $\rho = 5\ \text{Mg/m}^3$.



Prob. 9–113

9–114. Determine the surface area of the roof of the structure if it is formed by rotating the parabola about the y axis.



Prob. 9–114