SECTION 4.4 PERMUTATIONS AND COMBINATIONS

Permutations

Example 26 in Section 4.2 discussed the problem of counting all possibilities for the last four digits of a telephone number with no repeated digits. In this problem, the number 1259 is not the same as the number 2951 because the order of the four digits is important. An ordered arrangement of objects is called a **permutation**. Each of these numbers is a permutation of 4 distinct objects chosen from a set of 10 distinct objects (the digits). How many such permutations are there? The answer, found by using the multiplication principle, is $10 \cdot 9 \cdot 8 \cdot 7$—there are 10 choices for the first digit, then 9 for the next digit because repetitions are not allowed, 8 for the next digit, and 7 for the fourth digit. The number of permutations of r distinct objects chosen from n distinct objects is denoted by $P(n, r)$. Therefore the solution to the problem of the four-digit number without repeated digits can be expressed as $P(10, 4)$.

A formula for $P(n, r)$ can be written using the factorial function. For a positive integer n, ***n* factorial** is defined as $n(n-1)(n-2)\cdots 1$ and denoted by $n!$; also, $0!$ is defined to have the value 1. From the definition of $n!$, we see that

$$n! = n(n-1)!$$

and that for $r < n$,

$$\begin{aligned}\frac{n!}{(n-r)!} &= \frac{n(n-1)\cdots(n-r+1)(n-r)!}{(n-r)!}\\ &= n(n-1)\cdots(n-r+1)\end{aligned}$$

Using the factorial function,

$$\begin{aligned}P(10, 4) &= 10 \cdot 9 \cdot 8 \cdot 7\\ &= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{10!}{6!} = \frac{10!}{(10-4)!}\end{aligned}$$

In general, $P(n, r)$ is given by the formula

$$P(n, r) = \frac{n!}{(n-r)!} \text{ for } 0 \le r \le n$$

EXAMPLE 45 The value of $P(7, 3)$ is

$$\frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = 7 \cdot 6 \cdot 5 = 210$$

EXAMPLE 46 Three somewhat special cases that can arise when computing $P(n, r)$ are the two "boundary conditions" $P(n, 0)$ and $P(n, n)$, and also $P(n, 1)$. According to the formula,

$$P(n, 0) = \frac{n!}{(n - 0)!} = \frac{n!}{n!} = 1$$

This formula can be interpreted as saying that there is only one ordered arrangement of zero objects—the empty set.

$$P(n, 1) = \frac{n!}{(n - 1)!} = n$$

This formula reflects the fact that there are n ordered arrangements of 1 object. (Each arrangement consists of the 1 object, so this merely counts how many ways to get the 1 object.)

$$P(n, n) = \frac{n!}{(n - n)!} = \frac{n!}{0!} = n!$$

This formula states that there are $n!$ ordered arrangements of n distinct objects, which merely reflects the multiplication principle—n choices for the first object, $n - 1$ choices for the second object, and so on, with 1 choice for the nth object.

EXAMPLE 47 The number of permutations of 3 objects, say a, b, and c, is given by $P(3, 3) = 3! = 3 \cdot 2 \cdot 1 = 6$. The 6 permutations of a, b, and c are

$$abc, acb, bac, bca, cab, cba$$

EXAMPLE 48 How many three-letter words (not necessarily meaningful) can be formed from the word "compiler" if no letters can be repeated? Here the arrangement of letters matters, and we want to know the number of permutations of 3 distinct objects taken from 8 objects. The answer is $P(8, 3) = 8!/5! = 336$.

Note that we could have solved Example 48 just by using the multiplication principle—there are 8 choices for the first letter, 7 for the second, and 6 for the third, so the answer is $8 \cdot 7 \cdot 6 = 336$. $P(n, r)$ simply gives us a new way to think about the problem, as well as a compact notation.

EXAMPLE 49 Ten athletes compete in an Olympic event. Gold, silver, and bronze medals are awarded; in how many ways can the awards be made?

This problem is essentially the same as the one in Example 48. Order matters; given 3 winners A, B, and C, the arrangement A−gold, B−silver, C−bronze is different than the arrangement C−gold, A−silver, B−bronze. So we want the number of ordered arrangements of 3 objects from a pool of 10, or $P(10, 3)$. Using the formula for $P(n, r)$, $P(10, 3) = 10!/7! = 10 \cdot 9 \cdot 8 = 720$.

PRACTICE 30 In how many ways can a president and vice-president be selected from a group of 20 people? ■

PRACTICE 31 In how many ways can 6 people be seated in a row of 6 chairs? ■

Counting problems can have other counting problems as subtasks.

EXAMPLE 50 A library has 4 books on operating systems, 7 on programming, and 3 on data structures. Let's see how many ways these books can be arranged on a shelf, given that all books on the same subject must be together.

We can think of this problem as a sequence of subtasks. First we consider the subtask of arranging the 3 subjects. There are 3! outcomes to this subtask, that is, 3! different orderings of subject matter. The next subtasks are arranging the books on operating systems (4! outcomes), then arranging the books on programming (7! outcomes), and finally arranging the books on data structures (3! outcomes). Thus, by the multiplication principle, the final number of arrangements of all the books is $(3!)(4!)(7!)(3!) = 4{,}354{,}560$. ●

Combinations

Sometimes we want to select r objects from a set of n objects, but we don't care how they are arranged. Then we are counting the number of **combinations** of r distinct objects chosen from n distinct objects, denoted by $C(n, r)$. For each such combination, there are $r!$ ways to permute the r chosen objects. By the multiplication principle, the number of permutations of r distinct objects chosen from n objects is the product of the number of ways to choose the objects, $C(n, r)$, multiplied by the number of ways to arrange the objects chosen, $r!$ Thus,

$$C(n, r) \cdot r! = P(n, r)$$

or

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!} \text{ for } 0 \le r \le n$$

Other notations for $C(n, r)$ are

$$_nC_r, \qquad C^n_r, \qquad \binom{n}{r}$$

EXAMPLE 51 The value of $C(7, 3)$ is

$$\begin{aligned}\frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} &= \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\ &= \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 7 \cdot 5 = 35\end{aligned}$$

From Example 45, the value of $P(7, 3)$ is 210, and $C(7, 3) \cdot (3!) = 35(6) = 210 = P(7, 3)$. ●

EXAMPLE 52 The special cases for $C(n, r)$ are $C(n, 0)$, $C(n, 1)$, and $C(n, n)$. The formula for $C(n, 0)$,

$$C(n, 0) = \frac{n!}{0!(n - 0)!} = 1$$

reflects the fact that there is only one way to choose zero objects from n objects: Choose the empty set.

$$C(n, 1) = \frac{n!}{1!(n - 1)!} = n$$

Here the formula indicates that there are n ways to select 1 object from n objects.

$$C(n, n) = \frac{n!}{n!(n - n)!} = 1$$

Here we see that there is only one way to select n objects from n objects, and that is to choose all of the objects.

In the formula for $C(n, r)$, suppose n is held fixed and r is increased. Then $r!$ increases, which tends to make $C(n, r)$ smaller, but $(n - r)!$ decreases, which tends to make $C(n, r)$ larger. For small values of r, the increase in $r!$ is not as great as the decrease in $(n - r)!$, and so $C(n, r)$ increases from 1 to n to larger values. At some point, however, the increase in $r!$ overcomes the decrease in $(n - r)!$, and the values of $C(n, r)$ decrease back down to 1 by the time $r = n$, as we calculated in Example 52. Figure 4.9a illustrates the rise and fall of the values of $C(n, r)$ for a fixed n. For $P(n, r)$, as n is held fixed and r is increased, $n - r$ and therefore $(n - r)!$ decreases, so $P(n, r)$ increases. Values of $P(n, r)$ for $0 \le r \le n$ thus increase from 1 to n to $n!$, as we calculated in Example 46. See Figure 4.9b; note the difference in the vertical scale of Figures 4.9a and 4.9b.

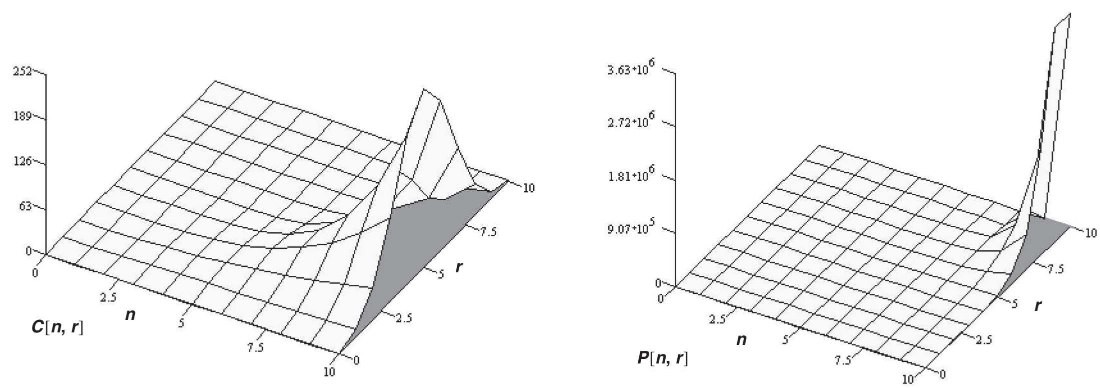

Figure 4.9 (a) (b)

EXAMPLE 53 How many 5-card poker hands are possible with a 52-card deck? Here order does not matter because we simply want to know which cards end up in the hand. We want the number of ways to choose 5 objects from a pool of 52, which is a combinations problem. The answer is $C(52, 5) = 52!/(5!47!) = 2{,}598{,}960$.

Unlike earlier problems, the answer to Example 53 cannot easily be obtained by applying the multiplication principle. Thus, $C(n, r)$ gives us a way to solve new problems.

EXAMPLE 54 Ten athletes compete in an Olympic event; 3 will be declared winners. How many sets of winners are possible?

Here, as opposed to Example 49, there is no order to the 3 winners, so we are simply choosing 3 objects out of 10. This is a combinations problem, not a permutations problem. The result is $C(10, 3) = 10!/(3!7!) = 120$. Notice that there are fewer ways to choose 3 winners (a combinations problem) than to award gold, silver, and bronze medals to 3 winners (a permutations problem—Example 49).

PRACTICE 32 How many committees of 3 are possible from a group of 12 people?

REMINDER

In a counting problem, first ask yourself if order matters. If it does, it's a permutations problem. If not, it's a combinations problem.

Remember that the distinction between permutations and combinations lies in whether the objects are to be merely selected or both selected and ordered. If ordering is important, the problem involves permutations; if ordering is not important, the problem involves combinations. For example, Practice 30 is a permutations problem—2 people are to be selected and ordered, the first as president, the second as vice-president—whereas Practice 32 is a combinations problem—3 people are selected but not ordered.

In solving counting problems, $C(n, r)$ can be used in conjunction with the multiplication principle or the addition principle.

EXAMPLE 55 A committee of 8 students is to be formed from a class consisting of 19 freshmen and 34 sophomores.

a. How many committees of 3 freshmen and 5 sophomores are possible?
b. How many committees with exactly 1 freshman are possible?
c. How many committees with at most 1 freshman are possible?
d. How many committees with at least 1 freshman are possible?

Because the ordering of the individuals chosen is not important, these are combinations problems.

For part (a), we have a sequence of two subtasks, selecting freshmen and selecting sophomores. The multiplication principle should be used. (Thinking of a sequence of subtasks may seem to imply ordering, but it just sets up the levels of the decision tree, the basis for the multiplication principle. There is no ordering of the students.) Because there are $C(19, 3)$ ways to choose the freshmen and $C(34, 5)$ ways to choose the sophomores, the answer is

$$C(19, 3) \cdot C(34, 5) = \frac{19!}{3!16!} \cdot \frac{34!}{5!29!} = (969)(278{,}256)$$

For part (b), we again have a sequence of subtasks: selecting the single freshman and then selecting the rest of the committee from among the sophomores. There are $C(19, 1)$ ways to select the single freshman and $C(34, 7)$ ways to select the remaining 7 members from the sophomores. By the multiplication principle, the answer is

$$C(19, 1) \cdot C(34, 7) = \frac{19!}{1!(19-1)!} \cdot \frac{34!}{7!(34-7)!} = 19(5{,}379{,}616)$$

For part (c), we get at most 1 freshman by having exactly 1 freshman or by having 0 freshmen. Because these are disjoint events, we use the addition principle. The number of ways to select exactly 1 freshman is the answer to part (b). The number of ways to select 0 freshmen is the same as the number of ways to select the entire 8-member committee from among the 34 sophomores, $C(34, 8)$. Thus the answer is

$$C(19, 1) \cdot C(34, 7) + C(34, 8) = \text{some big number}$$

We can attack part (d) in several ways. One way is to use the addition principle, thinking of the disjoint possibilities as exactly 1 freshman, exactly 2 freshmen, and so on, up to exactly 8 freshmen. We could compute each of these numbers and then add them. However, it is easier to do the problem by counting all the ways the committee of 8 can be selected from the total pool of 53 people and then eliminating (subtracting) the number of committees with 0 freshmen (all sophomores). Thus the answer is

$$C(53, 8) - C(34, 8)$$

REMINDER

"At least" counting problems are often best solved by subtraction.

The factorial function grows large quickly. *A* number like 100! cannot be computed on most calculators (or on most computers unless double−precision arithmetic is used), but expressions like

$$\frac{100!}{25!75!}$$

can nevertheless be computed by first canceling common factors.

Eliminating Duplicates

We mentioned earlier that counting problems can often be solved in different ways. Unfortunately, it is also easy to find so-called solutions that sound eminently reasonable but are incorrect. Usually they are wrong because they count something more than once (or sometimes they overlook counting something entirely).

EXAMPLE 56 Consider again part (d) of Example 55, the number of committees with at least 1 freshman. A bogus solution to this problem goes as follows: Think of a sequence of two subtasks, choosing a freshman and then choosing the rest of the committee.

There are $C(19, 1)$ ways to choose 1 freshman. Once a freshman has been selected, that guarantees that at least 1 freshman will be on the committee, so we are free to choose the remaining 7 members of the committee from the remaining 52 people without any restrictions, giving us $C(52, 7)$ choices. By the multiplication principle, this gives $C(19, 1) \cdot C(52, 7)$. However, this is a bigger number than the correct answer.

The problem is this: Suppose Derek and Felicia are both freshmen. In one of the choices we have counted, Derek is the one guaranteed freshman, and we pick the rest of the committee in such a way that Felicia is on it along with 6 others. But we have also counted the option of making Felicia the guaranteed freshman and having Derek and the same 6 others be the rest of the committee. This is the same committee as before, and we have counted it twice. ●

PRACTICE 33 A committee of 2 to be chosen from 4 math majors and 3 physics majors must include at least 1 math major. Compute the following 2 values.

a. $C(7, 2) - C(3, 2)$ (correct solution: all committees minus those with no math majors)
b. $C(4, 1) \cdot C(6, 1)$ (bogus solution: choose 1 math major and then choose the rest of the committee)

The expression $C(4, 1) \cdot C(6, 1) - C(4, 2)$ also gives the correct answer because $C(4, 2)$ is the number of committees with 2 math majors, and these are the committees counted twice in $C(4, 1) \cdot C(6, 1)$. ■

EXAMPLE 57

a. How many distinct permutations can be made from the characters in the word FLORIDA?
b. How many distinct permutations can be made from the characters in the word MISSISSIPPI?

Part (a) is a simple problem of the number of ordered arrangements of seven distinct objects, which is 7!. However, the answer to part (b) is not 11! because the 11 characters in MISSISSIPPI are not all distinct. This means that 11! counts some of the same arrangements more than once (the same arrangement meaning that we cannot tell the difference between MIS$_1$S$_2$ISSIPPI and MIS$_2$S$_1$ISSIPPI.)

Consider any one arrangement of the characters. The four *S*'s occupy certain positions in the string. Rearranging the *S*'s within those positions would result in no distinguishable change, so our one arrangement has 4! look-alikes. In order to avoid overcounting, we must divide 11! by 4! to take care of all the ways of moving the *S*'s around. Similarly, we must divide by 4! to take care of the four *I*'s and by 2! to take care of the two *P*'s. The number of distinct permutations is thus

$$\frac{11!}{4!4!2!}$$ ●

In general, suppose there are n objects of which a set of n_1 are indistinguishable from each other, another set of n_2 are indistinguishable from each other, and so on, down to n_k objects that are indistinguishable from each other. The number of distinct permutations of the n objects is

$$\frac{n!}{(n_1!)(n_2!)\cdots(n_k!)}$$

PRACTICE 34 How many distinct permutations are there of the characters in the word MONGOOSES? ■

Permutations and Combinations with Repetitions

Our formulas for $P(n, r)$ and $C(n, r)$ assume that we arrange or select r objects out of the n available using each object only once. Therefore $r \leq n$. Suppose, however, that the n objects are available for reuse as many times as desired. For example, we construct words using the 26 letters of the alphabet; the words may be as long as desired with letters used repeatedly. Or we may draw cards from a deck, replacing a card after each draw; we may draw as many cards as we like with cards used repeatedly. We can still talk about permutations or combinations of r objects out of n, but with repetitions allowed, r might be greater than n.

Counting the number of permutations of r objects out of n distinct objects with repetition is easy. We have n choices for the first object and, because we can repeat that object, n choices for the second object, n choices for the third, and so on. Hence, the number of permutations of r objects out of n distinct objects with repetition allowed is n^r.

To determine the number of combinations of r objects out of n distinct objects with repetition allowed, we use a rather clever idea.

EXAMPLE 58 A jeweler designing a pin has decided to use five stones chosen from a supply of diamonds, rubies, and emeralds. How many sets of stones are possible?

Because we are not interested in any ordered arrangement of the stones, this is a combinations problem rather than a permutations problem. We want the number of combinations of five objects out of three objects with repetition allowed. The pin might consist of 1 diamond, 3 rubies, and 1 emerald, for instance, or 5 diamonds. We can represent these possibilities by representing the stones chosen by 5 asterisks and placing markers between the asterisks to represent the distribution among the three types of gem, diamonds, rubies, and emeralds. For example, we could represent the choice of 1 diamond, 3 rubies, and 1 emerald by

$$*|***|*$$

while the choice of 5 diamonds, 0 rubies, and 0 emeralds would be represented by

$$*****||$$

Although we wrote the asterisks and markers in a row, there is no ordering implied. We are just looking at seven slots holding the five gems and the two markers, and the different choices are represented by which of the seven slots are occupied by asterisks. We therefore count the number of ways to choose five items out of seven, which is $C(7, 5)$ or

$$\frac{7!}{5!2!}$$ ●

In general, if we use the same scheme to represent a combination of r objects out of n distinct objects with repetition allowed, there must be $n - 1$ markers to indicate the number of copies of each of the n objects. This gives $r + (n - 1)$ slots to fill, and we want to know the number of ways to select r of these. Therefore we want

$$C(r + n - 1, r) = \frac{(r + n - 1)!}{r!(r + n - 1 - r)!} = \frac{(r + n - 1)!}{r!(n - 1)!}$$

This agrees with the result in Example 58, where $r = 5$, $n = 3$.

PRACTICE 35 Six children get one lollipop each from among a selection of red, yellow, and green lollipops. How many sets of lollipops are possible? (We do not care which child gets which.) ■

We have discussed a number of counting techniques in this chapter. Table 4.2 summarizes the techniques you can apply in various circumstances, although there may be several legitimate ways to solve any one counting problem.

TABLE 4.2

You Want to Count the Number of ...	Technique to Try
Subsets of an n-element set	Use formula 2^n.
Outcomes of successive events	Multiply the number of outcomes for each event.
Outcomes of disjoint events	Add the number of outcomes for each event.
Outcomes given specific choices at each step	Draw a decision tree and count the number of paths.
Elements in overlapping sections of related sets	Use principle of inclusion and exclusion formula.
Ordered arrangements of r out of n distinct objects	Use $P(n, r)$ formula.
Ways to select r out of n distinct objects	Use $C(n, r)$ formula.
Ways to select r out of n distinct objects with repetition allowed	Use $C(r + n - 1, r)$ formula.

Generating Permutations and Combinations

In a certain county, lottery ticket numbers consist of a sequence (a permutation) of the 9 digits 1, 2, …, 9. The ticket printing company may or may not know that $9! = 362{,}880$ distinct ticket numbers are possible, but it certainly needs a way to generate all possible ticket numbers. Or the county council (a group of 12 members) wants to form a subcommittee of 4 members but wants to pick the combination of council members it feels can best work together. The council could ask someone to generate all $C(12, 4) = 495$ potential subcommittees and examine the membership of each one. We see that in some situations, simply counting the number of permutations or combinations is not enough; it is useful to be able to list all the permutations or combinations.

EXAMPLE 59

Example 47 asked for the number of permutations of the three objects a, b, and c. The answer is given by the formula $P(3,3) = 3! = 6$. However Example 47 went on to list the six permutations:

$$abc, acb, bac, bca, cab, cba$$

This list presents the six permutations using **lexicographical ordering**, that is, the order in which they would be found in a dictionary if they were legitimate words. Thus *abc* precedes *acb* because although both words begin with the same first character, for the second character, b precedes c. If we had three integers, say 4, 6, and 7, instead of three alphabetical characters, the lexicographical ordering of all six permutations would present values in increasing numerical order:

$$467, 476, 647, 674, 746, 764$$

PRACTICE 36 Arrange the following list of permutations in lexicographical order:

$$scary, yarsc, scyra, cysar, scrya, yarcs$$

Words that are close in lexicographical order have the maximum number of matching leftmost characters or, equivalently, differ in the fewest rightmost characters. We use this characteristic to develop a process to generate all permutations of the integers $\{1, \ldots, n\}$ in lexicographical order.

EXAMPLE 60

Consider the set $\{1, 2, 3, 4, 5\}$. The smallest numerical value (the first permutation) is given by the increasing order of all the integers, namely,

$$12345$$

To generate the next number in lexicographical order, we want to retain as many of the leftmost digits as possible. Clearly we can't keep the leftmost four digits because this also determines the fifth digit. To keep the leftmost three digits, 123 – –, we must be able to rearrange the remaining two digits to represent a larger value than they do now. Reading 12345 from right to left, we find in the last two digits that $4 < 5$, which means we can reverse the 4 and the 5 to get

$$12354$$

which is the next permutation in the list. That's all that can be done with the last two digits; in particular, since 54 is a decreasing sequence, we can't use these two values to generate anything larger.

For the next number, we keep 12− − − and consider how to arrange the last three digits. Reading 12354 from right to left, we find in the last three digits that $3 < 5$, but we know that everything from 5 to the right is a decreasing sequence. The next permutation should replace 3 with the next largest value to its right. Reading from right to left in the number 12354, the first value larger than 3, in this case 4, is the least value larger than 3. Swapping 3 and 4 gives 12453, which puts 4 in the correct order; the digits to the right are now in descending order, so reversing them gives

$$12435$$

which is the next permutation.

EXAMPLE 61 To continue Example 60, let's jump ahead. Suppose we have just generated permutation

$$25431$$

and we want the next permutation. Reading from right to left, everything increases until we get to 2, where we have $2 < 5$. Starting again from right to left, we stop at the first (and smallest) value greater than 2, which is 3. Swapping 2 and 3 gives 35421, giving the correct first digit. The digits after 3 are in descending order, so reversing them involves swapping 5 and 1, and also swapping 4 and 2, giving the next permutation

$$31245$$

From the preceding examples, we can construct an algorithm to generate all permutations of the integers from 1 to n in lexicographical order.

ALGORITHM ***PERMUTATION GENERATOR***

PermGenerator(integer $n \geq 2$)

//generates in lexicographical order all permutations
//of the integers in the set $\{1, \ldots, n\}$
Local variables:
integers i, j //indices of permutation elements
integer k //for loop counter
integers $d_1, d_2, \ldots, d_n$ //left to right elements of a permutation

```
    //create and write out smallest permutation
    for k = 1 to n do
        d_k = k
    end for
    write d_1 d_2 ... d_n

    //create and write out remaining permutations
    for k = 2 to n! do
        //look right to left for first break in increasing sequence
        i = n − 1
        j = n
        while d_i > d_j do //still increasing right to left
            i = i − 1
            j = j − 1
        end while
        //now d_i < d_j, need to replace d_i with next largest integer

        //look right to left for smallest value greater than d_i
        j = n
        while d_i > d_j do
            j = j − 1
        end while
        //now d_j is smallest value > d_i

        swap d_i and d_j

        //reverse the digits to the right of index i
        i = i + 1
        j = n
        while i < j do
            swap d_i and d_j
            i = i + 1
            j = j − 1
        end while

        write d_1 d_2 ... d_n
    end for
end function PermGenerator
```

PRACTICE 37 Walk through the steps in the algorithm that generate the next permutation following 51432. ■

Another algorithm for generating (not in lexicographical order) all permutations of the integers $\{1, \ldots, n\}$ is suggested in Exercise 7 of On the Computer at the end of this chapter. Both of these algorithms can also be used to generate all permutations of any n distinct elements; simply assign each of the n elements a unique integer from 1 to n, generate the permutations of the integers, and then reverse the assignment.

Our second problem is to generate the $C(n, r)$ combinations of r distinct integers chosen from $\{1, \dots, n\}$. Such a combination does not involve order, it is merely a subset of r elements. Nonetheless we will represent the subset $\{3, 5, 7\}$ as the sequence 357, and generate the subsets in lexicographical order. Once we generate 357, we can't also generate 375 or 753 or any of the other permutations of the elements in this set. Each legitimate representation is an increasing sequence.

EXAMPLE 62 Consider the lexicographical ordering of the combinations of 4 integers from $\{1, \dots, 7\}$. If the combination

$$2346$$

has just been generated, then the next combination would be

$$2347$$

obtained by incrementing the last digit of the sequence. However, in 2347, the last digit is already at its maximum allowable value. Moving to the left, the 4 can be bumped up to 5, but then the last digit has to be reduced to its minimum value, which is 6 (one more than 5). Therefore,

$$2356$$

is the next combination. The next two values are

$$2357, 2367$$

at which point both 7 and 6 are at their maximum values. The 3 can be bumped up, but the two digits to its right have to be reset to their lowest possible values. The next few values are

$$2456, 2457, 2467, 2567 \dots$$

Based on the ideas of Example 62, given a combination sequence the algorithm should bump up the rightmost digit that is not at its maximum allowable value. The sequence of digits to the right of the newly incremented digit v should have the values $v + 1$, $v + 2$, and so on. The initial (smallest) combination is $12 \dots r$.

ALGORITHM COMBINATION GENERATOR

CombGenerator(integer $n \geq 2$, integer $r \geq 1$)
//generates in lexicographical order all combinations
//of r integers from the set $\{1, \ldots, n\}$
Local variables:
integers i, j //indices of combination elements
integer k //for loop counter
integer *max* //maximum allowable value for a digit
integers $d_1, d_2, \ldots, d_r$ //left to right elements of a combination

  //create and write out smallest combination
  for $k = 1$ to r **do**
    $d_k = k$
  end for
  write $d_1 d_2 \ldots d_r$

  //create and write out remaining combinations
  for $k = 2$ to $C(n, r)$ **do**
    //look right to left for first non-max value
    $max = n$
    $i = r$
    while $d_i = max$ **do** //look left
      $i = i - 1$
      $max = max - 1$
    end while
    //now $d_i < max$, need to increment d_i

    $d_i = d_i + 1$

    //reset values right of d_i
    for $j = i + 1$ to r **do**
      $d_j = d_{j-1} + 1$
    end for
    write $d_1 d_2 \ldots d_r$
  end for
end function *CombGenerator*

PRACTICE 38 Using this algorithm, find the next combination of five items from $\{1, \ldots, 9\}$ after 24589. ■