In Exercises 101–104, use algorithm permutation generator to generate the next permutation after the given permutation in the set of all permutations of the numbers $\{1, \ldots, 7\}$.

101. 7431652
102. 4365127
103. 3675421
104. 2756431
105. In generating all combinations of five items from the set $\{1, \ldots, 9\}$, find the next five values in the list after 24579.
106. In generating all combinations of four items from the set $\{1, \ldots, 6\}$, find the next five values in the list after 1234.
107. Describe an algorithm to generate all permutations of the integers $\{1, \ldots, n\}$ in reverse lexicographical order.
108. Describe an algorithm to generate all permutations of r elements from the set $\{1, \ldots, n\}$.

SECTION 4.5 BINOMIAL THEOREM

The expression for squaring a binomial is a familiar one:

$$(a + b)^2 = a^2 + 2ab + b^2$$

This is a particular case of raising a binomial to a nonnegative integer power n. The formula for $(a + b)^n$ involves combinations of n objects. Before we prove this formula, we'll look at a historically interesting array of numbers that suggests a fact we will need in the proof.

Pascal's Triangle

Pascal's triangle is named for the seventeenth-century French mathematician Blaise Pascal (for whom the programming language Pascal was also named), although it was apparently known several centuries earlier. Row n of the triangle ($n \geq 0$) consists of all the values $C(n, r)$ for $0 \leq r \leq n$. Thus the triangle looks like this:

$$\begin{array}{ccccccccccccl}
 & & & & & & & & & & & & \text{Row} \\
 & & & & & & C(0,0) & & & & & & 0 \\
 & & & & & C(1,0) & & C(1,1) & & & & & 1 \\
 & & & & C(2,0) & & C(2,1) & & C(2,2) & & & & 2 \\
 & & & C(3,0) & & C(3,1) & & C(3,2) & & C(3,3) & & & 3 \\
 & & C(4,0) & & C(4,1) & & C(4,2) & & C(4,3) & & C(4,4) & & 4 \\
 & C(5,0) & & C(5,1) & & C(5,2) & & C(5,3) & & C(5,4) & & C(5,5) & 5 \\
 & & & & & & \vdots & & & & & & \vdots \\
C(n,0) & & C(n,1) & & & & \ldots & & & & C(n,n-1) & C(n,n) & n
\end{array}$$

If we compute the numerical values of the expressions, we see that Pascal's triangle has the form

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}$$

Observing this figure, it is clear that the outer edges are all 1*s*. But it also seems that any element not on the outer edge can be obtained by adding together the two elements directly above it in the preceding row (for example, the first 10 in row five is below the first 4 and the 6 of row four). If this relationship is indeed always true, it means that

$$C(n, k) = C(n - 1, k - 1) + C(n - 1, k) \text{ for } 1 \le k \le n - 1 \tag{1}$$

Equation (1) is known as **Pascal's formula**.

To prove Pascal's formula, we begin with the right side:

$$\begin{aligned}
C(n-1, k-1) + C(n-1, k) &= \frac{(n-1)!}{(k-1)![n-1-(k-1)]!} + \frac{(n-1)!}{k!(n-1-k)!} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!} \\
&= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-1)!(n-k)}{k!(n-k)!}
\end{aligned}$$

(multiplying the first term by k/k and the second term by $(n - k)/(n - k)$)

$$= \frac{k(n-1)! + (n-1)!(n-k)}{k!(n-k)!}$$

(adding fractions)

$$= \frac{(n-1)![k + (n-k)]}{k!(n-k)!}$$

(factoring the numerator)

$$\begin{aligned}
&= \frac{(n-1)!(n)}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} \\
&= C(n, k)
\end{aligned}$$

Another, less algebraic way to prove Pascal's formula involves a counting argument; hence it is called a **combinatorial proof**. We want to compute $C(n, k)$, the number of ways to choose k objects from n objects. There are two disjoint categories of such choices—item 1 is one of the k objects or it is not. If item 1 is one of the k objects, then the remaining $k - 1$ objects must come from the remaining $n - 1$ objects exclusive of item 1, and there are $C(n - 1, k - 1)$ ways for this to happen. If item 1 is not one of the k objects, then all k objects must come from the remaining $n - 1$ objects, and there are $C(n - 1, k)$ ways for this to happen. The total number of outcomes is the sum of the number of outcomes from these two disjoint cases.

Once we have Pascal's formula for our use, we can develop the formula for $(a + b)^n$, known as the **binomial theorem**.

Binomial Theorem and Its Proof

In the expansion of $(a + b)^2$, $a^2 + 2ab + b^2$, the coefficients are 1, 2, and 1, which is row 2 in Pascal's triangle.

PRACTICE 39 Compute the expansion for $(a + b)^3$ and $(a + b)^4$ and compare the coefficients with rows 3 and 4 of Pascal's triangle. ■

Looking at the coefficients in the expansion of $(a + b)^2$, $(a + b)^3$, and $(a + b)^4$ suggests a general result, which is that the coefficients in the expansion of $(a + b)^n$ look like row n in Pascal's triangle. This is indeed the binomial theorem.

THEOREM **BINOMIAL THEOREM**

For every nonnegative integer n,

$$\begin{aligned}(a + b)^n &= C(n, 0)a^n b^0 + C(n, 1)a^{n-1}b^1 + C(n, 2)a^{n-2}b^2 \\ &\quad + \cdots + C(n, k)a^{n-k}b^k + \cdots + C(n, n-1)a^1 b^{n-1} + C(n, n)a^0 b^n \\ &= \sum_{k=0}^{n} C(n, k)a^{n-k}b^k\end{aligned}$$

REMINDER

The expansion of $(a + b)^n$ starts with $a^n b^0$. From there the power of a goes down and the power of b goes up, but for each term the powers of a and b add up to n. The coefficients are all of the form $C(n$, the power of $b)$.

Because the binomial theorem is stated "for every nonnegative integer n," a proof by induction seems appropriate. For the basis step, $n = 0$, the theorem states

$$(a + b)^0 = C(0, 0)a^0 b^0$$

which is

$$1 = 1$$

Since this is certainly true, the basis step is satisfied.

As the inductive hypothesis, we assume that

$$(a+b)^k = C(k,0)a^kb^0 + C(k,1)a^{k-1}b^1 + \cdots + C(k,k-1)a^1b^{k-1} + C(k,k)a^0b^k$$

Now consider

$$\begin{aligned}(a+b)^{k+1} &= (a+b)^k(a+b) = (a+b)^ka + (a+b)^kb \\ &= [C(k,0)a^kb^0 + C(k,1)a^{k-1}b^1 + \cdots + C(k,k-1)a^1b^{k-1} \\ &\quad + C(k,k)a^0b^k]a + [C(k,0)a^kb^0 + C(k,1)a^{k-1}b^1 \\ &\quad + \cdots + C(k,k-1)a^1b^{k-1} + C(k,k)a^0b^k]b\end{aligned}$$

(by the inductive hypothesis)

$$\begin{aligned}&= C(k,0)a^{k+1}b^0 + C(k,1)a^kb^1 + \cdots + C(k,k-1)a^2b^{k-1} \\ &\quad + C(k,k)a^1b^k + C(k,0)a^kb^1 + C(k,1)a^{k-1}b^2 \\ &\quad + \cdots + C(k,k-1)a^1b^k + C(k,k)a^0b^{k+1} \\ &= C(k,0)a^{k+1}b^0 + [C(k,0) + C(k,1)]a^kb^1 + [C(k,1) + C(k,2)]a^{k-1}b^2 \\ &\quad + \cdots + [C(k,k-1) + C(k,k)]a^1b^k + C(k,k)a^0b^{k+1}\end{aligned}$$

(collecting like terms)

$$\begin{aligned}&= C(k,0)a^{k+1}b^0 + C(k+1,1)a^kb^1 + C(k+1,2)a^{k-1}b^2 \\ &\quad + \cdots + C(k+1,k)a^1b^k + C(k,k)a^0b^{k+1}\end{aligned}$$

(using Pascal's formula)

$$\begin{aligned}&= C(k+1,0)a^{k+1}b^0 + C(k+1,1)a^kb^1 + C(k+1,2)a^{k-1}b^2 \\ &\quad + \cdots + C(k+1,k)a^1b^k + C(k+1,k+1)a^0b^{k+1}\end{aligned}$$

(because $C(k, 0) = 1 = C(k + 1, 0)$ and $C(k, k) = 1 = C(k + 1, k + 1)$)

This completes the inductive proof of the binomial theorem.

The binomial theorem also has a combinatorial proof. Writing $(a + b)^n$ as $(a + b)(a + b) \cdots (a + b)$ (n factors), we know that the answer (using the distributive law of numbers) is the sum of all values obtained by multiplying each term in a factor by a term from every other factor. For example, using b as the term from k factors and a as the term from the remaining $n - k$ factors produces the expression $a^{n-k}b^k$. Using b from a different set of k factors and a from the $n - k$ remaining factors also produces $a^{n-k}b^k$. How many such terms are there? There are $C(n, k)$ different ways to select k factors from which to use b; hence there are $C(n, k)$ such terms. After adding these terms together, the coefficient of $a^{n-k}b^k$ is $C(n, k)$. As k ranges from 0 to n, the result of summing the terms is the binomial theorem.

Because of its use in the binomial theorem, the expression $C(n, r)$ is also known as a **binomial coefficient**.

Applying the Binomial Theorem

EXAMPLE 63 Using the binomial theorem, we can write the expansion of $(x - 3)^4$. To match the form of the binomial theorem, think of this expression as $(x + (-3))^4$ so that b equals -3. Remember that a negative number raised to a power is positive for an even power, negative for an odd power. Thus

$$\begin{aligned}(x-3)^4 &= C(4,0)x^4(-3)^0 + C(4,1)x^3(-3)^1 + C(4,2)x^2(-3)^2 \\ &\quad + C(4,3)x^1(-3)^3 + C(4,4)x^0(-3)^4 \\ &= x^4 + 4x^3(-3) + 6x^2(9) + 4x(-27) + 81 \\ &= x^4 - 12x^3 + 54x^2 - 108x + 81\end{aligned}$$

PRACTICE 40 Expand $(x + 1)^5$ using the binomial theorem.

The binomial theorem tells us that term $k + 1$ in the expansion of $(a + b)^n$ is $C(n, k)a^{n-k}b^k$. This allows us to find individual terms in the expansion without computing the entire expression.

PRACTICE 41 What is the fifth term in the expansion of $(x + y)^7$?

By using various values for a and b in the binomial theorem, certain identities can be obtained.

EXAMPLE 64 Let $a = b = 1$ in the binomial theorem. Then

$$(1+1)^n = C(n,0) + C(n,1) + \cdots + C(n,k) + \cdots + C(n,n)$$

or

$$2^n = C(n,0) + C(n,1) + \cdots + C(n,k) + \cdots + C(n,n) \tag{2}$$

This result says that the sum of all the entries in row n of Pascal's triangle equals 2^n. Actually, Equation (2) can be proved on its own using a combinatorial proof. The number $C(n, k)$, the number of ways to select k items from a set of n items, can be thought of as the number of k-element subsets of an n-element set. The right side of Equation (2) therefore represents the total number of all the subsets (of all sizes) of an n-element set. But we already know that the number of such subsets is 2^n.

SECTION 4.5 REVIEW

TECHNIQUE

- Use the binomial theorem to expand a binomial.
- Use the binomial theorem to find a particular term in the expansion of a binomial.

MAIN IDEAS

- The binomial theorem provides a formula for expanding a binomial without multiplying it out.
- The coefficients of a binomial raised to a nonnegative integer power are combinations of n items as laid out in row n of Pascal's triangle.

EXERCISES 4.5

1. Expand the expression using the binomial theorem.
   a. $(a + b)^5$
   b. $(x + y)^6$
   c. $(a + 2)^5$
   d. $(a - 4)^4$
2. Expand the expression using the binomial theorem.
   a. $(2x + 3y)^3$
   b. $(3x - 1)^5$
   c. $(2p - 3q)^4$
   d. $(3x + \frac{1}{2})^5$

In Exercises 3–10, find the indicated term in the expansion.

3. The fourth term in $(a + b)^{10}$
4. The seventh term in $(x - y)^{12}$
5. The sixth term in $(2x - 3)^9$
6. The fifth term in $(3a + 2b)^7$
7. The last term in $(x - 3y)^8$
8. The last term in $(ab + 3x)^6$
9. The third term in $(4x - 2y)^5$
10. The fourth term in $(3x - \frac{1}{2})^8$
11. Use the binomial theorem (more than once) to expand $(a + b + c)^3$.
12. Expand $(1 + 0.1)^5$ in order to compute $(1.1)^5$.
13. What is the coefficient of x^3y^4 in the expansion of $(2x - y + 5)^8$?
14. What is the coefficient of $x^5y^2z^2$ in the expansion of $(x + y + 2z)^9$?
15. Prove that
$$C(n + 2, r) = C(n, r) + 2C(n, r - 1) + C(n, r - 2) \text{ for } 2 \le r \le n$$
(*Hint*: Use Pascal's formula.)
16. Prove that
$$C(k, k) + C(k + 1, k) + \cdots + C(n, k) = C(n + 1, k + 1) \text{ for } 0 \le k \le n$$
(*Hint*: Use induction on n for a fixed, arbitrary k, as well as Pascal's formula.)