Gaussian Elimination

Gaussian elimination for the solution of a linear system transforms the system $Sx = f$ into an equivalent system $Ux = c$ with upper triangular matrix U (that means all entries in U below the diagonal are zero).
This transformation is done by applying three types of transformations to the augmented matrix $(S \mid f)$.

Type 1: Interchange two equations; and

Type 2: Replace an equation with the sum of the same equation and a multiple of another equation.

Once the augmented matrix $(U \mid f)$ is transformed into $(U \mid c)$, where U is an upper triangular matrix, we can solve this transformed system $Ux = c$ using backsubstitution.

Example 1

Suppose that we want to solve

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}. \qquad (1)$$

We apply Gaussian elimination. To keep track of the operations, we use, e.g., $R_2 = R_2 - 2 * R_1$, which means that the new row 2 is computed by subtracting 2 times row 1 from row 2.

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{array}\right) \begin{array}{l} R_2 = R_2 - 2 * R_1 \\ R_3 = R_3 + R_1 \end{array}$$

$$\to \left(\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{array}\right) \quad R_3 = R_3 - R_2$$

The original system (1) is equivalent to

$$\begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}. \qquad (2)$$

The system (2) can be solved by backsubstitution. We get the solution

$$x_3 = 8/4 = 2, \quad x_2 = (4 - 1*2)/1 = 2, \quad x_1 = (2 - 4*2 - (-2)*2)/2 = -1.$$

Example 2

Suppose that we want to solve

$$\begin{pmatrix} 2 & 3 & -2 \\ 1 & -2 & 3 \\ 4 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}. \qquad (3)$$

We apply Gaussian elimination.

$$\left(\begin{array}{ccc|c} 2 & 3 & -2 & f_1 \\ 1 & -2 & 3 & f_2 \\ 4 & -1 & 4 & f_3 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 3 & -2 & f_1 \\ 0 & -7/2 & 4 & f_2 - f_1/2 \\ 0 & -7 & 8 & f_3 - 2f_1 \end{array}\right) \begin{array}{l} R_2 = R_2 - 0.5 * R_1 \\ R_3 = R_3 - 2 * R_1 \end{array}$$

$$\to \left(\begin{array}{ccc|c} 2 & 3 & -2 & f_1 \\ 0 & -7/2 & 4 & f_2 - f_1/2 \\ 0 & 0 & 0 & f_3 - 2f_2 - f_1 \end{array}\right) \quad R_3 = R_3 - 2R_2$$

The original system (3) is equivalent to

$$\begin{pmatrix} 2 & 3 & -2 \\ 0 & -7/2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 - f_1/2 \\ f_3 - 2f_2 - f_1 \end{pmatrix}. \qquad (4)$$

The last equation in system (4) reads $0x_1 + 0x_2 + 0x_3 = f_3 - 2f_2 - f_1$.
This can only be satisfied of the right hand side satisfies
$f_3 - 2f_2 - f_1 = 0$, for example if $f_1 = f_2 = f_3 = 1$.

Example 2 (cont.)

If $f_3 - 2f_2 - f_1 = 0$, then x_3 can be chosen arbitrarily and x_2, x_1 can be determined by backsubstitution.
If $f_3 - 2f_2 - f_1 = 0$, then

$$x_3 = \text{any scalar}, \quad x_2 = (f_2 - f_1/2 - 4x_3)*(-2/7), \quad x_1 = (f_1 - 3x_2 + 2x_3)/2.$$

For example if $f_1 = f_2 = f_3 = 1$, and if we choose $x_3 = 0$, then

$$x_3 = 0, \quad x_2 = -1/7, \quad x_1 = 5/7.$$

The Matrix Inverse

A square matrix $S \in \mathbb{R}^{n\times n}$ is invertible if there exists a matrix $X \in \mathbb{R}^{n\times n}$ such that

$$XS = I \quad \text{and} \quad SX = I.$$

The matrix X is called the inverse of S and is denoted by S^{-1}.

- An invertible matrix is also called non-singular.
  A matrix is called non-invertible or singular if it is not invertible.
- A matrix $S \in \mathbb{R}^{n\times n}$ cannot have two different inverses.
  In fact, if $X, Y \in \mathbb{R}^{n\times n}$ are two matrices with $XS = I$ and $SY = I$, then
  $$X = XI = X(SY) = (XS)Y = IY = Y.$$
- The property $SX = I$ (right inverse) is important for the existence of a solution. In fact, if there is a matrix X with $SX = I$, then $x = Xf$ satisfies $Sx = SXf = If = f$, i.e., $x = Xf$ is a solution to the linear system.
- The property $XS = I$ (left inverse) is important for the uniqueness of the solution. In fact, if there is a matrix X with $XS = I$ and if x and y satisfy $Sx = f$ and $Sy = f$, then $S(x-y) = Sx - Sy = f - f = 0$ and $x - y = X0 = 0$.
- It can be shown that if the square matrix S has a left inverse $XS = I$, then X is also a right inverse, $SX = I$, and vice versa.
- If S is invertible, then for every f the linear system $Sx = f$ has the unique solution $x = S^{-1}f$.
- We will see later that if for every f the linear system $Sx = f$ has a unique solution x, then S is invertible.

Computation of the Matrix Inverse

We want to find the inverse of $S \in \mathbb{R}^{n\times n}$, that is we want to find a matrix $X \in \mathbb{R}^{n\times n}$ such that $SX = I$.

- Let $X_{:,j}$ denote the jth column of X, i.e., $X = (X_{:,1}, \ldots, X_{:,n})$. Consider the matrix-matrix product SX. The jth column of SX is the matrix-vector product $SX_{:,j}$, i.e., $SX = (SX_{:,1}, \ldots, SX_{:,n})$. The jth column of the identity I is the jth unit vector $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)^T$.
  Hence $SX = (SX_{:,1}, \ldots, SX_{:,n}) = (e_1, \ldots, e_n) = I$ implies that we can compute the columns $X_{:,1}, \ldots, X_{:,n}$ of the inverse of S by solving n systems of linear equations
  $$\begin{aligned} SX_{:,1} &= e_1, \\ &\vdots \\ SX_{:,n} &= e_n. \end{aligned}$$
  Note that if for every f the linear system $Sx = f$ has a unique solution x, then there exists a unique $X = (X_{:,1}, \ldots, X_{:,n})$ with $SX = I$.

Example 3

Suppose that we want the inverse of

$$S = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}.$$

We can use Gaussian Elimination to solve the systems $SX_{:,1} = e_1, SX_{:,2} = e_2, SX_{:,3} = e_3$ for the three columns of $X = S^{-1}$

$$\left(\begin{array}{rrr|rrr} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{rrr|rrr} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{rrr|rrr} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{rrr|rrr} 2 & 4 & 0 & 5/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right) \to \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 27/4 & -11/4 & 3/4 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right).$$

$$S^{-1} = \frac{1}{4} \begin{pmatrix} 27 & -11 & 3 \\ -11 & 5 & -1 \\ 3 & -1 & 1 \end{pmatrix}.$$

Example 4

Suppose that we want the inverse of

$$S = \begin{pmatrix} 2 & 3 & -2 \\ 1 & -2 & 3 \\ 4 & -1 & 4 \end{pmatrix}.$$

We can use Gaussian Elimination to solve the systems $SX_{:,1} = e_1, SX_{:,2} = e_2, SX_{:,3} = e_3$ for the three columns of $X = S^{-1}$

$$\left(\begin{array}{rrr|rrr} 2 & 3 & -2 & 1 & 0 & 0 \\ 1 & -2 & 3 & 0 & 1 & 0 \\ 4 & -1 & 4 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{rrr|rrr} 2 & 3 & -2 & 1 & 0 & 0 \\ 0 & -7/2 & 4 & -1/2 & 1 & 0 \\ 0 & -7 & 8 & -2 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{rrr|rrr} 2 & 3 & -2 & 1 & 0 & 0 \\ 0 & -7/2 & 4 & -1/2 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right).$$

None of the linear systems $SX_{:,1} = e_1, SX_{:,2} = e_2, SX_{:,3} = e_3$ has a solution. Therefore, S is not invertible.