Narrowband FM

- FM wave is a nonlinear function of the modulating wave. This property makes the spectral analysis of the FM wave a much more difficult task than that of the corresponding AM wave.
- spectral analysis described
- above provides us with enough insight to propose a useful solution to the problem.

- Consider then a sinusoidal modulating wave defined by

$$m(t) = A_m \cos(2\pi f_m t)$$

The instantaneous frequency of the resulting FM wave is

$$\begin{aligned} f_i(t) &= f_c + k_f A_m \cos(2\pi f_m t) \\ &= f_c + \Delta f \cos(2\pi f_m t) \end{aligned}$$

where

$$\Delta f = k_f A_m$$

Δf is called the *frequency deviation*,

- the angle of the FM wave is obtained as

$$\theta_i(t) = 2\pi f_c t + \frac{\Delta f}{f_m}\sin(2\pi f_m t)$$

The ratio of the frequency deviation Δf to the modulation frequency f_m is commonly called the *modulation index* of the FM wave. We denote this new parameter by β, so we write

$$\beta = \frac{\Delta f}{f_m} \tag{4.13}$$

and

$$\theta_i(t) = 2\pi f_c t + \beta \sin(2\pi f_m t) \tag{4.14}$$

β is measured in radians.

$$s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)]$$

For the FM wave of to be narrow-band the modulation index must be small compared to one radian

$$s(t) = A_c \cos(2\pi f_c t) \cos[\beta \sin(2\pi f_m t)] - A_c \sin(2\pi f_c t) \sin[\beta \sin(2\pi f_m t)]$$

Remember that : $\cos(A + B) = \cos A \cos B - \sin A \sin B$

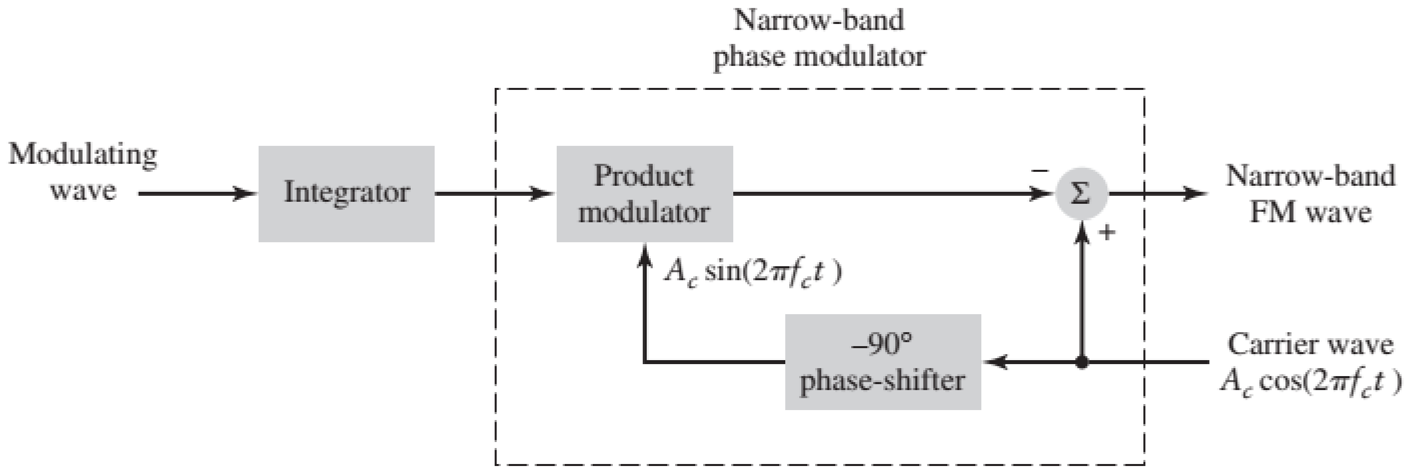
Then under the condition that the modulation index β is small compared to one radian, we may use the following two approximations for all times t:

$$\cos[\beta \sin(2\pi f_m t)] \approx 1$$

and

$$\sin[\beta \sin(2\pi f_m t)] \approx \beta \sin(2\pi f_m t)$$

the *approximate form* of a narrow-band FM $s(t) \approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_c t) \sin(2\pi f_m t)$


Narrow-band
phase modulator
Modulating
wave
Integrator
Product
modulator
Σ
Narrow-band
FM wave
$A_c \sin(2\pi f_c t)$
–90°
phase-shifter
Carrier wave
$A_c \cos(2\pi f_c t)$

► **Drill Problem 4.3** The *Cartesian representation* of band-pass signals discussed in Section 3.8 is well-suited for linear modulation schemes exemplified by the amplitude modulation family. On the other hand, the *polar representation*

$$s(t) = a(t) \cos[2\pi f_c t + \phi(t)]$$

is well-suited for nonlinear modulation schemes exemplified by the angle modulation family. The $a(t)$ in this new representation is the envelope of $s(t)$ and $\phi(t)$ is its phase.

Starting with the representation [see Eq. (3.39)]

$$s(t) = s_I(t) \cos(2\pi f_c t) - s_Q(t) \sin(2\pi f_c t)$$

where $s_I(t)$ is the in-phase component and $s_Q(t)$ is the quadrature component, we may write

$$a(t) = [s_I^2(t) + s_Q^2(t)]^{\frac{1}{2}}$$

and

$$\phi(t) = \tan^{-1}\left[\frac{s_Q(t)}{s_I(t)}\right]$$

Show that the polar representation of $s(t)$ in terms of $a(t)$ and $\phi(t)$ is exactly equivalent to its Cartesian representation in terms of $s_I(t)$ and $s_Q(t)$. ◄

Solution

We are given

$$a(t) = [s_I^2(t) + s_Q^2(t)]^{1/2}$$

and

$$\phi(t) = \tan^{-1}\left[\frac{s_Q(t)}{s_I(t)}\right]$$

Hence, expanding the polar representation of $s(t)$, we write

$$\begin{aligned} s(t) &= a(t)\cos[\theta t] \\ &= a(t)\cos[2\pi f_c t + \phi(t)] \end{aligned}$$

$$= a(t)\cos(\phi(t))\cos(2\pi f_c t) - a(t)\sin(\phi(t))\sin(2\pi f_c t)$$

Since $\tan[\phi(t)] = \left[\frac{s_Q(t)}{s_I(t)}\right]$, it follows that

$$\cos\phi(t) = \frac{s_I(t)}{[s_I^2(t) + s_Q^2(t)]^{1/2}} = \frac{s_I(t)}{a(t)}$$

and

$$\sin\phi(t) = \frac{s_Q(t)}{[s_I^2(t) + s_Q^2(t)]^{1/2}} = \frac{s_Q(t)}{a(t)}$$

Hence,

$$a(t)\cos\phi(t) = s_I(t)$$

and

$$a(t)\sin\phi(t) = s_Q(t)$$

Substituting Eqs. (2) and (3) into (1), we get

$$s(t) = s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t)$$

which is the Cartesian representation of $s(t)$.

Consider the narrow-band FM wave approximately defined by Eq. (4.17). Building on Problem 4.3, do the following:

(a) Determine the envelope of this modulated wave. What is the ratio of the maximum to the minimum value of this envelope?

(b) Determine the average power of the narrow-band FM wave, expressed as a percentage of the average power of the unmodulated carrier wave.

(c) By expanding the angular argument $\theta(t) = 2\pi f_c t + \phi(t)$ of the narrow-band FM wave $s(t)$ in the form of a power series and restricting the modulation index β to a maximum value of 0.3 radian, show that

$$\theta(t) \approx 2\pi f_c t + \beta \sin(2\pi f_m t) - \frac{\beta^3}{3}\sin^3(2\pi f_m t)$$

What is the value of the harmonic distortion for $\beta = 0.3$ radian?

Hint: For small x, the following power series approximation

$$\tan^{-1}(x) \approx x - \frac{1}{3}x^3$$

holds. In this approximation, terms involving x^5 and higher order ones are ignored, which is justified when x is small compared to unity.

Solution

(a) From Eq. (4.17), the narrow-band FM wave is approximately defined by

$$s(t) \approx A_c \cos((2\pi f_c t) - \beta A_c \sin(2\pi f_c t) \sin(2\pi f_m t))$$

The envelope of $s(t)$ is therefore

$$\begin{aligned} a(t) &= A_c(1 + \beta^2 \sin^2(2\pi f_m t))^{1/2} \\ &\approx A_c\left(1 + \frac{1}{2}\beta^2 \sin^2(2\pi f_m t)\right)^{1/2} \qquad \text{for small } \beta \end{aligned}$$

The maximum value of $a(t)$ occurs when $\sin^2(2\pi f_m t) = 1$, yielding

$$A_{\max} \approx A_c\left(1 + \frac{1}{2}\beta^2\right)$$

The minimum value of $a(t)$ occurs when $\sin^2(2\pi f_m t) = 0$, yielding

$$A_{\min} = A_c$$

The ratio of the maximum to the minimum value is therefore

$$\frac{A_{\max}}{A_{\min}} \approx \left(1 + \frac{1}{2}\beta^2\right)$$

(b) Expanding Eq. (1) into its individual frequency components, we may write

$$s(t) \approx A_c \cos(2\pi f_c t) + \frac{1}{2}\beta A_c \cos(2\pi(f_c + f_m)t) - \frac{1}{2}\beta A_c \cos(2\pi(f_c - f_m)t)$$

The average power of $s(t)$ is therefore

$$\begin{aligned} P_{\text{av}} &= \frac{1}{2}A_c^2 + \left(\frac{1}{2}\beta A_c\right)^2 + \left(\frac{1}{2}\beta A_c\right)^2 \\ &= \frac{1}{2}A_c^2(1 + \beta^2) \end{aligned}$$

The average power of the unmodulated carrier is

$$P_c = \frac{1}{2}A_c^2$$

Hence,

$$\frac{P_{\text{av}}}{P_c} = 1 + \beta^2$$

(c) The angle $\theta(t)$ is defined by

$$\theta(t) = 2\pi f_c t + \phi(t)$$

$$= 2\pi f_c t + \tan^{-1}(\beta \sin(2\pi f_m t))$$

Setting $\beta = \sin(2\pi f_m t)$

and using the approximation (based on the Hint), we may approximate $\theta(t)$ as

$$\theta(t) \approx 2\pi f_c t + \beta \sin(2\pi f_m t) - \frac{1}{3}\beta^3 \sin(2\pi f_m t)$$

Ideally, we should have (see Eq. (4.15))

$$\theta(t) = 2\pi f_c t + \beta \sin(2\pi f_m t)$$

The harmonic distortion produced by using the narrow-band approximation is therefore

$$D(t) = \frac{\beta^3}{3} \sin^3(2\pi f_m t)$$

The maximum absolute value of $D(t)$ for $\beta = 0.3$ is therefore

$$D_{\max} = \frac{\beta^3}{3}$$

$$= \frac{0.3^3}{3} = 0.009 \approx 1\%$$

which is small enough for it to be ignored in practice.

The important point to note from Problem 4.4 is that by restricting the modulation index to $\beta \leq 0.3$ radian, the effects of residual amplitude modulation and harmonic distortion are limited to negligible levels. We are therefore emboldened to proceed further with the use of Eq. (4.17), provided $\beta \leq 0.3$ radian. In particular, we may expand the modulated wave further into three frequency components:

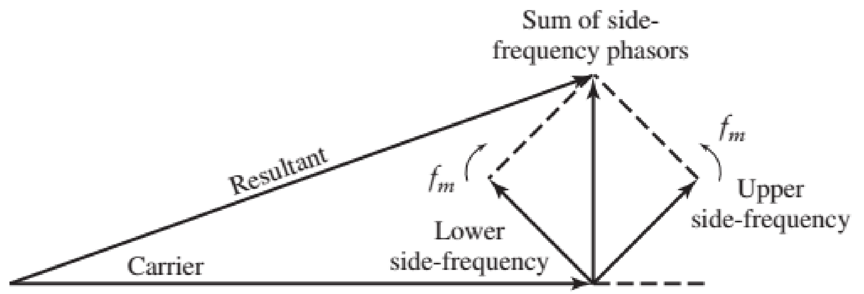
$$s(t) \approx A_c \cos(2\pi f_c t) + \frac{1}{2}\beta A_c\{\cos[2\pi(f_c + f_m)t] - \cos[2\pi(f_c - f_m)t]\} \qquad (4.18)$$

This expression is somewhat similar to the corresponding one defining an AM wave, which is reproduced from Example 3.1 of Chapter 3 as follows:

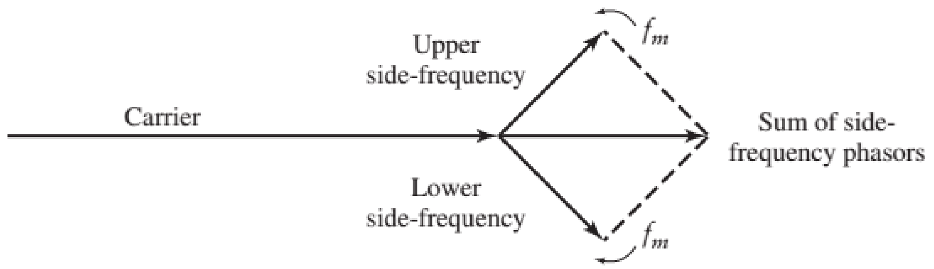
$$s_{\mathrm{AM}}(t) = A_c \cos(2\pi f_c t) + \frac{1}{2}\mu A_c\{\cos[2\pi(f_c + f_m)t] + \cos[2\pi(f_c - f_m)t]\} \qquad (4.19)$$

■ Phasor Interpretation

We may represent the narrow-band FM wave with a phasor diagram as shown in Fig. 4.5(*a*), where we have used the carrier phasor as reference. We see that the resultant of the two side-frequency phasors is always at right angles to the carrier phasor. The effect of this geometry is to produce a resultant phasor representing the narrow-band FM wave that is approximately of the same amplitude as the carrier phasor, but out of phase with respect to it.



(a)



(b)

Figure 4.5 Phasor comparison of narrow-band FM and AM waves for sinusoidal modulation. (*a*) Narrow-band FM wave. (*b*) AM wave.

Wide-Band Frequency Modulation

- how can we simplify the spectral analysis of the wide-band
- FM wave defined in Eq. (4.15)?
- The answer lies in using the complex baseband representation of a modulated (i.e., bandpass) signal, which was discussed in Section 3.8. Specifically, assume that the carrier frequency is large enough (compared to the bandwidth of the FM wave) to justify rewriting Eq. (4.15) in the form

$$\begin{aligned} s(t) &= \mathbf{Re}[A_c \exp(j2\pi f_c t + j\beta \sin(2\pi f_m t))] \\ &= \mathbf{Re}[\tilde{s}(t) \exp(j2\pi f_c t)] \end{aligned}$$

the complex envelope of the FM wave s(t)

$$\tilde{s}(t) = A_c \exp[j\beta \sin(2\pi f_m t)]$$

introduced in Eq. (4.21) is the *complex envelope* of the FM wave $s(t)$. The important point tc note from Eq. (4.21) is that unlike the original FM wave $s(t)$, the complex envelope $\tilde{s}(t)$ is a periodic function of time with a fundamental frequency equal to the modulation frequency f_m Specifically, replacing time t in Eq. (4.21) with $t + k/f_m$ for some integer k, we have

$$\begin{aligned}\tilde{s}(t) &= A_c \exp[j\beta \sin(2\pi f_m(t + k/f_m))] \\ &= A_c \exp[j\beta \sin(2\pi f_m t + 2k\pi)] \\ &= A_c \exp[j\beta \sin(2\pi f_m t)]\end{aligned}$$

which confirms f_m as the fundamental frequency of $\tilde{s}(t)$. We may therefore expand $\tilde{s}(t)$ in the form of a complex Fourier series as follows:

$$\tilde{s}(t) = \sum_{n=-\infty}^{\infty} c_n \exp(j2\pi n f_m t) \tag{4.22}$$

where the complex Fourier coefficient

$$\begin{aligned} c_n &= f_m \int_{-1/(2f_m)}^{1/(2f_m)} \tilde{s}(t) \exp(-j2\pi n f_m t)\, dt \\ &= f_m A_c \int_{-1/(2f_m)}^{1/(2f_m)} \exp[j\beta \sin(2\pi f_m t) - j2\pi n f_m t]\, dt \end{aligned} \tag{4.23}$$

Define the new variable:

$$x = 2\pi f_m t \tag{4.24}$$

Hence, we may redefine the complex Fourier coefficient c_n in Eq. (4.23) in the new form

$$c_n = \frac{A_c}{2\pi} \int_{-\pi}^{\pi} \exp[j(\beta \sin x - nx)]\, dx \tag{4.25}$$

The integral on the right-hand side of Eq. (4.25), except for the carrier amplitude A_c, is referred to as the nth *order Bessel function* of the first kind and argument β. This function is commonly denoted by the symbol $J_n(\beta)$, so we may write

$$J_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[j(\beta \sin x - nx)]\, dx \tag{4.26}$$

Accordingly, we may rewrite Eq. (4.25) in the compact form

$$c_n = A_c J_n(\beta) \tag{4.27}$$

Substituting Eq. (4.27) into (4.22), we get, in terms of the Bessel function $J_n(\beta)$, the following expansion for the complex envelope of the FM wave:

$$\tilde{s}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \exp(j2\pi n f_m t) \tag{4.28}$$

Next, substituting Eq. (4.28) into (4.20), we get

$$s(t) = \mathbf{Re}\left[A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \exp[j2\pi(f_c + nf_m)t]\right] \tag{4.29}$$

The carrier amplitude A_c is a constant and may therefore be taken outside the real-time operator **Re**[.]. Moreover, we may interchange the order of summation and real-part operation, as they are both linear operators. Accordingly, we may rewrite Eq. (4.29) in the simplified form

$$s(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos[2\pi(f_c + nf_m)t] \tag{4.30}$$

Equation (4.30) is the desired form for the Fourier series expansion of the single-tone FM signal $s(t)$ for an arbitrary value of modulation index β.

The discrete spectrum of $s(t)$ is obtained by taking the Fourier transforms of both sides of Eq. (4.30), which yields

$$S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)] \tag{4.31}$$

where $s(t) \rightleftharpoons S(f)$ and $\cos(2\pi f_i t) \rightleftharpoons \frac{1}{2}[\delta(f - f_i) + \delta(f + f_i)]$ for an arbitrary f_i. Equation (4.31) shows that the spectrum of $s(t)$ consists of an infinite number of delta functions spaced at $f = f_c \pm nf_m$ for $n = 0, +1, +2, \ldots$.

PROPERTIES OF SINGLE-TONE FM FOR ARBITRARY MODULATION INDEX β

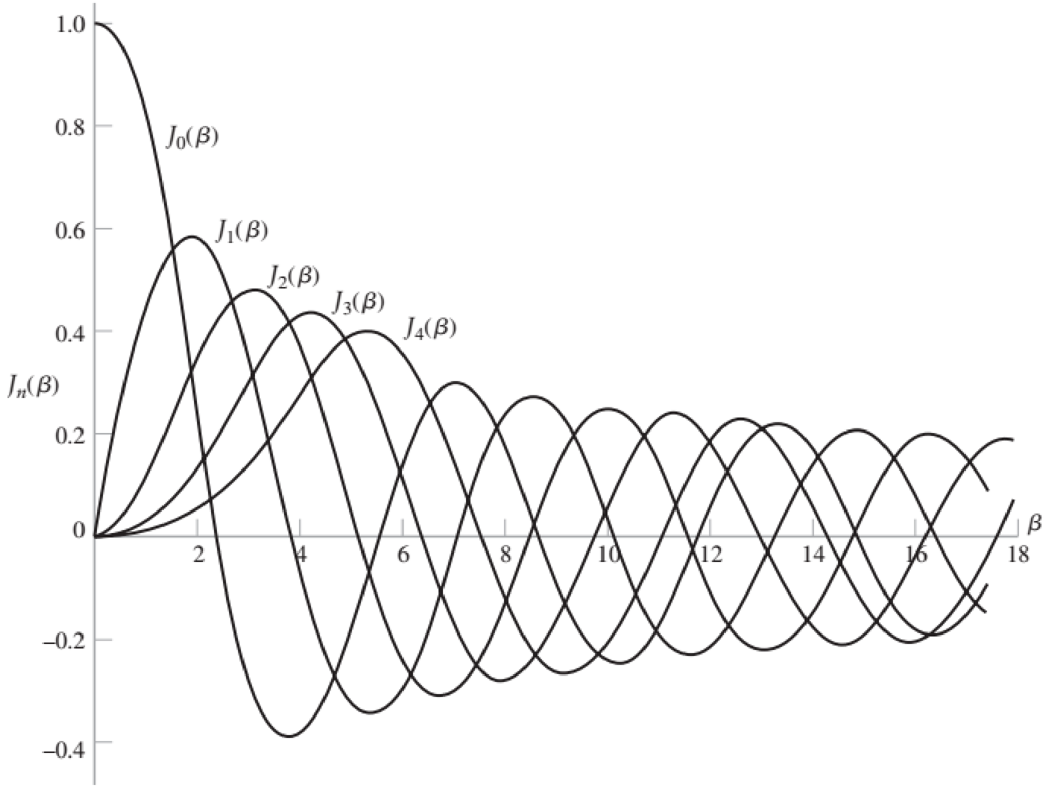


FIGURE 4.6 Plots of the Bessel function of the first kind, $J_n(\beta)$, for varying order n.

1. For different integer (positive and negative) values of n, we have

$$J_n(\beta) = J_{-n}(\beta), \qquad \text{for } n \text{ even}$$

   and

$$J_n(\beta) = -J_{-n}(\beta), \qquad \text{for } n \text{ odd}$$

2. For small values of the modulation index β, we have

$$\left.\begin{aligned} J_0(\beta) &\approx 1, \\ J_1(\beta) &\approx \frac{\beta}{2}, \\ J_n(\beta) &\approx 0, \qquad n > 2 \end{aligned}\right\}$$

3. The equality

$$\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$$

   holds exactly for arbitrary β.

1. The spectrum of an FM wave contains a carrier component and an infinite set of side frequencies located symmetrically on either side of the carrier at frequency separations of $f_m, 2f_m, 3f_m, \ldots$. In this respect, the result is unlike the picture that prevails in AM, since in the latter case a sinusoidal modulating wave gives rise to only one pair of side frequencies.

2. For the special case of β small compared with unity, only the Bessel coefficients $J_0(\beta)$ and $J_1(\beta)$ have significant values, so that the FM wave is effectively composed of a carrier and a single pair of side-frequencies at $f_c \pm f_m$. This situation corresponds to the special case of narrow-band FM that was considered in Section 4.4.

3. The amplitude of the carrier component varies with β according to $J_0(\beta)$. That is, unlike an AM wave, the amplitude of the carrier component of an FM wave is dependent on the modulation index β. The physical explanation for this property is that the envelope of an FM wave is constant, so that the average power of such a signal developed across a 1-ohm resistor is also constant, as in Eq. (4.8), which is reproduced here for convenience of presentation:

$$P_{\text{av}} = \frac{1}{2} A_c^2$$

When the carrier is modulated to generate the FM wave, the power in the side-frequencies may appear only at the expense of the power originally in the carrier, thereby making the amplitude of the carrier component dependent on β. Note that the average power of an FM wave may also be determined from Eq. (4.30), as shown by

$$P = \frac{1}{2} A_c^2 \sum_{n=-\infty}^{\infty} J_n^2(\beta) \tag{4.36}$$

Substituting Eq. (4.35) into (4.36), the expression for the average power P_{av} reduces to Eq. (4.8), and so it should.

Bessel function

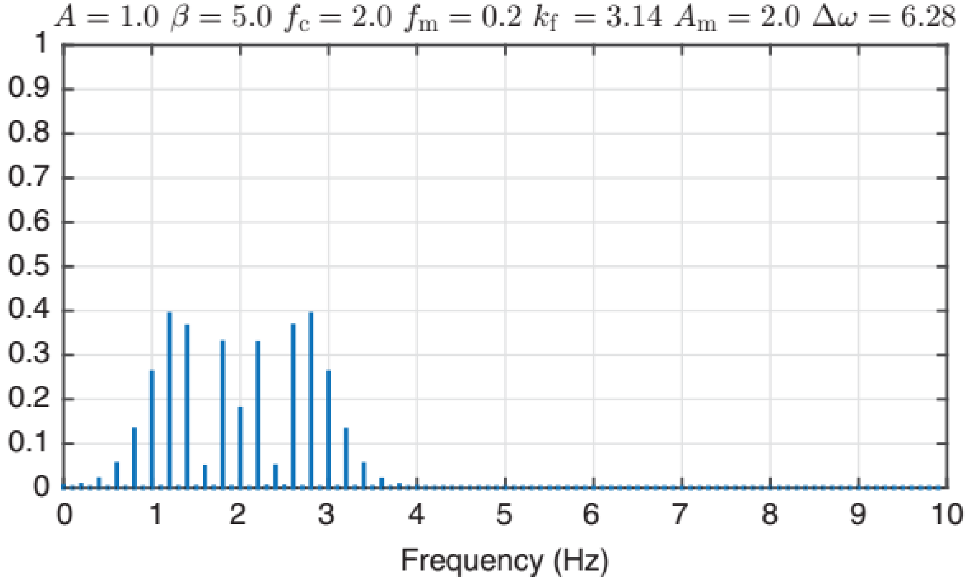
FM signal is

$$x_{\mathrm{FM}}(t) = A\cos(\omega_{\mathrm{c}}t + \beta\sin\omega_{\mathrm{m}}t)$$

From properties of the FM signal we can present it as

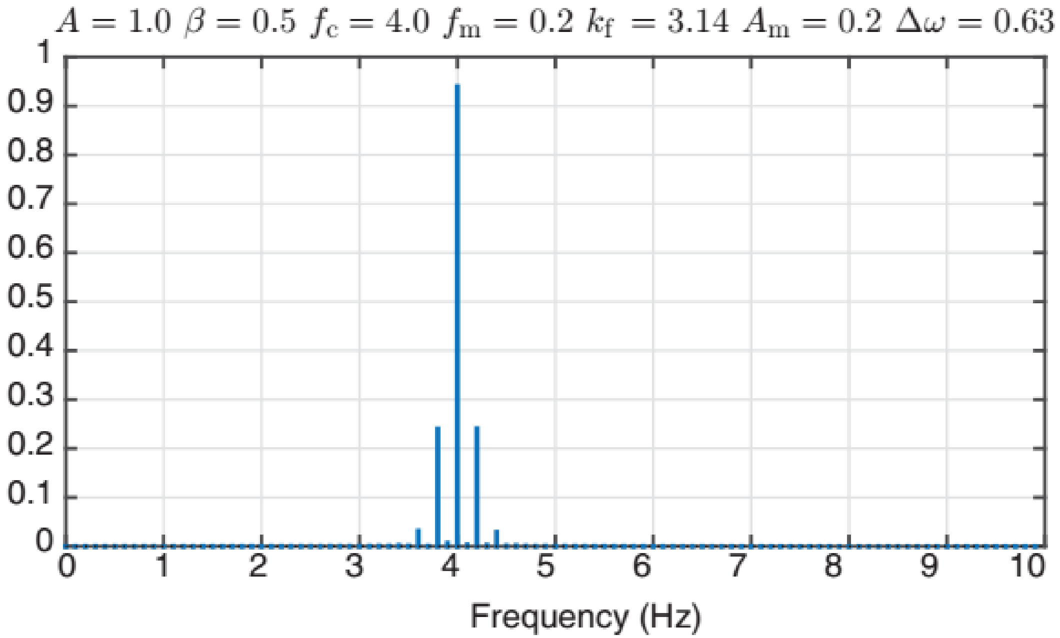
$$x_{\mathrm{FM}}(t) = A\sum_{n=-\infty}^{n=+\infty} J_n(\beta)\cos(\omega_{\mathrm{c}} + n\,\omega_{\mathrm{m}})t$$

$$
\begin{aligned}
x_{\mathrm{FM}}(t) = {} & J_0(\beta)A\cos\omega_{\mathrm{c}}t \\
& + J_1(\beta)A\cos(\omega_{\mathrm{c}} + \omega_{\mathrm{m}})t + J_{-1}(\beta)A\cos(\omega_{\mathrm{c}} - \omega_{\mathrm{m}})t \\
& + J_2(\beta)A\cos(\omega_{\mathrm{c}} + 2\omega_{\mathrm{m}})t + J_{-2}(\beta)A\cos(\omega_{\mathrm{c}} - 2\omega_{\mathrm{m}})t \\
& + J_3(\beta)A\cos(\omega_{\mathrm{c}} + 3\omega_{\mathrm{m}})t + J_{-3}(\beta)A\cos(\omega_{\mathrm{c}} - 3\omega_{\mathrm{m}})t \\
& + \cdots
\end{aligned}
$$



FM frequency spectrum

FM frequency spectrum



Bessel functions $J_n(\beta)$ Modulation index β											
β	J_0	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	J_9	J_{10}
0	1.00	—	—	—	—	—	—	—	—	—	—
0.25	0.98	0.12	—	—	—	—	—	—	—	—	—
0.5	0.94	0.24	0.03	—	—	—	—	—	—	—	—
1	0.77	0.44	0.12	0.02	—	—	—	—	—	—	—
1.5	0.51	0.56	0.23	0.06	0.01	—	—	—	—	—	—
2	0.22	0.58	0.35	0.13	0.03	—	—	—	—	—	—
2.4	0.00	0.52	0.43	0.20	0.06	0.02	—	—	—	—	—
2.5	−0.05	0.50	0.45	0.22	0.07	0.02	—	—	—	—	—
3	−0.26	0.34	0.49	0.31	0.13	0.04	0.01	—	—	—	—
4	−0.40	−0.07	0.36	0.43	0.28	0.13	0.05	0.01	—	—	—
5	−0.18	−0.33	0.05	0.36	0.39	0.26	0.13	0.05	0.02	—	—
6	0.15	−0.28	−0.24	0.11	0.36	0.36	0.25	0.13	0.06	0.02	—
7	0.30	0.0	−0.30	−0.17	0.16	0.35	0.34	0.23	0.13	0.06	0.02
8	0.17	0.23	−0.11	−0.29	−0.10	0.19	0.34	0.32	0.22	0.13	0.06
9	−0.09	0.24	0.15	−0.18	−0.26	−0.06	0.20	0.33	0.31	0.21	0.13
10	−0.25	0.04	0.26	0.06	−0.22	−0.23	−0.01	0.22	0.32	0.29	0.21
12	0.05	−0.22	−0.08	0.19	0.18	−0.07	−0.24	−0.17	0.05	0.23	0.30
15	−0.01	0.20	0.04	−0.19	−0.12	0.13	0.21	0.03	−0.17	−0.22	−0.09

Parameter name	Symbol	Value
Carrier frequency	f_c	600 kHz
Carrier amplitude	A_c	200 mVpp
Modulating frequency	f_m	20 kHz
Frequency deviation	Δf	20 kHz

$$\frac{\Delta f}{f_m} = 1$$

The sidebands are scaled according to the Bessel coefficient values. For $\beta = 1$,

$$J_0(\beta) = 0.77$$
$$J_1(\beta) = 0.44$$
$$J_2(\beta) = 0.12$$
$$J_3(\beta) = 0.02$$

EXAMPLE 4.2 FM Spectrum for Varying Amplitude and Frequency of Sinusoidal Modulating Wave

In this example, we wish to investigate the ways in which variations in the amplitude and frequency of a sinusoidal modulating wave affect the spectrum of the FM wave. Consider first the case when the frequency of the modulating wave is fixed, but its amplitude is varied, producing a corresponding variation in the frequency deviation Δf. Thus, keeping the modulation frequency f_m fixed, we find that the amplitude spectrum of the resulting FM wave is as shown plotted in Fig. 4.7 for $\beta = 1$, 2, and 5. In this diagram, we have normalized the spectrum with respect to the unmodulated carrier amplitude.

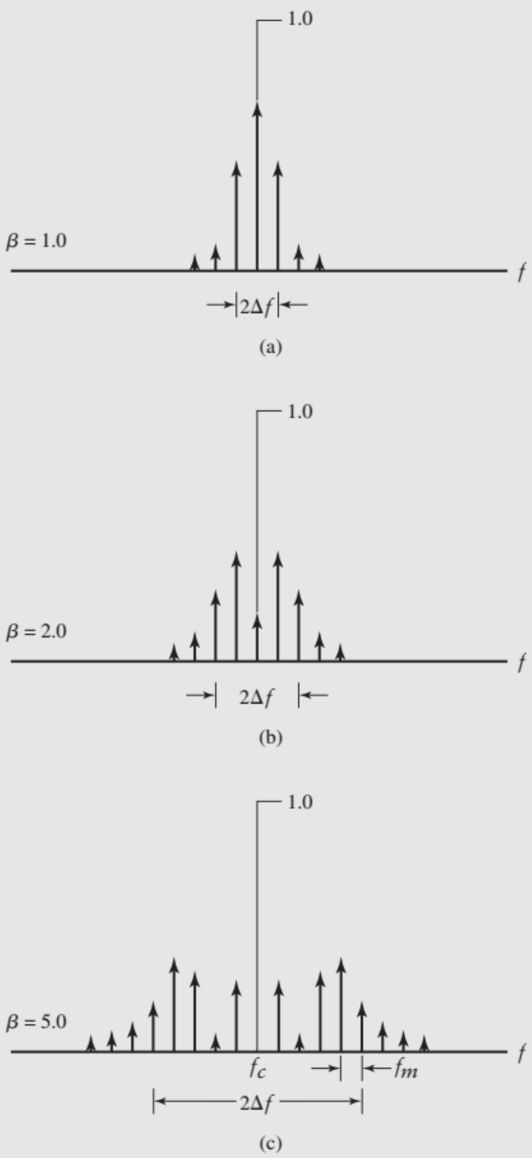
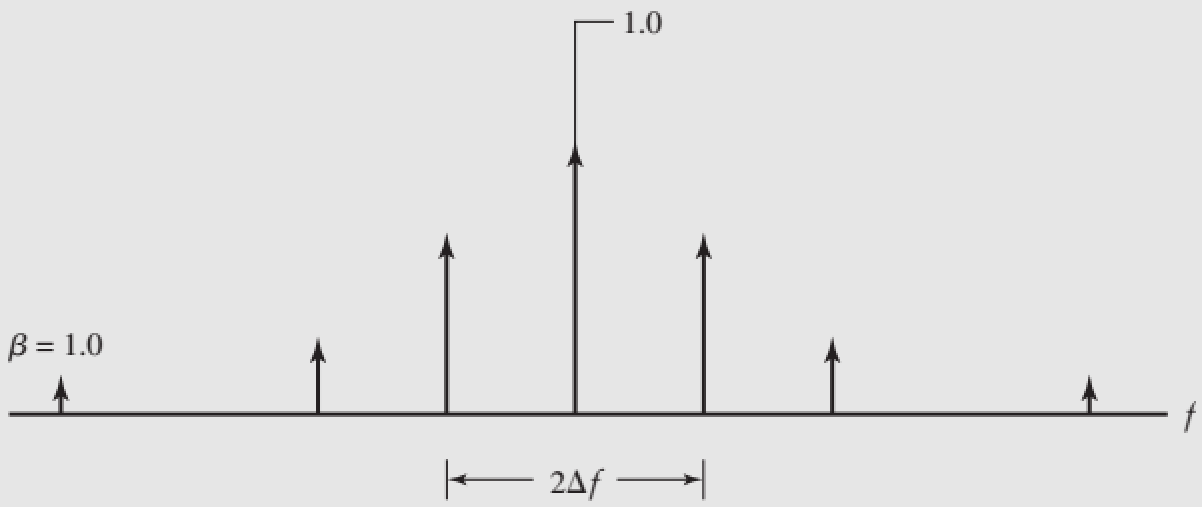


FIGURE 4.7 Discrete amplitude spectra of an FM wave, normalized with respect to the unmodulated carrier amplitude, for the case of sinusoidal modulation of fixed frequency and varying amplitude. Only the spectra for positive frequencies are shown.

Consider next the case when the amplitude of the modulating wave is fixed; that is, the frequency deviation Δf is maintained constant, and the modulation frequency f_m is varied. In this second case, we find that the amplitude spectrum of the resulting FM wave is as shown plotted in Fig. 4.8 for $\beta = 1, 2$, and 5. We now see that when Δf is fixed and β is increased, we have an increasing number of spectral lines crowding into the fixed frequency interval $f_c - \Delta f < |f| < f_c + \Delta f$. That is, when β approaches infinity, the bandwidth of the FM wave approaches the limiting value of $2\Delta f$, which is an important point to keep in mind.



(a)

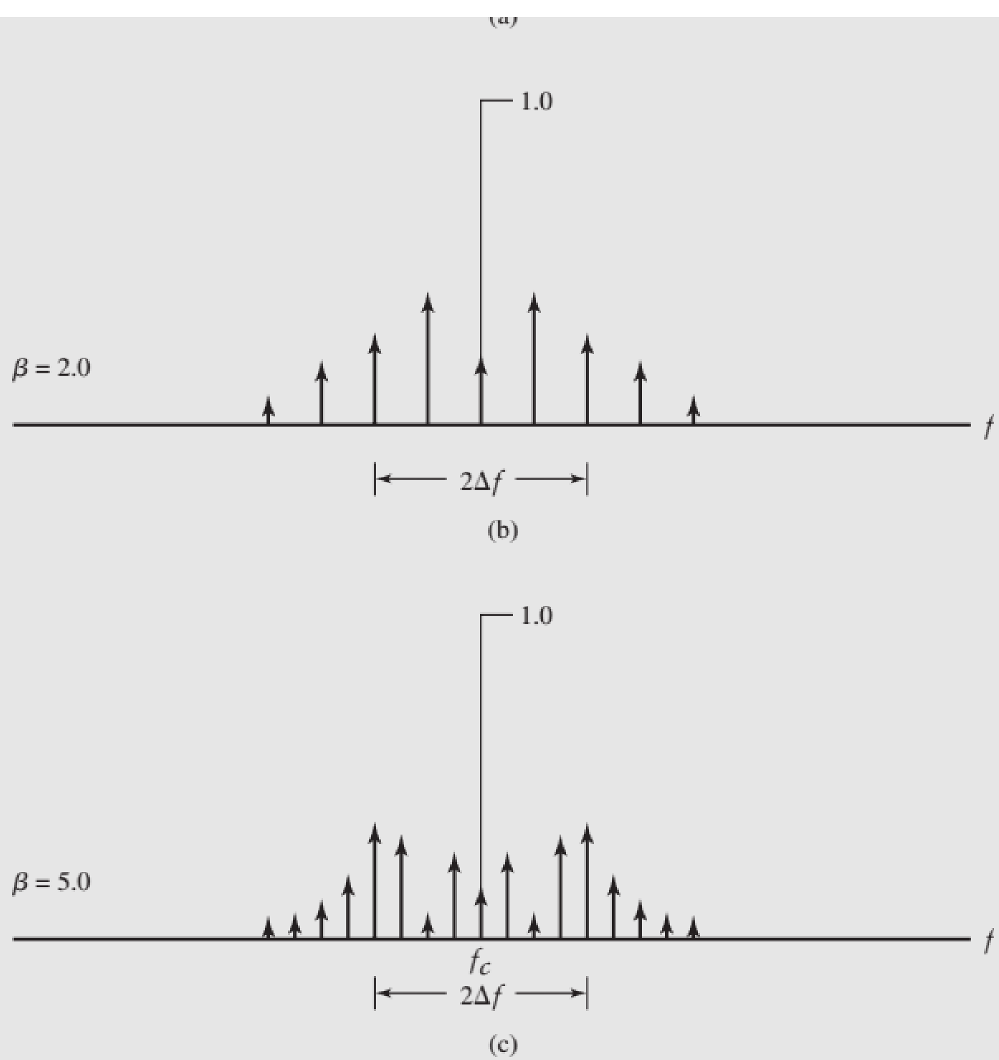


FIGURE 4.8 Discrete amplitude spectra of an FM wave, normalized with respect to the unmodulated carrier amplitude, for the case of sinusoidal modulation of varying frequency and fixed amplitude. Only the spectra for positive frequencies are shown.

Transmission Bandwidth of FM Waves

Carson's Rule

In theory, an FM wave contains an infinite number of side-frequencies so that the bandwidth required to transmit such a modulated wave is similarly infinite in extent. In practice, however, we find that the FM wave is *effectively limited to a finite number of significant side-frequencies compatible with a specified amount of distortion*. We may therefore build on this idea to specify an effective bandwidth required for the transmission of an FM wave. Consider first the case of an FM wave generated by a single-tone modulating wave of frequency f_m. In such an FM wave, the side-frequencies that are separated from the carrier frequency f_c by an amount greater than the frequency deviation Δf decrease rapidly toward zero, so that the bandwidth always exceeds the total frequency excursion, but nevertheless is limited. Specifically, we may identify two limiting cases:

1. For large values of the modulation index β, the bandwidth approaches, and is only slightly greater than the total frequency excursion $2\Delta f$, as illustrated in Fig. 4.8(c).
2. For small values of the modulation index β, the spectrum of the FM wave is effectively limited to the carrier frequency f_c and one pair of side-frequencies at $f_c \pm f_m$, so that the bandwidth approaches $2f_m$, as illustrated in Section 4.4.

In light of these two limiting scenarios, we may define an approximate rule for the transmission bandwidth of an FM wave generated by a single-tone modulating wave of frequency f_m as

$$B_T \approx 2\Delta f + 2f_m = 2\Delta f\left(1 + \frac{1}{\beta}\right) \tag{4.37}$$

This simple empirical relation is known as *Carson's rule*.

■ Universal Curve for FM Transmission Bandwidth

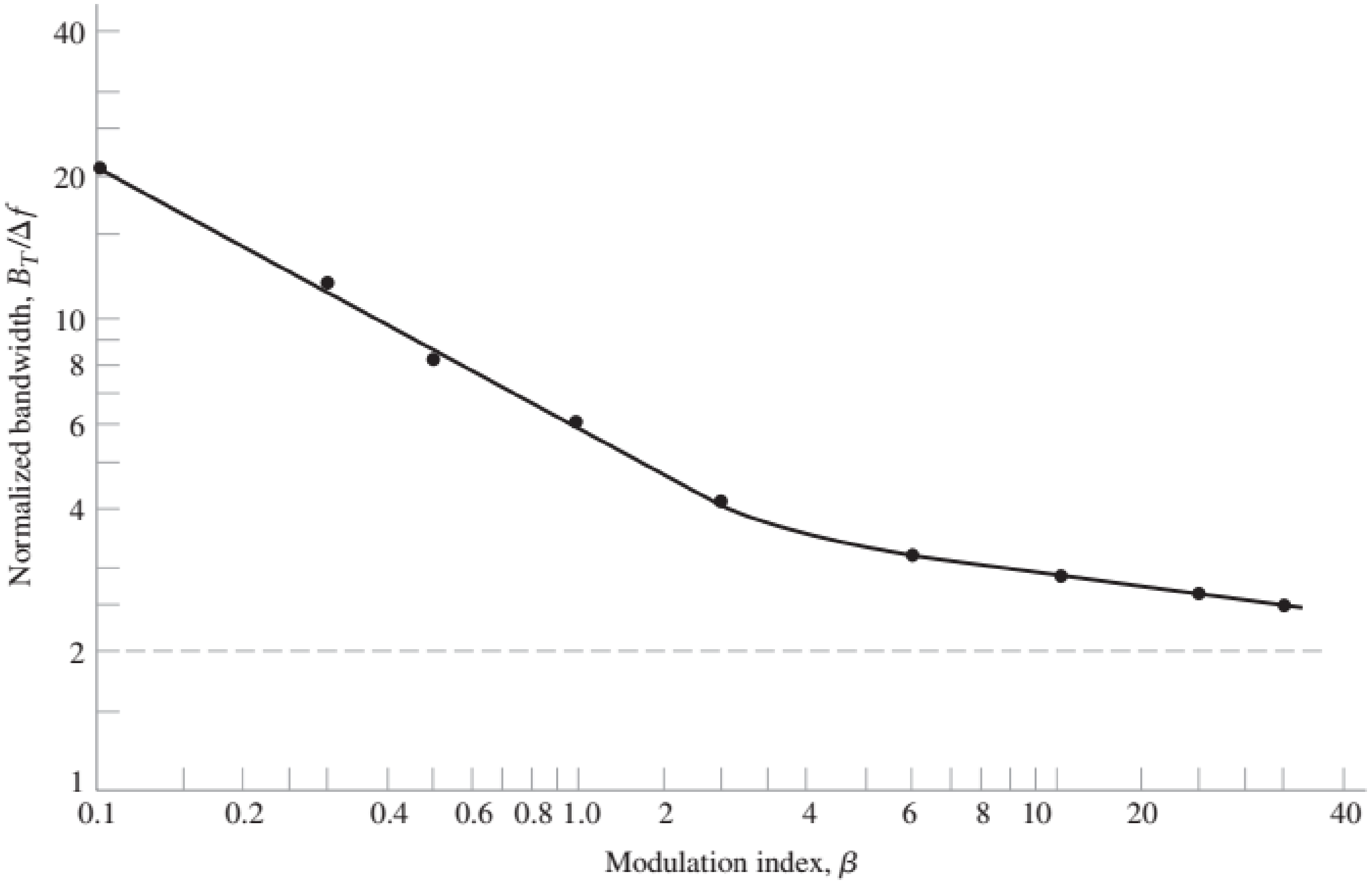


Figure 4.9 Universal curve for evaluating the one percent bandwidth of an FM wave.

ARBITRARY MODULATING WAVE

TABLE 4.2 ***Number of Significant Side-Frequencies of a Wide-Band FM Signal for Varying Modulation Index***

Modulation Index β	*Number of Significant Side-Frequencies* $2n_{max}$
0.1	2
0.3	4
0.5	4
1.0	6
2.0	8
5.0	16
10.0	28
20.0	50
30.0	70

the modulation wave $m(t)$, to the highest modulation frequency W. These conditions represent the extreme cases possible. We may thus formally write

$$D = \frac{\Delta f}{W} \tag{4.38}$$

The deviation ratio D plays the same role for nonsinusoidal modulation that the modulation index β plays for the case of sinusoidal modulation. Hence, replacing β by D and replacing f_m with W, we may generalize Eq. (4.37) as follows:

$$B_T = 2(\Delta f + W) \tag{4.39}$$

EXAMPLE 4.3 Commercial FM Broadcasting

In North America, the maximum value of frequency deviation Δf is fixed at 75 kHz for commercial FM broadcasting by radio. If we take the modulation frequency $W = 15$ kHz, which is typically the "maximum" audio frequency of interest in FM transmission, we find that the corresponding value of the deviation ratio is [using Eq. (4.38)]

$$D = \frac{75}{15} = 5$$

Using the values $\Delta f = 75$ kHz and $D = 5$ in the generalized Carson rule of Eq. (4.39), we find that the approximate value of the transmission bandwidth of the FM signal is obtained as

$$B_T = 2(75 + 15) = 180 \text{ kHz}$$

On the other hand, use of the universal curve of Fig. 4.9 gives the transmission bandwidth of the FM signal to be

$$B_T = 3.2\,\Delta f = 3.2 \times 75 = 240 \text{ kHz}$$

In this example, Carson's rule underestimates the transmission bandwidth by 25 percent compared with the result of using the universal curve of Fig. 4.9.

4.12 A carrier wave is frequency-modulated using a sinusoidal signal of frequency f_m and amplitude A_m.

(a) Determine the values of the modulation index β for which the carrier component of the FM wave is reduced to zero. For this calculation you may use the values of $J_0(\beta)$ given in Appendix 3.

(b) In a certain experiment conducted with $f_m = 1$ kHz and increasing A_m (starting from zero volt), it is found that the carrier component of the FM wave is reduced to zero for the first time when $A_m = 2$ volts. What is the frequency sensitivity of the modulator? What is the value of A_m for which the carrier component is reduced to zero for the second time?

(a) From Table A3.1 in Appendix 3, we find (by interpolation) that $J_0(\beta)$ is zero for the following values of modulation index:

$\beta = 2.44$,
$\beta = 5.52$,
$\beta = 8.65$,
$\beta = 11.8$,
and so on.

(b) The modulation index is defined by

$$\beta = \frac{\Delta f}{f_m} = \frac{k_f A_m}{f_m}$$

Therefore, the frequency sensitivity factor is

$$k_f = \frac{\beta f_m}{A_m} \tag{1}$$

We are given $f_m = 1$ kHz and $A_m = 2$ volts. Hence, with $J_0(\beta) = 0$ for the first time when $\beta = 2.44$, the use of Eq. (1) yields

$$k_f = \frac{2.44 \times 10^3}{2}$$

$$= 1.22 \times 10^3 \text{ hertz/volt}$$

Next, we note that $J_0(\beta) = 0$ for the second time when $\beta = 5.52$. Hence, the corresponding value of A_m for which the carrier component is reduced to zero is

$$A_m = \frac{\beta f_m}{k_f}$$

$$= \frac{5.52 \times 10^3}{1.22 \times 10^3}$$

$$= 4.52 \text{ volts}$$

4.13 A carrier wave of frequency 100 MHz is frequency-modulated by a sinusoidal wave of amplitude 20 V and frequency 100 kHz. The frequency sensitivity of the modulator is 25 kHz/V.

(a) Determine the approximate bandwidth of the FM wave, using Carson's rule.

(b) Determine the bandwidth obtained by transmitting only those side-frequencies with amplitudes that exceed one percent of the unmodulated carrier amplitude. Use the universal curve of Fig. 4.9 for this calculation.

(c) Repeat your calculations, assuming that the amplitude of the modulating wave is doubled.

(d) Repeat your calculations, assuming that the modulation frequency is doubled.

(a) The frequency deviation is

$$\Delta f = k_f A_m = 25 \times 10^3 \times 20 = 5 \times 10^5 \text{Hz}$$

The corresponding value of the modulation index is

$$\beta = \frac{\Delta f}{f_m} = \frac{5 \times 10^5}{10^5} = 5$$

Using Carson's rule, the transmission bandwidth of the FM wave is therefore

$$B_T = 2 f_m (1 + \beta) = 2 \times 100(1 + 5) = 1200 \text{kHz} = 1.2 \text{MHz}$$

(b) Using the universal curve of Fig. 4.9, we find that for $\beta = 5$:

$$\frac{B_T}{\Delta f} = 3$$

Therefore, the transmission bandwidth is

$$B_T = 3 \times 500 = 1500\text{kHz} = 1.5\text{MHz}$$

which is greater than the value calculated by Carson's rule.

(c) If the amplitude of the modulating wave is doubled, we find that

$$\Delta f = 1\text{MHz} \text{ and } \beta = 10$$

Thus, using Carson's rule we now obtain the transmission bandwidth

$$B_T = 2 \times 100(1 + 10) = 2200\text{kHz} = 2.2\text{MHz}$$

On the other hand, using the universal curve of Fig. 4.9, we get

$$\frac{B_T}{\Delta f} = 2.75$$

and $B_T = 2.75$ MHz.

(d) If f_m is doubled, $\beta = 2.5$. Then, using Carson's rule, $B_T = 1.4$ MHz. Using the universal curve, $(B_T/\Delta f) = 4$, and

$$B_T = 4\Delta f = 2\text{MHz}$$