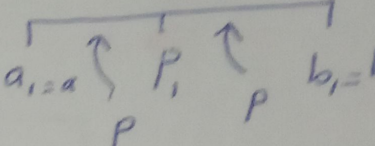
Chp [2] Lecture [3] Solution of equations in one variable

(1)

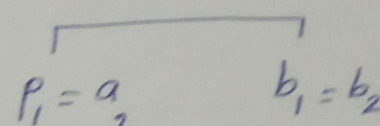
[2.1] Bisection method: The procedure of Bisection method on $[a, b]$:

① Let $a_1 = a$, $b = b_1$ and $P_1 = \frac{a_1 + b_1}{2}$



② If $f(P_1) = 0 \Rightarrow P_1 = P$

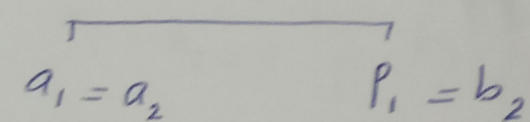
③ If $f(P_1) \neq 0$ with $f(a_1) f(P_1) > 0 \Rightarrow P \in [P_1, b_1]$ and set $a_2 = P_1$, $b_2 = b_1$



④ If $f(P_1) \neq 0$ with $f(a_1) f(P_1) < 0 \Rightarrow P \in [a_1, P_1]$

set $a_2 = a_1$, $b_2 = P_1$

(repeat)



⑤ we apply the same process on $[a_2, b_2]$.

Ex: Let $f(x) = \cos(x) - x$ on $[0, \pi]$

① show that f has a zero solution on $[0, \pi]$.

② Find P_3 by using bisection method to solve $f(p) = 0$ (use 4-digit chopping).

sl: ① $f \in C[0, \pi]$ because $\cos x$ is continuous and x is continuous so continuous − continuous ≡ continuous

② $f(0) = \cos(0) - 0 = 1$, $f(\pi) = -1 - \pi \Rightarrow f(0) f(\pi) < 0$

$\Rightarrow$ By Bolzano Thm $\exists\, d \in [0, \pi]$ s.t $f(d) = 0$. #

(2)

$a_1 = a = 0, \quad b_1 = b = \pi \Rightarrow p_1 = \frac{a_1 + b_1}{2} = \frac{0 + \pi}{2} = \frac{\pi}{2}$

(2)

$+ \quad - \quad [0, \pi]$

$f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} - \frac{\pi}{2} = \frac{-\pi}{2}$

$f(0)\, f\left(\frac{\pi}{2}\right) = 1 \times -\frac{\pi}{2} < 0$

$+ \quad - \quad [0, \frac{\pi}{2}]$

$p_2 = \frac{a_2 + b_2}{2} = \frac{0 + \frac{\pi}{2}}{2} = \frac{\pi}{4}$

$a_2 = a_1 = 0 \quad p \quad b_2 = p_1 = \frac{\pi}{2}$

$f(p_2) = f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{4}\right) - \frac{\pi}{4} = -0.078 < 0$

$f(a_2)\, f(p_2) = -.078 < 0$

$a_3 = a_2 = 0 \quad p \quad b_3 = p_2 = \frac{\pi}{4}$

$\Rightarrow p_3 = \frac{a_3 + b_3}{2} = \frac{0 + \frac{\pi}{4}}{2} = \frac{\pi}{8}$

Rules for a bound of absolute error of Bisection method

① $|p - p_n| < b_{n+1} - a_{n+1}$

Pf: $[a_n \quad p_n \quad b_n]$, $p_n = \frac{a_n + b_n}{2}$

If $f(a_n)\, f(p_n) > 0 \Rightarrow p \in [p_n, b_n]$

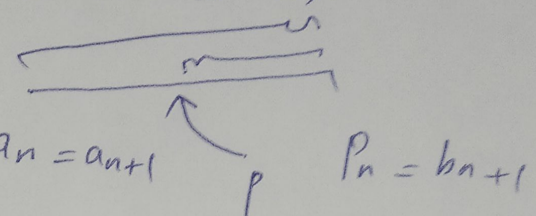
so $|p - p_n| \leq b_{n+1} - a_{n+1}$

$a_{n+1} = p_n \quad p \quad b_n = b_{n+1}$

(3)

If $f(a_n) f(p_n) < 0 \Rightarrow p \in [a_n, b_n] \Rightarrow$

$|p - p_n| \le b_{n+1} - a_{n+1}$.



(2) $|p - p_n| \le \dfrac{b_n - a_n}{2}$

Pf: From (1) $|p - p_n| \le b_{n+1} - a_{n+1}$

$$= \begin{cases} b_n - p_n & \text{if } f(a_n) f(p_n) > 0 \\ p_n - a_n & \text{if } f(a_n) f(p_n) < 0 \end{cases}$$

So $$|p - p_n| \le \begin{cases} \dfrac{2b_n}{2} - \dfrac{a_n + b_n}{2} & f(a_n) f(p_n) > 0 \\ \dfrac{a_n + b_n}{2} - \dfrac{2a_n}{2} & f(a_n) f(p_n) < 0 \end{cases}$$

$$\Rightarrow |p - p_n| \le \frac{b_n - a_n}{2}.$$

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(3) Thm: Let (1) $f \in C[a,b]$

(2) $f(a) f(b) < 0$

$\Rightarrow$ The bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with $|p - p_n| \le \dfrac{b-a}{2^n}$.

Pf. From ② $|P - P_n| \leq \frac{b_n - a_n}{2}$

(4)

Now, $b_1 - a_1 = b - a$

$b_2 - a_2 = \frac{b_1 - a_1}{2} = \frac{b - a}{2}$

$b_3 - a_3 = \frac{b_2 - a_2}{2} = \frac{b - a}{2^n}$

⋮

$b_n - a_n = \frac{b - a}{2^{n-1}}$

From ② $|P - P_n| \leq \frac{b-a}{2^{n-1}} / 2 = \frac{b-a}{2^n}$ #.

Ex: Let $f(x) = \cos x - x$ on $[0, \pi]$ find:

① approximate $|P - P_3|$ by two ways.

② Determine the number of iterations necessary to solve $f(x) = \cos(x) - x$ on $[0, \pi]$ with accuracy 10^{-3}.

Sl: ① From previous example $P_3 = \frac{\pi}{8}$ repeating bisection method

$b_4 = \frac{\pi}{4}$, $a_4 = \frac{\pi}{8} \Rightarrow |P - P_3| \leq b_4 - a_4$

$= \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8}$.

method 2: $|P - P_3| \le \frac{b-a}{2^n} = \frac{b-a}{2^3}$

$= \frac{\pi - 0}{8} = \frac{\pi}{8}$. #

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(2)

$|P - P_n| \le \frac{b-a}{2^n} \le 10^{-3}$

$\frac{\pi - 0}{2^n} \le 10^{-3} \Rightarrow 2^n \ge 1000\pi$

$\ln 2^n \ge \ln 1000\pi \Rightarrow n \ge \frac{\ln 1000\pi}{\ln 2}$

$n \ge 11.6$

$\Rightarrow \boxed{n = 12}$ #.