2.2 Properties of Determinants

In this section, we will study properties determinants have and we will see how these properties can help in computing the determinant of a matrix. We will also see how these properties can give us information about matrices.

2.2.1 Determinants and Elementary Row Operations

We study how performing an elementary row operation on a matrix affects its determinant. This, in turn, will give us a powerful tool to compute determinants. We give the main result as a theorem. Its proof will be given at the end of the section.

Theorem 151 *Let A and B be an $n \times n$ matrix.*

1. *If B is obtained by replacing one row of A by itself plus a multiple of another row, then $|B| = |A|$.*
2. *If B is obtained by multiplying a row of A by a nonzero constant k, then $|B| = k\,|A|$.*
3. *If B is obtained by interchanging two rows of A, then $|B| = -\,|A|$.*

We illustrate the theorem for 3×3 matrices. Assuming that the original matrix we have is $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, we see what happens to its determinant as we perform one of the elementary row operations.

Relationship	Operation
$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$	$(kR_1) \rightarrow (R_1)$
$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$	$(R_1) \longleftrightarrow (R_2)$
$\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$	$(R_1 + kR_2) \rightarrow (R_1)$

In particular, looking at the first row of this table, we see that we can "factor" a constant from any row.

This theorem is very important for computing determinants. recall from the previous section that the determinant of a triangular matrix is the product of the entries on its diagonal. A matrix in row-echelon form is a triangular matrix. So, a strategy to compute the determinant of a matrix is to transform the matrix into a row-echelon matrix using elementary row transformations, recording how these elementary row transformations affect the determinant of the matrix. More specifically, if A is a matrix and U a row-echelon form of A then

$$|A| = (-1)^r \alpha |U| \tag{2.2}$$

where r is the number of times we performed a row interchange and α is the product of all the constants k which appear in row operations of the form $(kR_i) \to (R_i)$.

We illustrate this with a few examples.

Example 152 *Compute* $|A|$ *for* $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$.

The strategy is to reduce A into row-echelon form and use the fact that the determinant of a triangular matrix is the product of the diagonal entries.

$$\begin{aligned} \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} &= \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix} \\ &= -\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} \\ &= -(1)(3)(-5) \\ &= 15 \end{aligned}$$

On the first line, we performed $(R_2 + 2R_1) \to (R_2)$ and $(R_3 + R_1) \to (R_3)$. These two transformations do not change the determinant. On the second line, we switched rows 2 and 3, this introduces the minus sign we see. On the third line, we simply used the fact that the determinant of a triangular matrix is the product of the diagonal entries.

Example 153 *Find* $|A|$ *for* $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$.

We proceed as above.

$$\begin{aligned} \begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} &= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} \\ &= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix} \\ &= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix} \end{aligned}$$

$$= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\begin{aligned} &= 2\,(1)\,(3)\,(-6)\,(1) \\ &= -36 \end{aligned}$$

On the first line, we factored out 2 from the first row. On line 2, we performed $(R_2 - 3R_1) \to (R_2)$, $(R_3 + 3R_1) \to (R_3)$ and $(R_4 - R_1) \to (R_4)$. These transformations do not change the determinant. On line 3, we performed $(R_3 + 4R_2) \to (R_3)$, again this leaves the determinant unchanged. On line 4, we performed $\left(R_4 - \frac{1}{2}R_3\right) \to (R_4)$ which, again, ;eaves the determinant unchanged. Once we have a triangular matrix, we compute its determinant by multiplying the diagonal entries.

Remark 154 *In the above examples, we actually did not obtain a row-echelon matrix. According to our definition, the first nonzero entry of each row also called a pivot element, should have been a 1. Doing this simply requires a transformation of the form $(kR_i) \to (R_i)$. But as we can see, it is not necessary. In fact, even what we did on the first line of the above example, factoring the 2, was not necessary. It simply made our computations easier. For the purpose of computing the determinant of a matrix A, we only need to transform it into a row-echelon matrix in which the leading entries on each row need not be 1. We can achieve this using the elementary row transformations $(R_i + kR_j) \to (R_i)$ and $(R_i) \longleftrightarrow (R_j)$. The first transformation does not change the determinant. The second one changes its sign. Thus we see that if U is a row-echelon form obtain from A using row replacements or row interchanges, then*

$$|A| = (-1)^r\,|U| \tag{2.3}$$

where r is the number of row interchanges.

Remark 155 *In addition, we know that if A is invertible, then all the diagonal entries of U in the previous remark will be nonzero entries since A is row equivalent to the identity matrix. Otherwise, if A is not invertible, at least one of the diagonal entries of U will be zero, hence $|A| = |U| = 0$.*

Combining the two remarks, we have the following proposition:

Proposition 156 *If U is a row-echelon form obtain from A using row replacements or row interchanges only, then assuming there are r row interchanges performed:*

$$\begin{aligned} |A| &= \begin{cases} (-1)^r\,|U| & \text{if } A \text{ is invertible} \\ 0 & \text{if } A \text{ is not invertible} \end{cases} \\ &= \begin{cases} (-1)^r\,(\text{product of the diagonal entries of } U) & \text{if } A \text{ is invertible} \\ 0 & \text{if } A \text{ is not invertible} \end{cases} \end{aligned}$$

An immediate consequence of this result is the following important theorem.

Theorem 157 *An $n \times n$ matrix A is invertible if and only if $|A| \neq 0$.*

We finish this subsection with a note on the determinant of elementary matrices.

Theorem 158 *Let E be an elementary $n \times n$ matrix.*

1. *If E is obtained by multiplying a row of I_n by k, then $|E| = k$.*
2. *If E is obtained by switching two rows of I_n, then $|E| = -1$.*
3. *If E is obtained by replacing a row of I_n by itself plus a multiple of another row of I_n, then $|E| = 1$.*

Remark 159 (Numerical Notes) *Earlier, we mentioned that computing the determinant of an $n \times n$ matrix using cofactor expansion involved $n!$ operations, which makes it impossible for fast computers to compute even the determinant of a 25×25 matrix (500 000 years for a machine which performs one trillion operations per second). If we use the method outlines in the proposition, it can be proven that it requires $\frac{2n^3}{3}$ operations. Thus, it would take $\frac{\frac{2(25^3)}{3}}{1000000000000} = 1.0417 \times 10^{-8}$ seconds for a computer performing one trillion operations per second. This is much faster.*

2.2.2 Additional Properties

We begin with a few useful theorems which will make computing determinants easier in certain cases.

Theorem 160 *Let A be an $n \times n$ matrix. If A has a row of zeros then $|A| = 0$.*
Proof. *The proof is straightforward. We simply do a cofactor expansion along the row containing zeros.* ■

Corollary 161 *Let A be an $n \times n$ matrix. If A has a row which is a multiple of another row, then $|A| = 0$.*
Proof. *Suppose that $R_i = kR_j$. Then, if we perform an A the elementary row operation $(R_i - kR_j) \to (R_i)$ and call B the resulting matrix, then the i^{th} row of B will consist of zeros. Since this transformation does not change the determinant, it follows that $|B| = |A|$. By the theorem, $|B| = 0$.* ■

Theorem 162 *Let A be an $n \times n$ matrix. $\left|A^T\right| = |A|$.*
Proof. *A^T is obtained from A by switching its rows and columns. Since we can compute the determinant by row or column cofactor expansion and get the same answer, we can compute $|A|$ by cofactor expansion along the first row of A which is the same as cofactor expansion along the first column of A^T. But the latter is $\left|A^T\right|$* ■

Remark 163 *This is a very important result. Everything we said above regarding rows can be restated using columns. For example a matrix with a column of zeros has a determinant equal to* 0*. Similarly, a matrix for which one column is a multiple of another has a determinant equal to* 0*.*

Next, we look at $|A+B|$, $|kA|$ and $|AB|$.

Theorem 164 *Let* A *be an* $n\times n$ *matrix and* k *a constant. Then* $|kA| = k^n\,|A|$*.*
Proof. *This is a repeated application of theorem 151, we have*

$$\begin{aligned}
\begin{vmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{n1} & ka_{n2} & \cdots & ka_{nn} \end{vmatrix} &= k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{n1} & ka_{n2} & \cdots & ka_{nn} \end{vmatrix} \\
&= k^2 \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{n1} & ka_{n2} & \cdots & ka_{nn} \end{matrix} \\
&\vdots \\
&= k^n \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix}
\end{aligned}$$

■

Theorem 165 *If* A *is an* $n\times n$ *matrix and* E *an* $n\times n$ *elementary matrix, then* $|EA| = |E|\,|A|$*.*
Proof. *We consider three cases.*

Case 1 E *is obtained from* I_n *by interchanging two rows. On one hand, by theorem 151,* $|EA| = -|A|$*. But by theorem 158,* $|E|\,|A| = -|A|$*. So, the two are equal.*

Case 2 E *is obtained from* I_n *replacing a row by a non-zero multiple* (k) *of itself. On one hand, by theorem 151,* $|EA| = k\,|A|$*. But by theorem 158,* $|E|\,|A| = k\,|A|$*. So, the two are equal.*

Case 3 E *is obtained from* I_n *replacing one row by itself plus a multiple of another row. On one hand, by theorem 151,* $|EA| = |A|$*. But by theorem 158,* $|E|\,|A| = |A|$*. So, the two are equal.*

■

Theorem 166 *If* A *and* B *are two* $n\times n$ *matrices, then* $|AB| = |A|\,|B|$*.*
Proof. *Again, we divide the proof in two case based on the invertibility of* A*.*

Case 1 *Suppose A is not invertible. Then, AB is not invertible. Thus, by theorem 157 we have $|A| = 0$ thus $|A||B| = 0$ and $|AB| = 0$.*

Case 2 *Suppose A is invertible. Then, A is row equivalent to I_n thus there exist a sequence of elementary matrices $E_1, E_2, ..., E_k$ such that $A = E_1 E_2 ... E_k$. Thus,*

$$AB = E_1 E_2 ... E_k B$$

Hence, by repeated application of theorem 165, we have

$$\begin{aligned} |AB| &= |E_1 E_2 ... E_k B| \\ &= |E_1||E_2|..|E_k||B| \\ &= |E_1 E_2 ... E_k||B| \\ &= |A||B| \end{aligned}$$

■

Theorem 167 *If A is invertible, then $\left|A^{-1}\right| = \dfrac{1}{|A|}$.*

Proof. *If A is invertible, then $A^{-1}A = I$. By theorem 166, we have*

$$\begin{aligned} \left|A^{-1}\right||A| &= |I| \\ &= 1 \end{aligned}$$

hence the result. ■

Using what we learned in this section, we can add to theorem 116. The new version is:

Theorem 168 *If A is an $n \times n$ matrix, then the following statements are equivalent:*

1. *A is invertible.*
2. *$A\mathbf{x} = \mathbf{b}$ has a unique solution for any $n \times 1$ column matrix $\mathbf{b}$.*
3. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
4. *A is row equivalent to I_n.*
5. *A can be written as the product of elementary matrices.*
6. *$|A| \neq 0$*

2.2.3 Introduction to Eigenvalues

Many applications in Linear Algebra involve solving an equation of the form $A\mathbf{x} = \lambda\mathbf{x}$ where A is an $n \times n$ matrix, $\mathbf{x}$ is an $n \times 1$ vector, and λ is a scalar. More specifically, these applications seek the values of λ for which the system has nontrivial solutions.

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \iff A\mathbf{x} - \lambda\mathbf{x} = 0 \\ &\iff (A - \lambda I)\,\mathbf{x} = 0 \end{aligned}$$

From theorem 168, this system has nontrivial solutions if and only if $|A - \lambda I| = 0$.

Definition 169 *Let A be an $n \times n$ matrix.*

1. *The values of λ such that $(A - \lambda I)\,\mathbf{x} = 0$ has nontrivial solutions are called* ***characteristic values*** *or* ***proper values*** *or* ***eigenvalues*** *of the matrix A.*

2. *If λ is an eigenvalue of A, the corresponding nontrivial solution of the system is called an* ***eigenvector****.*

3. *$|A - \lambda I| = 0$ is called the* ***characteristic equation*** *of A.*

Example 170 *Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$.*

- ***Finding the Eigenvalues:*** *We begin by writing $A - \lambda I$.*

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{bmatrix} \end{aligned}$$

Therefore,

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda)(2-\lambda) - 12 \\ &= 2 - 3\lambda + \lambda^2 - 12 \\ &= \lambda^2 - 3\lambda - 10 \end{aligned}$$

Hence, the characteristic equation is

$$\lambda^2 - 3\lambda - 10 = 0$$

Its solutions are

$$\begin{aligned} \lambda^2 - 3\lambda - 10 &= 0 \iff (\lambda - 5)(\lambda + 2) = 0 \\ &\iff \lambda = 5 \text{ or } \lambda = -2 \end{aligned}$$

These are the eigenvalues.

- ***Finding the Eigenvectors:*** *We do it for each eigenvalue.*
  - *If* $\lambda = 5$*, then the system becomes*

$$(A - 5I)\,\mathbf{x} = 0$$

$$\begin{bmatrix} 1-5 & 3 \\ 4 & 2-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We see that the two equations are the same, therefore the solution is $4x_1 = 3x_2$*. So, if* $x_2 = t$ *then* $x_1 = \frac{3}{4}t$*. So, the eigenvectors corresponding to* $\lambda = 5$ *are the non-zero solutions of* $\mathbf{x} = \begin{bmatrix} \frac{3}{4}t \\ t \end{bmatrix}$

  - *If* $\lambda = -2$*, then the system becomes*

$$(A + 2I)\,\mathbf{x} = 0$$

$$\begin{bmatrix} 1+2 & 3 \\ 4 & 2+2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Once again, the two equations are the same. The solutions are $x_1 = -x_2$*. If* $x_2 = t$*, then* $x_1 = -t$*. So, the eigenvectors corresponding to* $\lambda = -2$ *are the non-zero solutions of* $\mathbf{x} = \begin{bmatrix} -t \\ t \end{bmatrix}$

2.2.4 Concepts Review

- Know how the elementary row (columns) transformations affect the determinant of a matrix.
- Know how to compute determinants using elementary row transformations.
- Know the relationship between the determinant of a matrix and the determinant of its transpose.
- Know the relationship between the determinant of a matrix and the determinant of its inverse.
- Know what eigenvalues and eigenvectors are and be able to compute them.

2.2.5 Problems

1. On pages 101, 102, do # 2, 3, 4, 5, 8, 9, 11, 12, 13, 14.
2. On pages 109 - 111, do # 4, 5, 6, 12, 13, 14, 15, 20, 22, 23.
3. Citing theorems studied, explain why a matrix with a row or column of zeros is not invertible.
4. Citing theorems studied, explain why a matrix with a row or column which is a multiple of another row or column is not invertible.
5. Prove that if A and B are two $n \times n$ matrices, then $|AB| = |BA|$.