(a) Suppose that $W(y_1, \dots, y_n)(t_0) \neq 0$, and suppose that

$$c_1 y_1(t) + \cdots + c_n y_n(t) = 0 \tag{ii}$$

for all t in I. By writing the equations corresponding to the first $n-1$ derivatives of Eq. (ii) at t_0, show that $c_1 = \cdots = c_n = 0$. Therefore, $y_1, \dots, y_n$ are linearly independent.
(b) Suppose that $y_1, \dots, y_n$ are linearly independent solutions of Eq. (i). If $W(y_1, \dots, y_n)(t_0) = 0$ for some t_0, show that there is a nonzero solution of Eq. (i) satisfying the initial conditions

$$y(t_0) = y'(t_0) = \cdots = y^{(n-1)}(t_0) = 0.$$

Since $y = 0$ is a solution of this initial value problem, the uniqueness part of Theorem 4.1.1 yields a contradiction. Thus W is never zero.

26. Show that if y_1 is a solution of

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0,$$

then the substitution $y = y_1(t)v(t)$ leads to the following second order equation for v':

$$y_1 v''' + (3y_1' + p_1 y_1)v'' + (3y_1'' + 2p_1 y_1' + p_2 y_1)v' = 0.$$

In each of Problems 27 and 28 use the method of reduction of order (Problem 26) to solve the given differential equation.

27. $(2-t)y''' + (2t-3)y'' - ty' + y = 0, \quad t < 2; \qquad y_1(t) = e^t$
28. $t^2(t+3)y''' - 3t(t+2)y'' + 6(1+t)y' - 6y = 0, \quad t > 0; \qquad y_1(t) = t^2, \quad y_2(t) = t^3$

4.2 Homogeneous Equations with Constant Coefficients

Consider the nth order linear homogeneous differential equation

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0, \tag{1}$$

where $a_0, a_1, \dots, a_n$ are real constants. From our knowledge of second order linear equations with constant coefficients it is natural to anticipate that $y = e^{rt}$ is a solution of Eq. (1) for suitable values of r. Indeed,

$$L[e^{rt}] = e^{rt}(a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1}r + a_n) = e^{rt} Z(r) \tag{2}$$

for all r, where

$$Z(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1}r + a_n. \tag{3}$$

For those values of r for which $Z(r) = 0$, it follows that $L[e^{rt}] = 0$ and $y = e^{rt}$ is a solution of Eq. (1). The polynomial $Z(r)$ is called the **characteristic polynomial**, and the equation $Z(r) = 0$ is the **characteristic equation** of the differential equation (1).

A polynomial of degree n has n zeros,[1] say $r_1, r_2, \dots, r_n$, some of which may be equal; hence we can write the characteristic polynomial in the form

$$Z(r) = a_0(r - r_1)(r - r_2) \cdots (r - r_n). \tag{4}$$

Real and Unequal Roots. If the roots of the characteristic equation are real and no two are equal, then we have n distinct solutions $e^{r_1 t}, e^{r_2 t}, \dots, e^{r_n t}$ of Eq. (1). If these functions are linearly independent, then the general solution of Eq. (1) is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}. \tag{5}$$

One way to establish the linear independence of $e^{r_1 t}, e^{r_2 t}, \dots, e^{r_n t}$ is to evaluate their Wronskian determinant. Another way is outlined in Problem 40.

EXAMPLE 1

Find the general solution of

$$y'''' + y''' - 7y'' - y' + 6y = 0. \tag{6}$$

Also find the solution that satisfies the initial conditions

$$y(0) = 1, \qquad y'(0) = 0, \qquad y''(0) = -2, \qquad y'''(0) = -1 \tag{7}$$

and plot its graph.

Assuming that $y = e^{rt}$, we must determine r by solving the polynomial equation

$$r^4 + r^3 - 7r^2 - r + 6 = 0. \tag{8}$$

The roots of this equation are $r_1 = 1$, $r_2 = -1$, $r_3 = 2$, and $r_4 = -3$. Therefore the general solution of Eq. (6) is

$$y = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-3t}. \tag{9}$$

The initial conditions (7) require that $c_1, \dots, c_4$ satisfy the four equations

$$\begin{aligned} c_1 + c_2 + c_3 + c_4 &= 1, \\ c_1 - c_2 + 2c_3 - 3c_4 &= 0, \\ c_1 + c_2 + 4c_3 + 9c_4 &= -2, \\ c_1 - c_2 + 8c_3 - 27c_4 &= -1. \end{aligned} \tag{10}$$

By solving this system of four linear algebraic equations, we find that

$$c_1 = 11/8, \qquad c_2 = 5/12, \qquad c_3 = -2/3, \qquad c_4 = -1/8.$$

Therefore the solution of the initial value problem is

$$y = \tfrac{11}{8} e^t + \tfrac{5}{12} e^{-t} - \tfrac{2}{3} e^{2t} - \tfrac{1}{8} e^{-3t}. \tag{11}$$

The graph of the solution is shown in Figure 4.2.1.

[1] An important question in mathematics for more than 200 years was whether every polynomial equation has at least one root. The affirmative answer to this question, the fundamental theorem of algebra, was given by Carl Friedrich Gauss in his doctoral dissertation in 1799, although his proof does not meet modern standards of rigor. Several other proofs have been discovered since, including three by Gauss himself. Today, students often meet the fundamental theorem of algebra in a first course on complex variables, where it can be established as a consequence of some of the basic properties of complex analytic functions.

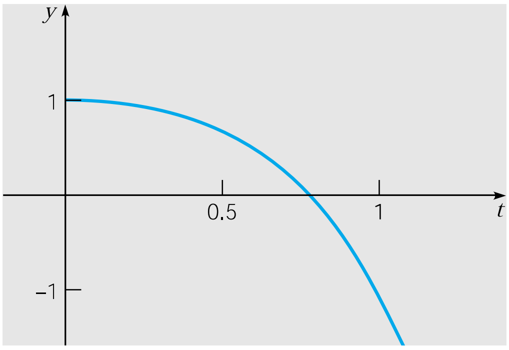

FIGURE 4.2.1 Solution of the initial value problem of Example 1.

As Example 1 illustrates, the procedure for solving an nth order linear differential equation with constant coefficients depends on finding the roots of a corresponding nth degree polynomial equation. If initial conditions are prescribed, then a system of n linear algebraic equations must be solved to determine the proper values of the constants $c_1, \ldots, c_n$. While each of these tasks becomes much more complicated as n increases, they can often be handled without difficulty with a calculator or computer.

For third and fourth degree polynomials there are formulas,[2] analogous to the formula for quadratic equations but more complicated, that give exact expressions for the roots. Root-finding algorithms are readily available on calculators and computers. Sometimes they are included in the differential equation solver, so that the process of factoring the characteristic polynomial is hidden and the solution of the differential equation is produced automatically.

If you are faced with the need to factor the characteristic polynomial by hand, here is one result that is sometimes helpful. Suppose that the polynomial

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0 \tag{12}$$

has integer coefficients. If $r = p/q$ is a rational root, where p and q have no common factors, then p must be a factor of a_n and q must be a factor of a_0. For example, in Eq. (8) the factors of a_0 are ± 1 and the factors of a_n are ± 1, ± 2, ± 3, and ± 6. Thus, the only possible rational roots of this equation are ± 1, ± 2, ± 3, and ± 6. By testing these possible roots, we find that 1, -1, 2, and -3 are actual roots. In this case there are no other roots, since the polynomial is of fourth degree. If some of the roots are irrational or complex, as is usually the case, then this process will not find them, but at least the degree of the polynomial can be reduced by dividing out the factors corresponding to the rational roots.

If the roots of the characteristic equation are real and different, we have seen that the general solution (5) is simply a sum of exponential functions. For large values of t the

[2] The method for solving the cubic equation was apparently discovered by Scipione dal Ferro (1465–1526) about 1500, although it was first published in 1545 by Girolamo Cardano (1501–1576) in his *Ars Magna*. This book also contains a method for solving quartic equations that Cardano attributes to his pupil Ludovico Ferrari (1522–1565). The question of whether analogous formulas exist for the roots of higher degree equations remained open for more than two centuries, until in 1826 Niels Abel showed that no general solution formulas can exist for polynomial equations of degree five or higher. A more general theory was developed by Evariste Galois (1811–1832) in 1831, but unfortunately it did not become widely known for several decades.

solution will be dominated by the term corresponding to the algebraically largest root. If this root is positive, then solutions will become exponentially unbounded, while if it is negative, then solutions will tend exponentially to zero. Finally, if the largest root is zero, then solutions will approach a nonzero constant as t becomes large. Of course, for certain initial conditions the coefficient of the otherwise dominant term will be zero; then the nature of the solution for large t is determined by the next largest root.

Complex Roots. If the characteristic equation has complex roots, they must occur in conjugate pairs, $\lambda \pm i\mu$, since the coefficients $a_0, \ldots, a_n$ are real numbers. Provided that none of the roots is repeated, the general solution of Eq. (1) is still of the form (4). However, just as for the second order equation (Section 3.4), we can replace the complex-valued solutions $e^{(\lambda+i\mu)t}$ and $e^{(\lambda-i\mu)t}$ by the real-valued solutions

$$e^{\lambda t}\cos\mu t, \qquad e^{\lambda t}\sin\mu t \tag{13}$$

obtained as the real and imaginary parts of $e^{(\lambda+i\mu)t}$. Thus, even though some of the roots of the characteristic equation are complex, it is still possible to express the general solution of Eq. (1) as a linear combination of real-valued solutions.

EXAMPLE 2

Find the general solution of

$$y^{\text{iv}} - y = 0. \tag{14}$$

Also find the solution that satisfies the initial conditions

$$y(0) = 7/2, \qquad y'(0) = -4, \qquad y''(0) = 5/2, \qquad y'''(0) = -2 \tag{15}$$

and draw its graph.

Substituting e^{rt} for y, we find that the characteristic equation is

$$r^4 - 1 = (r^2 - 1)(r^2 + 1) = 0.$$

Therefore the roots are $r = 1, -1, i, -i$, and the general solution of Eq. (14) is

$$y = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t.$$

If we impose the initial conditions (15), we find that

$$c_1 = 0, \qquad c_2 = 3, \qquad c_3 = 1/2, \qquad c_4 = -1;$$

thus the solution of the given initial value problem is

$$y = 3e^{-t} + \tfrac{1}{2}\cos t - \sin t. \tag{16}$$

The graph of this solution is shown in Figure 4.2.2.

Observe that the initial conditions (15) cause the coefficient c_1 of the exponentially growing term in the general solution to be zero. Therefore this term is absent in the solution (16), which describes an exponential decay to a steady oscillation, as Figure 4.2.2 shows. However, if the initial conditions are changed slightly, then c_1 is likely to be nonzero and the nature of the solution changes enormously. For example, if the first three initial conditions remain the same, but the value of $y'''(0)$ is changed from -2 to $-15/8$, then the solution of the initial value problem becomes

$$y = \tfrac{1}{32}e^t + \tfrac{95}{32}e^{-t} + \tfrac{1}{2}\cos t - \tfrac{17}{16}\sin t. \tag{17}$$

The coefficients in Eq. (17) differ only slightly from those in Eq. (16), but the exponentially growing term, even with the relatively small coefficient of 1/32, completely dominates the solution by the time t is larger than about 4 or 5. This is clearly seen in Figure 4.2.3, which shows the graphs of the two solutions (16) and (17).

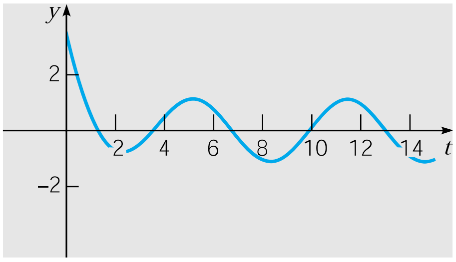

FIGURE 4.2.2 A plot of the solution (16).

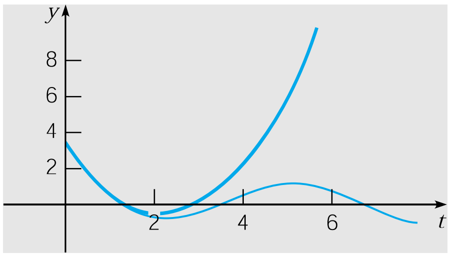

FIGURE 4.2.3 Plots of the solutions (16) (light curve) and (17) (heavy curve).

Repeated Roots. If the roots of the characteristic equation are not distinct, that is, if some of the roots are repeated, then the solution (5) is clearly not the general solution of Eq. (1). Recall that if r_1 is a repeated root for the second order linear equation $a_0y'' + a_1y' + a_2y = 0$, then the two linearly independent solutions are e^{r_1t} and te^{r_1t}. For an equation of order n, if a root of $Z(r) = 0$, say $r = r_1$, has multiplicity s (where $s \le n$), then

$$e^{r_1t}, \quad te^{r_1t}, \quad t^2e^{r_1t}, \quad \ldots, \quad t^{s-1}e^{r_1t} \tag{18}$$

are corresponding solutions of Eq. (1).

If a complex root $\lambda + i\mu$ is repeated s times, the complex conjugate $\lambda - i\mu$ is also repeated s times. Corresponding to these $2s$ complex-valued solutions, we can find $2s$ real-valued solutions by noting that the real and imaginary parts of $e^{(\lambda+i\mu)t}$, $te^{(\lambda+i\mu)t}, \ldots, t^{s-1}e^{(\lambda+i\mu)t}$ are also linearly independent solutions:

$$e^{\lambda t}\cos\mu t, \qquad e^{\lambda t}\sin\mu t, \qquad te^{\lambda t}\cos\mu t, \qquad te^{\lambda t}\sin\mu t,$$
$$\ldots, t^{s-1}e^{\lambda t}\cos\mu t, \qquad t^{s-1}e^{\lambda t}\sin\mu t.$$

Hence the general solution of Eq. (1) can always be expressed as a linear combination of n real-valued solutions. Consider the following example.

EXAMPLE 3

Find the general solution of

$$y^{\text{iv}} + 2y'' + y = 0. \tag{19}$$

The characteristic equation is

$$r^4 + 2r^2 + 1 = (r^2 + 1)(r^2 + 1) = 0.$$

The roots are $r = i, i, -i, -i$, and the general solution of Eq. (19) is

$$y = c_1\cos t + c_2\sin t + c_3t\cos t + c_4t\sin t.$$

In determining the roots of the characteristic equation it may be necessary to compute the cube roots, or fourth roots, or even higher roots of a (possibly complex) number. This can usually be done most conveniently by using Euler's formula $e^{it} = \cos t + i \sin t$ and the algebraic laws given in Section 3.4. This is illustrated in the following example.

EXAMPLE 4

Find the general solution of

$$y^{\text{iv}} + y = 0. \tag{20}$$

The characteristic equation is

$$r^4 + 1 = 0.$$

To solve the equation we must compute the fourth roots of -1. Now -1, thought of as a complex number, is $-1 + 0i$. It has magnitude 1 and polar angle π. Thus

$$-1 = \cos \pi + i \sin \pi = e^{i\pi}.$$

Moreover, the angle is determined only up to a multiple of 2π. Thus

$$-1 = \cos(\pi + 2m\pi) + i \sin(\pi + 2m\pi) = e^{i(\pi+2m\pi)},$$

where m is zero or any positive or negative integer. Thus

$$(-1)^{1/4} = e^{i(\pi/4+m\pi/2)} = \cos\left(\frac{\pi}{4} + \frac{m\pi}{2}\right) + i \sin\left(\frac{\pi}{4} + \frac{m\pi}{2}\right).$$

The four fourth roots of -1 are obtained by setting $m = 0$, 1, 2, and 3; they are

$$\frac{1+i}{\sqrt{2}}, \qquad \frac{-1+i}{\sqrt{2}}, \qquad \frac{-1-i}{\sqrt{2}}, \qquad \frac{1-i}{\sqrt{2}}.$$

It is easy to verify that for any other value of m we obtain one of these four roots. For example, corresponding to $m = 4$, we obtain $(1 + i)/\sqrt{2}$. The general solution of Eq. (20) is

$$y = e^{t/\sqrt{2}}\left(c_1 \cos\frac{t}{\sqrt{2}} + c_2 \sin\frac{t}{\sqrt{2}}\right) + e^{-t/\sqrt{2}}\left(c_3 \cos\frac{t}{\sqrt{2}} + c_4 \sin\frac{t}{\sqrt{2}}\right). \tag{21}$$

In conclusion, we note that the problem of finding all the roots of a polynomial equation may not be entirely straightforward, even with computer assistance. For instance, it may be difficult to determine whether two roots are equal, or merely very close together. Recall that the form of the general solution is different in these two cases.

If the constants $a_0, a_1, \ldots, a_n$ in Eq. (1) are complex numbers, the solution of Eq. (1) is still of the form (4). In this case, however, the roots of the characteristic equation are, in general, complex numbers, and it is no longer true that the complex conjugate of a root is also a root. The corresponding solutions are complex-valued.

PROBLEMS

In each of Problems 1 through 6 express the given complex number in the form $R(\cos\theta + i\sin\theta) = Re^{i\theta}$.

1. $1 + i$
2. $-1 + \sqrt{3}i$
3. -3
4. $-i$
5. $\sqrt{3} - i$
6. $-1 - i$

In each of Problems 7 through 10 follow the procedure illustrated in Example 4 to determine the indicated roots of the given complex number.

7. $1^{1/3}$
8. $(1-i)^{1/2}$
9. $1^{1/4}$
10. $[2(\cos\pi/3 + i\sin\pi/3)]^{1/2}$

In each of Problems 11 through 28 find the general solution of the given differential equation.

11. $y''' - y'' - y' + y = 0$
12. $y''' - 3y'' + 3y' - y = 0$
13. $2y''' - 4y'' - 2y' + 4y = 0$
14. $y^{\text{iv}} - 4y''' + 4y'' = 0$
15. $y^{\text{vi}} + y = 0$
16. $y^{\text{iv}} - 5y'' + 4y = 0$
17. $y^{\text{vi}} - 3y^{\text{iv}} + 3y'' - y = 0$
18. $y^{\text{vi}} - y'' = 0$
19. $y^{\text{v}} - 3y^{\text{iv}} + 3y''' - 3y'' + 2y' = 0$
20. $y^{\text{iv}} - 8y' = 0$
21. $y^{\text{viii}} + 8y^{\text{iv}} + 16y = 0$
22. $y^{\text{iv}} + 2y'' + y = 0$
23. $y''' - 5y'' + 3y' + y = 0$
24. $y''' + 5y'' + 6y' + 2y = 0$
25. $18y''' + 21y'' + 14y' + 4y = 0$
26. $y^{\text{iv}} - 7y''' + 6y'' + 30y' - 36y = 0$
27. $12y^{\text{iv}} + 31y''' + 75y'' + 37y' + 5y = 0$
28. $y^{\text{iv}} + 6y''' + 17y'' + 22y' + 14y = 0$

In each of Problems 29 through 36 find the solution of the given initial value problem and plot its graph. How does the solution behave as $t \to \infty$?

29. $y''' + y' = 0; \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 2$
30. $y^{\text{iv}} + y = 0; \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = -1, \quad y'''(0) = 0$
31. $y^{\text{iv}} - 4y''' + 4y'' = 0; \quad y(1) = -1, \quad y'(1) = 2, \quad y''(1) = 0, \quad y'''(1) = 0$
32. $y''' - y'' + y' - y = 0; \quad y(0) = 2, \quad y'(0) = -1, \quad y''(0) = -2$
33. $2y^{\text{iv}} - y''' - 9y'' + 4y' + 4y = 0; \quad y(0) = -2, \quad y'(0) = 0, \quad y''(0) = -2,$ $y'''(0) = 0$
34. $4y''' + y' + 5y = 0; \quad y(0) = 2, \quad y'(0) = 1, \quad y''(0) = -1$
35. $6y''' + 5y'' + y' = 0; \quad y(0) = -2, \quad y'(0) = 2, \quad y''(0) = 0$
36. $y^{\text{iv}} + 6y''' + 17y'' + 22y' + 14y = 0; \quad y(0) = 1, \quad y'(0) = -2, \quad y''(0) = 0,$ $y'''(0) = 3$
37. Show that the general solution of $y^{\text{iv}} - y = 0$ can be written as

$$y = c_1 \cos t + c_2 \sin t + c_3 \cosh t + c_4 \sinh t.$$

Determine the solution satisfying the initial conditions $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$, $y'''(0) = 1$. Why is it convenient to use the solutions $\cosh t$ and $\sinh t$ rather than e^t and e^{-t}?

38. Consider the equation $y^{\text{iv}} - y = 0$.
(a) Use Abel's formula [Problem 20(d) of Section 4.1] to find the Wronskian of a fundamental set of solutions of the given equation.
(b) Determine the Wronskian of the solutions e^t, e^{-t}, $\cos t$, and $\sin t$.
(c) Determine the Wronskian of the solutions $\cosh t$, $\sinh t$, $\cos t$, and $\sin t$.

39. Consider the spring–mass system, shown in Figure 4.2.4, consisting of two unit masses suspended from springs with spring constants 3 and 2, respectively. Assume that there is no damping in the system.
(a) Show that the displacements u_1 and u_2 of the masses from their respective equilibrium positions satisfy the equations

$$u_1'' + 5u_1 = 2u_2, \qquad u_2'' + 2u_2 = 2u_1. \tag{i}$$

(b) Solve the first of Eqs. (i) for u_2 and substitute into the second equation, thereby obtaining the following fourth order equation for u_1:

$$u_1^{\text{iv}} + 7u_1'' + 6u_1 = 0. \tag{ii}$$

Find the general solution of Eq. (ii).