CHAPTER

6

The Laplace Transform

Many practical engineering problems involve mechanical or electrical systems acted on by discontinuous or impulsive forcing terms. For such problems the methods described in Chapter 3 are often rather awkward to use. Another method that is especially well suited to these problems, although useful much more generally, is based on the Laplace transform. In this chapter we describe how this important method works, emphasizing problems typical of those arising in engineering applications.

6.1 Definition of the Laplace Transform

Among the tools that are very useful for solving linear differential equations are **integral transforms**. An integral transform is a relation of the form

$$F(s) = \int_{\alpha}^{\beta} K(s,t) f(t)\, dt, \tag{1}$$

where $K(s,t)$ is a given function, called the **kernel** of the transformation, and the limits of integration α and β are also given. It is possible that $\alpha = -\infty$ or $\beta = \infty$, or both. The relation (1) transforms the function f into another function F, which is called the transform of f. The general idea in using an integral transform to solve a differential equation is as follows: Use the relation (1) to transform a problem for an unknown function f into a simpler problem for F, then solve this simpler problem to find F, and finally recover the desired function f from its transform F. This last step is known as "inverting the transform."

There are several integral transforms that are useful in applied mathematics, but in this chapter we consider only the Laplace[1] transform. This transform is defined in the following way. Let $f(t)$ be given for $t \geq 0$, and suppose that f satisfies certain conditions to be stated a little later. Then the Laplace transform of f, which we will denote by $\mathcal{L}\{f(t)\}$ or by $F(s)$, is defined by the equation

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt. \tag{2}$$

The Laplace transform makes use of the kernel $K(s,t) = e^{-st}$. Since the solutions of linear differential equations with constant coefficients are based on the exponential function, the Laplace transform is particularly useful for such equations.

Since the Laplace transform is defined by an integral over the range from zero to infinity, it is useful to review some basic facts about such integrals. In the first place, an integral over an unbounded interval is called an **improper integral**, and is defined as a limit of integrals over finite intervals; thus

$$\int_a^\infty f(t)\, dt = \lim_{A\to\infty} \int_a^A f(t)\, dt, \tag{3}$$

where A is a positive real number. If the integral from a to A exists for each $A > a$, and if the limit as $A \to \infty$ exists, then the improper integral is said to **converge** to that limiting value. Otherwise the integral is said to **diverge**, or to fail to exist. The following examples illustrate both possibilities.

EXAMPLE 1

Let $f(t) = e^{ct}$, $t \geq 0$, where c is a real nonzero constant. Then

$$\begin{aligned}\int_0^\infty e^{ct}\, dt &= \lim_{A\to\infty} \int_0^A e^{ct}\, dt = \lim_{A\to\infty} \left.\frac{e^{ct}}{c}\right|_0^A \\ &= \lim_{A\to\infty} \frac{1}{c}(e^{cA} - 1).\end{aligned}$$

It follows that the improper integral converges if $c < 0$, and diverges if $c > 0$. If $c = 0$, the integrand $f(t)$ is the constant function with value 1, and the integral again diverges.

EXAMPLE 2

Let $f(t) = 1/t$, $t \geq 1$. Then

$$\int_1^\infty \frac{dt}{t} = \lim_{A\to\infty} \int_1^A \frac{dt}{t} = \lim_{A\to\infty} \ln A.$$

Since $\lim_{A\to\infty} \ln A = \infty$, the improper integral diverges.

[1] The Laplace transform is named for the eminent French mathematician P. S. Laplace, who studied the relation (2) in 1782. However, the techniques described in this chapter were not developed until a century or more later. They are due mainly to Oliver Heaviside (1850–1925), an innovative but unconventional English electrical engineer, who made significant contributions to the development and application of electromagnetic theory.

EXAMPLE 3

Let $f(t) = t^{-p}$, $t \geq 1$, where p is a real constant and $p \neq 1$; the case $p = 1$ was considered in Example 2. Then

$$\int_1^\infty t^{-p}\, dt = \lim_{A\to\infty} \int_1^A t^{-p}\, dt = \lim_{A\to\infty} \frac{1}{1-p}(A^{1-p} - 1).$$

As $A \to \infty$, $A^{1-p} \to 0$ if $p > 1$, but $A^{1-p} \to \infty$ if $p < 1$. Hence $\int_1^\infty t^{-p}\, dt$ converges for $p > 1$, but (incorporating the result of Example 2) diverges for $p \leq 1$. These results are analogous to those for the infinite series $\sum_{n=1}^{\infty} n^{-p}$.

Before discussing the possible existence of $\int_a^\infty f(t)\, dt$, it is helpful to define certain terms. A function f is said to be **piecewise continuous** on an interval $\alpha \leq t \leq \beta$ if the interval can be partitioned by a finite number of points $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ so that

1. f is continuous on each open subinterval $t_{i-1} < t < t_i$.
2. f approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.

In other words, f is piecewise continuous on $\alpha \leq t \leq \beta$ if it is continuous there except for a finite number of jump discontinuities. If f is piecewise continuous on $\alpha \leq t \leq \beta$ for every $\beta > \alpha$, then f is said to be piecewise continuous on $t \geq \alpha$. An example of a piecewise continuous function is shown in Figure 6.1.1.

If f is piecewise continuous on the interval $a \leq t \leq A$, then it can be shown that $\int_a^A f(t)\, dt$ exists. Hence, if f is piecewise continuous for $t \geq a$, then $\int_a^A f(t)\, dt$ exists for each $A > a$. However, piecewise continuity is not enough to ensure convergence of the improper integral $\int_a^\infty f(t)\, dt$, as the preceding examples show.

If f cannot be integrated easily in terms of elementary functions, the definition of convergence of $\int_a^\infty f(t)\, dt$ may be difficult to apply. Frequently, the most convenient way to test the convergence or divergence of an improper integral is by the following comparison theorem, which is analogous to a similar theorem for infinite series.

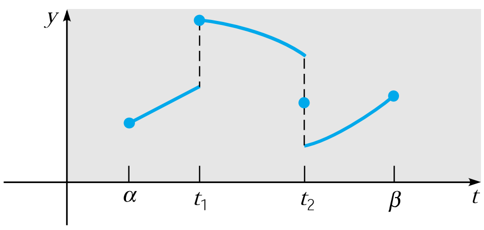

FIGURE 6.1.1 A piecewise continuous function.

Theorem 6.1.1 If f is piecewise continuous for $t \geq a$, if $|f(t)| \leq g(t)$ when $t \geq M$ for some positive constant M, and if $\int_M^\infty g(t)\,dt$ converges, then $\int_a^\infty f(t)\,dt$ also converges. On the other hand, if $f(t) \geq g(t) \geq 0$ for $t \geq M$, and if $\int_M^\infty g(t)\,dt$ diverges, then $\int_a^\infty f(t)\,dt$ also diverges.

The proof of this result from the calculus will not be given here. It is made plausible, however, by comparing the areas represented by $\int_M^\infty g(t)\,dt$ and $\int_M^\infty |f(t)|\,dt$. The functions most useful for comparison purposes are e^{ct} and t^{-p}, which were considered in Examples 1, 2, and 3.

We now return to a consideration of the Laplace transform $\mathcal{L}\{f(t)\}$ or $F(s)$, which is defined by Eq. (2) whenever this improper integral converges. In general, the parameter s may be complex, but for our discussion we need consider only real values of s. The foregoing discussion of integrals indicates that the Laplace transform F of a function f exists if f satisfies certain conditions, such as those stated in the following theorem.

Theorem 6.1.2 Suppose that

1. f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A.
2. $|f(t)| \leq Ke^{at}$ when $t \geq M$. In this inequality K, a, and M are real constants, K and M necessarily positive.

Then the Laplace transform $\mathcal{L}\{f(t)\} = F(s)$, defined by Eq. (2), exists for $s > a$.

To establish this theorem it is necessary to show only that the integral in Eq. (2) converges for $s > a$. Splitting the improper integral into two parts, we have

$$\int_0^\infty e^{-st} f(t)\,dt = \int_0^M e^{-st} f(t)\,dt + \int_M^\infty e^{-st} f(t)\,dt. \tag{4}$$

The first integral on the right side of Eq. (4) exists by hypothesis (1) of the theorem; hence the existence of $F(s)$ depends on the convergence of the second integral. By hypothesis (2) we have, for $t \geq M$,

$$|e^{-st} f(t)| \leq Ke^{-st}e^{at} = Ke^{(a-s)t},$$

and thus, by Theorem 6.1.1, $F(s)$ exists provided that $\int_M^\infty e^{(a-s)t}\,dt$ converges. Referring to Example 1 with c replaced by $a - s$, we see that this latter integral converges when $a - s < 0$, which establishes Theorem 6.1.2.

Unless the contrary is specifically stated, in this chapter we deal only with functions satisfying the conditions of Theorem 6.1.2. Such functions are described as piecewise continuous, and of **exponential order** as $t \to \infty$. The Laplace transforms of some important elementary functions are given in the following examples.

EXAMPLE 4

Let $f(t) = 1$, $t \geq 0$. Then

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\, dt = \frac{1}{s}, \qquad s > 0.$$

EXAMPLE 5

Let $f(t) = e^{at}$, $t \geq 0$. Then

$$\begin{aligned}\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt \\ &= \frac{1}{s-a}, \qquad s > a.\end{aligned}$$

EXAMPLE 6

Let $f(t) = \sin at$, $t \geq 0$. Then

$$\mathcal{L}\{\sin at\} = F(s) = \int_0^\infty e^{-st} \sin at\, dt, \qquad s > 0.$$

Since

$$F(s) = \lim_{A\to\infty} \int_0^A e^{-st} \sin at\, dt,$$

upon integrating by parts we obtain

$$\begin{aligned}F(s) &= \lim_{A\to\infty} \left[-\frac{e^{-st} \cos at}{a}\bigg|_0^A - \frac{s}{a} \int_0^A e^{-st} \cos at\, dt \right] \\ &= \frac{1}{a} - \frac{s}{a} \int_0^\infty e^{-st} \cos at\, dt.\end{aligned}$$

A second integration by parts then yields

$$\begin{aligned}F(s) &= \frac{1}{a} - \frac{s^2}{a^2} \int_0^\infty e^{-st} \sin at\, dt \\ &= \frac{1}{a} - \frac{s^2}{a^2} F(s).\end{aligned}$$

Hence, solving for $F(s)$, we have

$$F(s) = \frac{a}{s^2 + a^2}, \qquad s > 0.$$

Now let us suppose that f_1 and f_2 are two functions whose Laplace transforms exist for $s > a_1$ and $s > a_2$, respectively. Then, for s greater than the maximum of a_1 and a_2,

$$\begin{aligned}\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} &= \int_0^\infty e^{-st}[c_1 f_1(t) + c_2 f_2(t)]\, dt \\ &= c_1 \int_0^\infty e^{-st} f_1(t)\, dt + c_2 \int_0^\infty e^{-st} f_2(t)\, dt;\end{aligned}$$

hence

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}. \qquad (5)$$

Equation (5) is a statement of the fact that the Laplace transform is a *linear operator*. This property is of paramount importance, and we make frequent use of it later.

PROBLEMS

In each of Problems 1 through 4 sketch the graph of the given function. In each case determine whether f is continuous, piecewise continuous, or neither on the interval $0 \le t \le 3$.

1. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 2+t, & 1 < t \le 2 \\ 6-t, & 2 < t \le 3 \end{cases}$

2. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ (t-1)^{-1}, & 1 < t \le 2 \\ 1, & 2 < t \le 3 \end{cases}$

3. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 1, & 1 < t \le 2 \\ 3-t, & 2 < t \le 3 \end{cases}$

4. $f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 3-t, & 1 < t \le 2 \\ 1, & 2 < t \le 3 \end{cases}$

5. Find the Laplace transform of each of the following functions:
   (a) t
   (b) t^2
   (c) t^n, where n is a positive integer
6. Find the Laplace transform of $f(t) = \cos at$, where a is a real constant.

Recall that $\cosh bt = (e^{bt} + e^{-bt})/2$ and $\sinh bt = (e^{bt} - e^{-bt})/2$. In each of Problems 7 through 10 find the Laplace transform of the given function; a and b are real constants.

7. $\cosh bt$
8. $\sinh bt$
9. $e^{at}\cosh bt$
10. $e^{at}\sinh bt$

In each of Problems 11 through 14 recall that $\cos bt = (e^{ibt} + e^{-ibt})/2$ and $\sin bt = (e^{ibt} - e^{-ibt})/2i$. Assuming that the necessary elementary integration formulas extend to this case, find the Laplace transform of the given function; a and b are real constants.

11. $\sin bt$
12. $\cos bt$
13. $e^{at}\sin bt$
14. $e^{at}\cos bt$

In each of Problems 15 through 20, using integration by parts, find the Laplace transform of the given function; n is a positive integer and a is a real constant.

15. te^{at}
16. $t\sin at$
17. $t\cosh at$
18. $t^n e^{at}$
19. $t^2\sin at$
20. $t^2\sinh at$

In each of Problems 21 through 24 determine whether the given integral converges or diverges.

21. $\displaystyle\int_0^\infty (t^2+1)^{-1}\,dt$
22. $\displaystyle\int_0^\infty te^{-t}\,dt$
23. $\displaystyle\int_1^\infty t^{-2}e^{t}\,dt$
24. $\displaystyle\int_0^\infty e^{-t}\cos t\,dt$
25. Suppose that f and f' are continuous for $t \ge 0$, and of exponential order as $t \to \infty$. Show by integration by parts that if $F(s) = \mathcal{L}\{f(t)\}$, then $\lim_{s\to\infty} F(s) = 0$. The result is actually true under less restrictive conditions, such as those of Theorem 6.1.2.
26. **The Gamma Function.** The gamma function is denoted by $\Gamma(p)$ and is defined by the integral

$$\Gamma(p+1) = \int_0^\infty e^{-x}x^p\,dx. \qquad \text{(i)}$$