Value Problems

The integral converges as $x \to \infty$ for all p. For $p < 0$ it is also improper because the

integrand becomes unbounded as $x \to 0$. However, the integral can be shown to converge

(a) Show that for $p > 0$
 Value Problems

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(a) Show that for $p > 0$
 Value Problems

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The integral converges as $x \to \infty$ for all p. For $p < 0$ it is also improper because the

integral deconnes unboded as $x \to 0$. However, the integral can be shown to converge
 $\text{at } x = 0$ for $p > -1$.
 Value Problems

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Value Problems

The integral converges as $x \to \infty$ for all p. For $p < 0$ it is also improper because the

integrand becomes unbounded as $x \to 0$. However, the integral can be shown to converge

(a) Sh Value Problems

The integral converges as $x \to \infty$ for all p. For $p < 0$ it is also improper because the

integrand becomes unbounded as $x \to 0$. However, the integral can be shown to converge

(a) Show that for $p > 0$

(

$$
\Gamma(p+1) = p\Gamma(p).
$$

-
-

$$
\Gamma(n+1)=n!.
$$

Value Problems

The integral converges as $x \to \infty$ for all p. For $p < 0$ it is also improper because the

integrand hecomes unbounded as $x \to 0$. However, the integral can be shown to converge

(a) Show that I (1) = 1.

(Value Problems

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integrand becomes unbounded as $x \to 0$. However, the integral can be shown to converge

(a) Show that for $p > 0$
 Value Problems

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 $\text{at } x = 0$ for $p > -1$.
 Example 11.1
 Consider the Example System Alternation (Somewheat of $p \ge -1$.

The integral decomes unbounded as $x \to 0$. However, the integral can be shown to converge $ax = 0$ for $p > -1$.

(a) Show that for $p > 0$
 Γ **Value Problems**
 299

The integral converges as $x \to \infty$ for all p. For $p < 0$ it is also improper because the

integrand becomes unbounded as $x \to 0$. However, the integral can be shown to converge

(a) Show that for Value Problems

The integral converges as $x \to \infty$ for all p. For $p < 0$ it is also improper because the

tingerand becomes unbounded as $x \to 0$. However, the integral can be shown to converge

(a) Show that for $p > 0$
 Value Problems
 Example 10 converges as $x \to \infty$ for all p. For $p < 0$ it is also improper because the integral decomes unbounded as $x \to 0$. However, the integral can be shown to converge tax $= 0$ for $p > -1$.

(a) S 299
because the
to converge
extension of
hat it is also
in a single
Find $\Gamma(\frac{3}{2})$ **EXECUTE:** The integral converges as $x \to \infty$ for all p . For $p < 0$ it is also improper because the integral decomes unbounded as $x \to 0$. However, the integral can be shown to converge at $x = 0$ for $p > -1$.

(a) Show t **299**

or all *p*. For $p < 0$ it is also improper because the
 $\rightarrow 0$. However, the integral can be shown to converge
 $(p + 1) = p\Gamma(p)$.

that

that
 $\Gamma(n + 1) = n!$.

not an integer, this function provides an extension of
 \cd Value Problems

The integral converges as $x \to \infty$ for all p. For $p < 0$ it is also improper because the

integrand becomes unbounded as $x \to 0$. However, the integral can be shown to converge

(a) Show that $\Gamma(1) = 1$.
 at $x = 0$ for $p > -1$.

(a) Show that $\Gamma(f) = 1$.

(b) If p is a positive integer n , show that
 $\Gamma(n + 1) = n!$.

Since $\Gamma(p)$ is also defined when p is not an integer, this function provides an extension of

the factori (b) Show that $\Gamma(1) = 1$.

(c) If p is a positive integer n, show that
 $\Gamma(n + 1) = n!$.

(c) If p is a bosotive integer n, show that
 $\Gamma(n + 1) = n!$.

Since $\Gamma(p)$ is also defined when p is not an integra, this function provi Since $\Gamma(p)$ is also defined when p is not an integer, this function provides an extension of
the factorial function to nonintegral values of the independent variable. Note that it is also
consistent to define $0! = 1$.

$$
p(p+1)(p+2)\cdots(p+n-1)=\Gamma(p+n)/\Gamma(p).
$$

interval of unit length, say, $0 < p \le 1$. It is possible to show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Find $\Gamma(\frac{3}{2})$ $\overline{2}$ and $\Gamma(\frac{11}{2})$. Thus $\Gamma(\rho)$ can be determined for all positive values of p if $\Gamma(\rho)$ is known in a single
interval of unit length, say, $0 < p \le 1$. It is possible to show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Find $\Gamma(\frac{3}{2})$
consider the Laplace tra

- 27. Consider the Laplace transform of t^p , where $p > -1$.
	-

$$
\mathcal{L}{t^p} = \int_0^\infty e^{-st} t^p dt = \frac{1}{s^{p+1}} \int_0^\infty e^{-x} x^p dx
$$

= $\Gamma(p+1)/s^{p+1}$, $s > 0$.

$$
\mathcal{L}\lbrace t^n\rbrace = n!/s^{n+1}, \qquad s > 0.
$$

$$
\mathcal{L}\{t^{-1/2}\} = \frac{2}{\sqrt{s}} \int_0^\infty e^{-x^2} \, dx, \qquad s > 0.
$$

$$
\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2};
$$

hence **hence** the control of the control o

$$
\mathcal{L}\lbrace t^{-1/2}\rbrace = \sqrt{\pi/s}, \qquad s > 0.
$$

$$
\mathcal{L}\{t^{1/2}\} = \sqrt{\pi}/2s^{3/2}, \qquad s > 0.
$$

(b) Let *p* be a positive integer *n* in (a); show that
 $\mathcal{L}\{t^{-1}\} = n!/s^{n+1}, \quad s > 0.$

(c) Show that
 $\mathcal{L}\{t^{-1/2}\} = \frac{2}{\sqrt{s}} \int_0^\infty e^{-s^2} dx, \quad s > 0.$

It is possible to show that
 $\int_0^\infty e^{-s^2} dx = \frac{\sqrt{\pi}}{2}$;

hence
 $\mathcal{L}{t^{n}} = n!/s^{n+1},$ $s > 0.$

(c) Show that
 $\mathcal{L}{t^{-1/2}} = \frac{2}{\sqrt{s}} \int_0^\infty e^{-s^2} dx,$ $s > 0.$

It is possible to show that
 $\int_0^\infty e^{-s^2} dx = \frac{\sqrt{\pi}}{2}$;

hence

(d) Show that
 $\mathcal{L}{t^{-1/2}} = \sqrt{\pi}/s,$ $s > 0.$

(d) Show that (c) Show that
 $\mathcal{L}\lbrace t^{-1/2}\rbrace = \frac{2}{\sqrt{s}} \int_0^\infty e^{-x^2} dx, \qquad s > 0.$

It is possible to show that
 $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$;

hence
 $\mathcal{L}\lbrace t^{-1/2}\rbrace = \sqrt{\pi/s}, \qquad s > 0.$

(d) Show that
 $\mathcal{L}\lbrace t^{1/2}\rbrace = \sqrt{\pi/2}x^{3/2}, \qquad s > 0.$

Chapter 6. The Laplace Transform

of f' is related in a simple way to the transform of f. The relationship is expressed in

the following theorem.

Suppose that f is continuous and f' is piecewise continuous on any inte **Chapter 6. The Laplace Transform**

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Suppose that f is continuous and f' is piecewise continuous on any inte **Chapter 6. The Laplace Transform**

ed in a simple way to the transform of f . The relationship is expressed in

that f is continuous and f' is piecewise continuous on any interval

Suppose further that there exist c Theorem 6.2.1

$$
\mathcal{L}\lbrace f'(t)\rbrace = s\mathcal{L}\lbrace f(t)\rbrace - f(0). \tag{1}
$$

$$
\int_0^A e^{-st} f'(t) dt.
$$

 t_1, t_2, \ldots, t_n . Then we can write this integral as

$$
\int_0^A e^{-st} f'(t) dt = \int_0^{t_1} e^{-st} f'(t) dt + \int_{t_1}^{t_2} e^{-st} f'(t) dt + \cdots + \int_{t_n}^A e^{-st} f'(t) dt.
$$

Chapter 6. The Laplace Transform
\nof
$$
f'
$$
 is related in a simple way to the transform of f . The relationship is expressed in
\nthe following theorem.
\nSuppose that f is continuous and f' is piecewise continuous on any interval
\n $0 \le t \le A$. Suppose further that there exist constants K , a , and M such that $|f(t)| \le$
\n Ke^{at} for $t \ge M$. Then $L{f'(t)}$ (to) exists for $s > a$, and moreover
\n $L{f'(t)} = sL{f(t)} - f(0)$.
\nTo prove this theorem we consider the integral
\n
$$
\int_0^4 e^{-st} f'(t) dt.
$$
\nIf f' has points of discontinuity in the interval $0 \le t \le A$, let them be denoted by
\n $t_1, t_2, ..., t_n$. Then we can write this integral as
\n
$$
\int_0^4 e^{-st} f'(t) dt = \int_0^{t_1} e^{-st} f'(t) dt + \int_{t_1}^{t_2} e^{-st} f'(t) dt + \cdots + \int_{t_n}^A e^{-st} f'(t) dt.
$$
\nIntegrating each term on the right by parts yields
\n
$$
\int_0^A e^{-st} f'(t) dt = e^{-st} f(t) \int_{t_1}^{t_2} + \cdots + e^{-st} f(t) \Big|_{t_a}^A
$$
\n
$$
+ s \left[\int_0^{t_1} e^{-st} f(t) dt + \int_{t_1}^{t_2} e^{-st} f(t) dt + \cdots + \int_{t_n}^A e^{-st} f(t) dt \right].
$$
\nSince f is continuous, the contributions of the integrated terms at $t_1, t_2, ..., t_n$ cancel.
\nCombining the integrals gives
\n
$$
\int_0^A e^{-st} f'(t) dt = e^{-st} f(A) - f(0) + s \int_0^A e^{-st} f(t) dt.
$$
\nAs $A \rightarrow \infty$, $e^{-st} f'(A) \rightarrow 0$ whenever $s > a$. Hence, for $s > a$,
\n
$$
L{f'(t)} = sL{f'(t)} - f(0),
$$
\nwhich establishes the theorem.
\n
$$
\int_0^A e^{-st} f'(t) dt = e^{-st}
$$

 t_2, \ldots, t_n cancel.

$$
\int_0^A e^{-st} f'(t) dt = e^{-sA} f(A) - f(0) + s \int_0^A e^{-st} f(t) dt.
$$

As $A \to \infty$, $e^{-sA} f(A) \to 0$ whenever $s > a$. Hence, for $s > a$,

$$
\mathcal{L}{f'(t)} = s\mathcal{L}{f(t)} - f(0),
$$

 $+ s \left[\int_0^{t_1} e^{-st} f(t) dt + \int_{t_1}^{t_2} e^{-st} f(t) dt + \cdots + \int_{t_n}^{t_n} e^{-st} f(t) dt \right].$
Since f is continuous, the contributions of the integrated terms at t_1, t_2, \ldots, t_n cancel.
Combining the integrals gives
 $\int_0^A e^{-st} f'(t) dt = e^{-sA} f(A) \int_{t_1}^{t_2} e^{-st} f(t) dt + \cdots + \int_{t_n}^{A} e^{-st} f(t) dt$

he integrated terms at t_1, t_2, \ldots, t_n cancel.
 $-f(0) + s \int_0^A e^{-st} f(t) dt$.

a. Hence, for $s > a$,
 $f(t) = f(0)$,

hat are imposed on f and f' , respectively,

place transform of is piecewise Table F is continuous, the contributions of the integrated terms at $t_1, t_2, ..., t_n$ cancel.
Combining the integrats gives
Combining the integrats gives
 $\int_0^A e^{-st} f'(t) dt = e^{-s \cdot t} f'(A) - f(0) + s \int_0^A e^{-st} f'(t) dt$.
As $A \to \infty, e^{-s \cdot t}$

$$
\mathcal{L}\{f''(t)\} = s^2 \mathcal{L}\{f(t)\} - sf(0) - f'(0). \tag{2}
$$

Corollary 6.2.2

itial Value Problems

\n**301**

\nand *M* such that
$$
|f(t)| \leq Ke^{at}, |f'(t)| \leq Ke^{at}, \ldots, |f^{(n-1)}(t)| \leq Ke^{at} \text{ for } t \geq M.
$$

\nThen $\mathcal{L}\{f^{(n)}(t)\}$ exists for $s > a$ and is given by

\n
$$
\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0).
$$
\n(3)

\nWe now show how the Laplace transform can be used to solve initial value problems.

\nIt is most useful for problems involving nonhomogeneous differential equations as we

$$
\mathcal{L}\lbrace f^{(n)}(t)\rbrace = s^n \mathcal{L}\lbrace f(t)\rbrace - s^{n-1} f(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0). \tag{3}
$$

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 $\leq Ke^{at}, \ldots, |f^{(n-1)}(t)| \leq Ke^{at}$ for $t \geq M$.

given by
 $f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$. (3)

orm can be used to solve initial value problems.

monhomogeneous differential equations, as we

chapter. However, we be M Value Problems

M Value Problems

M Such that $|f(t)| \leq Ke^{at}$, $|f'(t)| \leq Ke^{at}$, \ldots , $|f^{(m-1)}(t)| \leq Ke^{at}$ for $t \geq M$,
 $\mathcal{L}{f^{(m)}(t)} = s^n \mathcal{L}{f(t)} = s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$. (3)

We now show how the Laplace **301**
 Ital Value Problems
 Innolving that $|f(t)| \leq Ke^{at}$ **,** $|f''(t)| \leq Ke^{at}$, $\dots, |f^{(n-1)}(t)| \leq Ke^{at}$ for $t \geq M$.

Then $\mathcal{L}(f^{(n)}(t))$ exists for $s > a$ and is given by
 $\mathcal{L}(f^{(n)}(t)) = s^n \mathcal{L}(f(t)) = s^{n-1} f(0) - \dots - s f^{(n-$ **301**

and *M* such that $|f(t)| \leq Ke^{gt}$, $|f'(t)| \leq Ke^{gt}$, ..., $|f^{(n-1)}(t)| \leq Ke^{at}$ for $t \geq M$.

Then $\mathcal{L}{f^{(n)}(t)}$ exists for $s > a$ and is given by
 $\mathcal{L}{f^{(n)}(t)} = s^n \mathcal{L}{f(t)} = s^{n-1}f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$. (**SO1**
 SO1
 Comography
 Comography equation **301**
 $f'(t) \leq Ke^{at}, \dots, |f^{(n-1)}(t)| \leq Ke^{at}$ for $t \geq M$.

and is given by
 $s^{n-1} f(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)$. (3)

ransform can be used to solve initial value problems.

this chapter. However, we begin by looking at so **SOI**
 Also Problems
 Also Problems
 Also Problems
 Also Problems
 Also Problems
 Also Problems
 $\mathcal{L}{f^{(n)}(t)} \leq Ke^{at}, {f^{(r)}(t)} \leq Ke^{at}, \dots, {f^{(n-1)}(t)} \leq Ke^{at}$ for $t \geq M$.

Then $\mathcal{L}{f^{(n)}(t)} = s^n \mathcal{L}{f(t)} = s^{n-$ **Solution**
 Solution
 Solution
 Solution
 Solution
 Solution
 C($f^{(n)}(t)$) exists for $s > a$ and is given by
 $\mathcal{L}(f^{(n)}(t)) = s^n \mathcal{L}(f(t)) - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$. (3)

We now show how the Laplac **SO1**
 Solution *M* such that $|f(t)| \leq Ke^{at}$, $|f''(t)| \leq Ke^{at}$, ..., $|f^{(n-1)}(t)| \leq Ke^{at}$ for $t \geq M$.

Then $\mathcal{L}{f^{(n)}(t)}$ exists for $s > a$ and is given by
 $\mathcal{L}{f^{(n)}(t)} = s^n \mathcal{L}{f(t)} - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1$ **301**
 and M such that $|f(t)| \leq Ke^{at}, |f'(t)| \leq Ke^{at}, \ldots, |f^{(n-1)}(t)| \leq Ke^{at}$ for $t \geq M$.

Then $\mathcal{L}{f^{(n)}(t)}$ exists for $s > a$ and is given by
 $\mathcal{L}{f^{(n)}(t)} = s^n \mathcal{L}{f(t)} = s^{n-1}f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$. (3)

We no Then $\mathcal{L}{f^{(n)}(t)}$ exists for $s > a$ and is given by
 $\mathcal{L}{f^{(n)}(t)} = s^n \mathcal{L}{f(f)} = s^{n-1}f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$. (3)

We now show how the Laplace transform can be used to solve initial value problems.

It is mo $f^{(n-1)}(0)$. (3)

initial value problems.

ential equations, as we

egin by looking at some

onsider the differential

(4)

(5)

3.1. The characteristic

(6)

(7)

1 and $-c_1 + 2c_2 = 0$;

problem (4) and (5) is

(8)

orm. $\mathcal{E}(f(t)) = s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0).$ (3)

Laplace transform can be used to solve initial value problems.

ems involving nonhomogeneous differential equations, as we

exections of this chapter. However, we beg We now show how the Laplace transform can be used to solve initial value problems.

In demonstrate in later screions of this ehapter. However, we begin by looking at some

mogeneous equations, which are a bit simpler. For We now show the Laplace transform can be used to solve mital value problems,

We not susted in for problems involving monhomogeneous differential equations, as we

will demonstrate in later sections of this chapter. Howev

$$
y'' - y' - 2y = 0 \tag{4}
$$

$$
y(0) = 1, \qquad y'(0) = 0. \tag{5}
$$

$$
r^2 - r - 2 = (r - 2)(r + 1) = 0,\t\t(6)
$$

$$
y = c_1 e^{-t} + c_2 e^{2t}.
$$
 (7)

 $c_2 = 1$ and $-c_1 + 2c_2 = 0$; 0; hence $c_1 = \frac{2}{3}$ and $c_2 = \frac{1}{3}$, so that the solution of the initial value

$$
y = \phi(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}.
$$
 (8)

It is most useful for problems involving nonhomogeneous differential equations, as we
will demonstrate in late reccions of this chapter. However, we begin by looking at some
homogeneous equations, which are a bit simpler. will demonstrate in later sectoros of this chapter. However, we begin by looking at some
homogeneous equations, which are a bit simpler. For example, consider the differential
equation
and the initial conditions
 $y(0) = 1$ and the initial conditions
 $y'' - y' - 2y = 0$ (4)

This problem is easily solved by the methods of Section 3.1. The characteristic

equation is
 $r^2 - r - 2 = (r - 2)(r + 1) = 0$, (6)

and consequently the general solution of Eq. (4 and the initial conditions
 $y(0) = 1$, $y'(0) = 0$. (5)

This problem is easily solved by the methods of Section 3.1. The characteristic

equation is
 $r^2 - r - 2 = (r - 2)(r + 1) = 0$, (6)

and consequently the general solution of and the mittal conditions
 $y(0) = 1$, $y'(0) = 0$. (5)

This problem is easily solved by the methods of Section 3.1. The characteristic

equation is
 $r^2 - r - 2 = (r - 2)(r + 1) = 0$, (6)

and consequently the general solution of and consequently the general solution of Eq. (4) is
 $y = c_1e^{-t} + c_2e^{2t}$. (7)

To satisfy the initial conditions (5) we must have $c_1 + c_2 = 1$ and $-c_1 + 2c_2 = 0$;

hence $c_1 = \frac{2}{3}$ and $c_2 = \frac{1}{3}$, so that the solut and consequently the general solution of Eq. (4) is
 $y = c_1 e^{-t} + c_2 e^{-2t}$. (7)

To satisfy the initial conditions (5) we must have $c_1 + c_2 = 1$ and $-c_1 + 2c_2 = 0$;

hence $c_1 = \frac{2}{3}$ and $c_2 = \frac{1}{3}$, so that the solut hence $c_1 = \frac{2}{3}$ and $c_2 = \frac{1}{3}$, so that the solution of the initial value problem (4) and (5) is
 $y = \phi(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}$. (8)

Now let us solve the same have howein by using the Laplace transform. To do $y = \phi(t) = \frac{2}{3}e^{-t} + \frac{1}{2}e^{2t}$. (8)

Show let us solve the same problem hy using the Laplace transform. To do this we must

assume that the problem has a solution $y = \phi(t)$, which with its first two derivatives

differe Now let us solve the same problem by window the transform. To do this we must
assume that the problem has a solution $y = \phi(t)$, which with its first two derivatives
assistics the conditions of Corollary 6.2.2. Then, taking

$$
\mathcal{L}{y''} - \mathcal{L}{y'} - 2\mathcal{L}{y} = 0,\t\t(9)
$$

$$
s^{2}\mathcal{L}{y} - sy(0) - y'(0) - [s\mathcal{L}{y} - y(0)] - 2\mathcal{L}{y} = 0,
$$

$$
s2 - s - 2Y(s) + (1 - s)y(0) - y'(0) = 0,
$$
\n(10)

$$
Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}.
$$
\n(11)

Chapter 6. The Laplace Transform
This can be done most easily by expanding the right side of Eq. (11) in partial fractions. Thus we write

$$
Y(s) = \frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1)+b(s-2)}{(s-2)(s+1)},
$$
(12)
where the coefficients *a* and *b* are to be determined. By equating numerators of the second and fourth members of Eq. (12), we obtain

$$
s - 1 = a(s + 1) + b(s - 2),
$$

Chapter 6. The Laplace Transform
This can be done most easily by expanding the right side of Eq. (11) in partial
fractions. Thus we write
 $Y(s) = \frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1)+b(s-2)}{(s-2)(s+1)}$, (12)
where the **Chapter 6. The Laplace Transform**

This can be done most easily by expanding the right side of Eq. (11) in partial

fractions. Thus we write
 $Y(s) = \frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1) + b(s-2)}{(s-2)(s+1)}$, (12)

where **Chapter 6. The Laplace Transform**

This can be done most easily by expanding the right side of Eq. (11) in partial

fractions. Thus we write
 $Y(s) = \frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1)+b(s-2)}{(s-2)(s+1)}$, (12)

where $a = \frac{1}{3}$. Similarly, if we set $s = -1$, then we find that $b = \frac{2}{3}$. By substituting these **Chapter 6. The Laplace Transform**

Similar Similar Similar Similarly, by expanding the right side of Eq. (11) in partial

Similarly, if $S(s) = \frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1)+b(s-2)}{(s-2)(s+1)}$, (12)

the coeffici The Laplace Transform

b. of Eq. (11) in partial
 $\frac{b+b(s-2)}{2(s+1)}$, (12)

thing numerators of the

= 2, then it follows that

By substituting these

(13)

bllows that $\frac{1}{3}e^{2t}$ has the **Chapter 6. The Laplace Transform**

This can be done most easily by expanding the right side of Eq. (11) in partial

fractions. Thus we write
 $Y(s) = \frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1)+b(s-2)}{(s-2)(s+1)}$, (12)

where

$$
Y(s) = \frac{1/3}{s - 2} + \frac{2/3}{s + 1}.
$$
 (13)

Chapter 6. The Laplace Transform

This can be done most easily by expanding the right side of Eq. (11) in partial
 $Y(s) = \frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1)+b(s-2)}{(s-2)(s+1)}$. (12)

where the coefficients a and b a transform $\frac{1}{3}(s-2)^{-1}$; similarly, $\frac{2}{3}e^{-t}$ has the transform $\frac{2}{3}(s+1)^{-1}$. Hence, by the Fransform

in partial
 \therefore (12)

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(13)
 e^{2t} has the

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em (4), (5). **Chapter 6. The Laplace Transform**

most easily by expanding the right side of Eq. (11) in partial
 $\frac{is-1}{s-2(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1)+b(s-2)}{(s-2)(s+1)}$, (12)
 s a and *b* are to be determined. By equating numera **Chapter 6. The Laplace Transform**

expanding the right side of Eq. (11) in partial
 $\frac{a}{-2} + \frac{b}{s+1} = \frac{a(s+1) + b(s-2)}{(s-2)(s+1)}$, (12)

be determined. By equating numerators of the

be determined. By equating numerators **Example 12**
 $\frac{(-2)}{1}$, (12)

Interactors of the

Interactors of the

Interactors of the

Interactors the
 $\frac{1}{3}e^{2t}$ has the

Hence, by the

Problem (4), (5). **Chapter 6. The Laplace Transform**

This can be done most easily by expanding the right side of Eq. (11) in partial

fractions. Thus we write
 $Y(s) = \frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1)+b(s-2)}{(s-2)(s+1)}$, (12)

where **Chapter 6. The Laplace Transform**

This can be donc most easily by expanding the right side of Eq. (11) in partial
 $Y(s) = \frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1)+b(s-2)}{(s-2)(s+1)}$. (12)

where the coefficients at and b This can be done most easily by expanding the right side of Eq. (11) in partial
fractions. Thus we write
 $Y(s) = \frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1)+b(s-2)}{(s-2)(s+1)}$, (12)
where the coefficients *a* and *b* are to be This can be done most easily by expanding the right side of Eq. (11) in partial

ticitions. Thus we write
 $Y(s) = \frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1)+b(s-2)}{(s-2)(s+1)}$, (12)

here the coefficients a and b are to be de fractions. Thus we write
 $Y(s) = \frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1)+b(s-2)}{(s-2)(s+1)}$, (12)

where the coefficients a and b are to be determined. By equating numerators of the

second and fourth members of Eq. (12), where the coefficients a and b are to be determined. By equating numerators of the
second and fourth members of Eq. (12), we obtain
an equation that must hold for all s. In particular, if we set $s = 2$, then it follows th where the coefments a and θ are to be determined. By equating numerators of the second and fourth members of Eq. (12), we obtain

an equation that must hold for all s. In particular, if we set $s = 2$, then it follows t an equation that must hold for all s. In particular, if we set $s = 2$, then it follows that $a = \frac{1}{3}$. Similarly, if we set $s = -1$, then we find that $b = \frac{2}{3}$. By substituting these values for a and b, respectively,

$$
y = \phi(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}
$$

$$
ay'' + by' + cy = f(t).
$$
 (14)

$$
a[s2Y(s) - sy(0) - y'(0)] + b[sY(s) - y(0)] + cY(s) = F(s),
$$
 (15)

$$
Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}.
$$
 (16)

Finally, if we use the result of Example 5 of Section 6.1, it follows that $\frac{1}{4}e^{2t}$ has the
transform $\frac{1}{3}(s-2)^{-1}$; similarly, $\frac{2}{3}e^{-t}$ has the transform $\frac{2}{3}(s+1)^{-1}$. Hence, by the
tinearity of the Lap Finally, if we use the result of Example 5 of Section 6.1, it follows that $\frac{1}{2}e^{2t}$ has the transform $\frac{1}{3}(s-2)^{-1}$; similarly, $\frac{2}{3}e^{-t}$ has the transform $\frac{2}{3}(s+1)^{-1}$. Hence, by the linearity of the Lap Francy, If we use the research Examples 5 of Section 6.1, it follows that $\frac{2}{3}e^{-x}$ has the transform $\frac{1}{3}(s-2)^{-1}$; similarly, $\frac{2}{3}e^{-x}$ has the transform $\frac{2}{3}(s+1)^{-1}$. Hence, by the linearity of the Lapla transform $\frac{1}{3}(s-2)^{-1}$; similarly, $\frac{1}{3}e^{-1}$ has the transform $\frac{2}{3}(s+1)^{-1}$. Hence, by the linearity of the Laplace transform,
 $y = \phi(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$

has the transform (13) and is therefore the so linearity of the Laplace transform,

linearity of the Laplace transform (4), and is therefore the solution of the initial value problem (4), (5).

Of course, this is the same solution that we obtained earlier.

The same p $y = \phi(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$

has the transform (13) and is therefore the solution of the initial value problem (4), (5).

Of course, this is the same solution that we obtained earlier.

The same procedure can be app has the transform (13) and is therefore the solution of the initial value problem (4), (5). Of course, this is the same solution that we obtained earlier.

The same procedure can be applied to the general second order lin that the transform is 3 (s). The sime of $\cos \theta$ the minimal value problem (4), (5). Of ourse, this is the same solution that we obtained earlier.

The same procedure can be applied to the general second order linear equ Of course, this is the same solution that we obtained carlier

The same procedure can be applied to the general second order linear equation with

constant coefficients,
 $ay'' + by' + cy = f(t)$. (14)

Assuming that the solution y The same procedure can be applied to the general second order linear equation with
constant coefficients,
 $ay'' + by' + cy = f(t)$. (14)
Assuming that the solution $y = \phi(t)$ satisfies the conditions of Corollary 6.2.2 for
 $n = 2$, we constant coefficients,
 $ay'' + by' + cy = f(t)$. (14)

Assuming that the solution $y = \phi(t)$ satisfies the conditions of Corollary 6.2.2 for
 $n = 2$, we can take the transform of Eq. (14) and thereby obtain
 $a[s^2Y(s) - sy(0) - y'(0)] + b[sY(s)$ $ay'' + by' + cy = f(t)$. (14)
 $n = 2$, we can take the transform of F_4 (14) and thereby obtain
 $n = 2$, we can take the transform of F_4 (14) and thereby obtain
 $a[s^2Y(s) - sy(0) - y'(0)] + b[s'Y(s) - y(0)] + cY(s) = F(s)$, (15)

where $F(s)$ is t Assuming that the solution $y = \phi(t)$ satisfies the conditions of Corollary 6.2.2 for $n = 2$, we can take the transform of Eq. (14) and thereby obtain $a[s^2Y(s) - sy(0) - y'(0)] + b[sY(s) - y(0)] + cY(s) = F(s)$. (15) where $F(s)$ is the transfo Assumming that the solution $y = \phi(t)$ satisfies the conditions of Corollary 6.2.2 for $n = 2$, we can take the transform of Eq. (14) and thereby obtain $a[s^2Y(s) - sy(0) - y'(0)] + b[sY(s) - y(0)] + cY(s) = F(s)$. (15) where $F(s)$ is the transf

11 Value Problems
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 Observe that the polynomial $as^2 + bs + c$ **in the denominator on the right side of**

1. (16) is precisely the characteristic polynomial associated with Eq. (14). Since the

1. (16) is precisely the **303**
bs + c in the denominator on the right side of
polynomial associated with Eq. (14). Since the
 $c(s)$ to determine $\phi(t)$ requires us to factor this
ms does not avoid the necessity of finding roots
tions of higher tha **Exercise 16)**
 Exerce that the polynomial $as^2 + bs + c$ **in the denominator on the right side of**

Eq. (16) is precisely the characteristic polynomial associated with Eq. (14). Since the

use of a partial fraction expansion **Userable of a particular conduct** $\mathbf{B}^{2} + bs + c$ in the denominator on the right side of C by is precisely the characteristic polynomial associated with Eq. (14). Since the BEq. (16) is precisely the characteristic poly **Example 1908**
 Conserve that the polynomial $as^2 + bs + c$ in the denominator on the right side of

Eq. (16) is precisely the characteristic polynomial associated with Eq. (14). Since the

the use of a partial fraction expa **Solution**
 Solution Value Problems
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Observe that the polynomial $as^2 + bs + c$ in the denominator on the right side of

Eq. (16) is precisely the characteristic polynomial associated with Eq. (14). Since the

su **303**
 Conserve that the polynomial $as^2 + bs + c$ **in the denominator on the right side of**

Eq. (16) is precisely the characteristic polynomial associated with Eq. (14). Since the

tase of a partial fraction expansion of $Y(s$

Solution Solution Solution $ds^2 + bs + c$ in the denominator on the right side of (. (16) is precisely the characteristic polynomial associated with Eq. (14). Since the of a partial fraction expansion of $Y(s)$ to determi **303**
 Conserve that the polynomial $as^2 + bs + c$ in the denominator on the right side of

Eq. (16) is precisely the characteristic polynomial associated with Eq. (14). Since the

use of a partial fraction expansion of $Y(s)$ **303**
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 51. (16) is precisely the elaracteristic polynomial associated with Eq. (14). Since the Eq. (16) is precisely the elaracteristic polynomial associated with Eq. (14). Since the polyno **303**
 Conserve that the polynomial $as^2 + bs + c$ **in the denominator on the right side of**

Eq. (16) is precisely the characteristic polynomial associated with Eq. (14). Since the

use of a partial fraction expansion of $Y(s)$ **SOS**
 CONSECT TO THE CONSECT TERM THE TRANSIFY (19) the state of particle is the notation expected to the notation of $Y(s)$ to determine $\phi'(t)$ requires us to fact and from the polynomial, the use of Laplace transform **1 Value Problems** 3008
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 Sobserve that the polynomial $as^2 + bs + c$ **in the denominator on the right side of** (16) **is precisibly the characteristic polynomial associated with Eq. (14). Since the is** (16) **is itial Value Problems 303**
 Conserve that the polynomial $as^2 + bs + c$ **in the denominator on the right side of** Eq. (16) is precisely the characteristic polynomial associated with Eq. (14). Since the suse of a partial frac **Compleme**
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 Software that the polynomial $as^2 + bs + c$ **in the denominator on the right side of**
Eq. (16) is precisely the characteristic polynomial associated with Eq. (14). Since the

Buc of a partial f **303**
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 Observe that the polynomial $as^2 + bs + c$ **in the denominator on the right side of**

Eq. (16) is precisely the characteristic polynomial associated with Eq. (14). Since the
 S108 Buese of a partial fr **303**
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 51. (16) is precisely the characteristic polynomial associated with Eq. (14). Since the Eq. (16) is precisely the characteristic polynomial associated with Eq. (14). Since the u **In Value Problems 303**
 IDENT CAST (10) is precisely the characteristic polynomial associated with Eq. (10), Since the considerably the characteristic polynomial associated with Eq. (14), Since the constrained problem, whether there may be functions other than the one given by Eq. (8) that also have **Consert of the problems 303**
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 Conserve that the polynomial $as^2 + bs + c$ **in the denominator on the right side of the Eq. (16) is precisely the characteristic colynomial associated with Eq. (14). Since the suese of SO3**
 Conserve that the polynomial $as^2 + bs + c$ **in the denominator on the right side of** Eq. (16) is precisely the characteristic polynomial associated with Eq. (14). Since the Suse of a partial fraction expansion of $Y(s)$

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 Starl Value Problems

Observe that the polynomial $as^2 + bs + c$ in the denominator on the right side of

Eq. (16) is precisely the characteristic polynomial associated with Eq. (14). Since the

there is essent of a Observe that the polynomial $as^2 + bs + c$ in the denominator on the right side of
Eq. (16) is precisely the characteristic polynomial associated with Eq. (14). Since the
sust of a partial fraction expansion of Y (s) to determ Observe that the polynomial $as^2 + bs + c$ in the denominator on the right is de of
Eq. (16) is precisely the characteristic polynomial associated with Eq. (14). Since the
sus of a partial fraction expansion of $Y(s)$ to determ Eq. (16) is precisely the characteristic polymomial associated with Eq. (14). Since the entries in eq. (16) is precise to a find proportional, the use of Laplace transforms does not avoid the necessity of finding roots of use of a partial fraction expansion of Y (s) to determine $\phi(t)$ requires us to factor this column of the characteristic equation. For equations of higher than second order this may be a The characteristic equation. For e polynomial, the use of Laplace transforms does not avoid the necessivy of finding roots of the characteristic equation. For equations of higher than second order this may be a difficult algebraic problem, particularly if of the characteristic equation. For equations of higher than second order this may be a differential equation. For equations of higher than exceptoding to technique lies in the problem of determining the fenomical $y = \phi(t)$ difficult algebraic problem, particularly if the roots are irrational or complex.
The main difficulty that occurs in solving initial value problems by the transform
technique lies in the problem of determining the functio The main fifficulty that occurs in solving initial value problems by the transform
technique lies in the problem of determining the function $y = \phi(t)$ corresponding to
the transform $V(s)$. This problem is known as the inve tchnique lies in the problem of determining the function $y = \phi(t)$ corresponding to $Y(\text{shm})$. This problem is known as the inversion problem for the laplace transform, $\phi(t)$ is called the inverse transform corresponding the transform $V(s)$. This problem is known as the inversion problem for the Laplace transform, $\phi(t)$ is called the inverse transform corresponding to $Y(s)$, and the process of finding $\phi(t)$ from $Y(s)$ is known as invert transform. $\phi(t)$ is called the inverse transform corresponding to $V(s)$, and the process
of finding $\phi(t)$ if could the inverse transform of $Y(s)$. There is a general formula for the
 $\mathcal{L}^{-1}(Y(s))$ to denote the inverse of finding $\phi(t)$ from $Y(s)$ is known as inverting the transform. We also use the notation
($\pi^{-1}(Y(s))$ to denote the inverse transform of $Y(s)$. There is a general formula for the
inverse Laplace transform, but its use re $\mathcal{L}^{-1}(Y(s))$ to denote the inverse transform of $Y(s)$. There is a general formula for the inverse Laplace transform, but its use requires a knowledge of the theory of functions of a complex variable, and we do not consi reres Laplace transform, but its use requires a knowledge of the theory of functions and more in the sum primerating problems, without the use of complex variable, and we do not consider it in this book. However, it is st may increasing processins, wundow the use to comprex variances.
In solving the initial value problem (4), (5) we did not consider the question of
the transform (13). In fact, it can be shown that if f is a continuous func valuables.

Ind consider the question of

ten by Eq. (8) that also have

s a continuous function with

ous function having the same

one correspondence between

frequently encountered, and

1.1 are the transforms of those the Laplace transform F, then there is no other continuous function having the same
transform. In other words, there is essentially a one-to-one correspondence between
functions and their Laplace transforms. This fact sug transform. In other words, there is essentially a one-to-one correspondence between
functions and their Laphice turasforms. This field suggests the compilation of a table,
such as Table 6.2.1, giving the transforms of fun vice versa. The entires in the second column of Table 6.2.1 are the transforms of those or the particles for the second column. Thus, for example, if the transform is the second column. Thus, for example, if the ransform the first column. Perhaps more important, the functions in the first column are the present transform
the solution of a differential equation is known, the solution itself can often be found
the solution of a differential inverse transforms of those in the second column. Thus, for example, if the transform
is relation of a differential equation is known, the solution itself can often be found
merely by looking it up in the table. Some of t of the solution of a differential equation is known, the solution itself can often or for the calismic particular on the found in the function of the stamples, or appear as problems in Section 6.1, while others will be de

$$
F(s) = F_1(s) + F_2(s) + \dots + F_n(s).
$$
 (17)

 $f(t) = \mathcal{L}^{-1}{F_1(s)}, \dots, f_n(t) = \mathcal{L}^{-1}{F_n(s)}$. Then the function

$$
f(t) = f_1(t) + \dots + f_n(t)
$$

$$
\mathcal{L}^{-1}{F(s)} = \mathcal{L}^{-1}{F_1(s)} + \dots + \mathcal{L}^{-1}{F_n(s)};
$$
 (18)

		Chapter 6. The Laplace Transform
TABLE 6.2.1 Elementary Laplace Transforms		
$f(t) = \mathcal{L}^{-1}{F(s)}$	$F(s) = \mathcal{L}{f(t)}$	Notes
1.1	$\frac{1}{s}$, $\sqrt{s} > 0$	Sec. 6.1; Ex. 4
$2. e^{at}$	s > a $s - a$	Sec. 6.1; Ex. 5
$n =$ positive integer $3. t^n$;	$\frac{n!}{s^{n+1}},$ $\sqrt{s} > 0$	Sec. 6.1; Prob. 27
4. t^p , $p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}},$ $\sqrt{s} > 0$	Sec. 6.1; Prob. 27
5. \sin{at}	$\frac{a}{s^2 + a^2},$ $\sqrt{s} > 0$	Sec. 6.1; Ex. 6
$6. \cos at$	$\sqrt{s} > 0$ s^2 + $3 + u$	Sec. 6.1; Prob. 6
7. $sinh at$	$rac{a}{s^2 - a^2}$, $s > a $	Sec. 6.1; Prob. 8
8. $\cosh at$	$\frac{s}{s^2-a^2}$, $s > a $	Sec. 6.1; Prob. 7
9. e^{at} sin bt	$\frac{b}{(s-a)^2 + b^2}$, $s > a$	Sec. 6.1; Prob. 13
10. e^{at} cos bt	$\frac{s-a}{(s-a)^2 + b^2}$, $s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}$, $n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$rac{e^{-cs}}{s}$, $s > 0$	Sec. 6.3
13. $u_c(t) f(t - c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct} f(t)$	$F(s - c)$	Sec. 6.3
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \qquad c>0$	Sec. 6.3; Prob. 19
16. $\int_0^t f(t-\tau)g(\tau) d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t - c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 28

EXAMPLE 1

Connection, and a general result covering many cases is given in Problem 38. Other
suseful properties of Laplace transforms are derived later in this chapter.
As further illustrations of the technique of solving initial **1997**
 1997 As further as further illustrations of the technique of solving initial value problem 38. Other
the properties of Laplace transforms are derived later in this chapter.
As further illustrations of the technique of solvin the Using Problems

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connection, and a general result covering many cases is given in Problem 38. Other

useful properties of Laplace transforms are derived later in this chapter.

As further illustrat **305**
 Example 10
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sovering many cases is given in Problem 38. Other

proms are derived later in this chapter.

mique of solving initial value problems by means of

action expansions, consider the following examples.

lequation
 $y'' +$

$$
y'' + y = \sin 2t,\tag{19}
$$

$$
y(0) = 2, \qquad y'(0) = 1. \tag{20}
$$

SIDS
 SIDS We allow **Problems**

We are approximated result covering many cases is given in Problem 38. Other

trial properties of Laplace transforms are derived later in this chapter.

As further illustrations of the technique of so **first Value Problems**
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connection, and a general result covering many cases is given in Problem 38. Other

useful properties of Laplace transforms are derived later in this chapter.

As further illustrations of th **305**
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 **accredifferential of Laplace transforms are derived later in this chapter.

As further illustrations of the technique of solving initial value problems by m Solution**
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 Example 6.1. Caplace transform and partial fraction **Solution**
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 Solution: **Solution**: **Solution**: **Solution**: **Solution**: **Solution**
 Solution: **Solution**: **Solution**
 Find the solution of the differential eq Subsettion that is electron to the entropy the section of the the form of the form of the technique of solving initial value problems by means of the Leplace transform and partial fraction expansions, consider the follow Find the solution of the differential equation
 $y'' + y = \sin 2t$, (19)

satisfying the initial conditions
 $y(0) = 2$, $y'(0) = 1$. (20)

We assume that this initial value problem has a solution $y = \phi(t)$, which with its

first t Find the solution of the differential equation
 $y'' + y = \sin 2t$, (19)

satisfying the initial conditions
 $y(0) = 2$, $y'(0) = 1$. (20)

We assume that this initial value problem has a solution $y = \phi(t)$, which with its

first t satisfying the initial conditions
 $y(0) = 2$, $y'(0) = 1$. (20)

We assume that this initial value problem has a solution $y = \phi(t)$, which with its

first two derivatives satisfies the conditions of Corollary 6.2.2. Then, ta

$$
s^{2}Y(s) - sy(0) - y'(0) + Y(s) = 2/(s^{2} + 4),
$$

$$
Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2 + 1)(s^2 + 4)}.
$$
\n(21)

first two derivatives satisfies the conditions of Corollary 6.2.2. Then, taking the Laplace transform of the differential equation, we have
\n
$$
s^2Y(s) - sy(0) - y'(0) + Y(s) = 2/(s^2 + 4)
$$
,
\nwhere the transform of sin 2t has been obtained from line 5 of Table 6.2.1. Substituting
\nfor y(0) and y'(0) from the initial conditions and solving for Y(s), we obtain
\n
$$
Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2 + 1)(s^2 + 4)}
$$
\n
$$
Y(s) = \frac{as + b}{s^2 + 1} + \frac{cs + d}{s^2 + 4} = \frac{(as + b)(s^2 + 4) + (cs + d)(s^2 + 1)}{(s^2 + 1)(s^2 + 4)}
$$
\nBy expanding the numerator on the right side of Eq. (22) and equating it to the numerator
\nin Eq. (21) we find that
\n
$$
2s^3 + s^2 + 8s + 6 = (a + c)s^3 + (b + d)s^2 + (4a + c)s + (4b + d)
$$
\nfor all s. Then, comparing coefficients of like powers of s, we have
\n
$$
a + c = 2, \qquad b + d = 1,
$$
\n
$$
4a + c = 8, \qquad 4b + d = 6.
$$
\nConsequently, $a = 2, c = 0, b = \frac{5}{3}$, and $d = -\frac{2}{5}$, from which it follows that
\n
$$
Y(s) = \frac{2s}{s^2 + 1} + \frac{5/3}{s^2 + 1} - \frac{2/3}{s^2 + 4}.
$$
\n(23)
\nFrom lines 5 and 6 of Table 6.2.1, the solution of the given initial value problem is
\n
$$
y = \phi(t) = 2 \cos t + \frac{5}{3} \sin t - \frac{1}{3} \sin 2t.
$$
\n(24)
\nFind the solution of the initial value problem
\n
$$
y^w - y = 0,
$$
\n(25)
\n
$$
y(0) = 0, \qquad y'(0) = 1, \qquad y''(0) = 0, \qquad y'''(0) = 0.
$$

 $2s^3 + s^2 + 8s + 6 = (a + c)s^3 + (b + d)s^2 + (4a + c)s + (4b + d)$

$$
a + c = 2,
$$
 $b + d = 1,$
\n $4a + c = 8,$ $4b + d = 6.$

Consequently, $a = 2$, $c = 0$, $b = \frac{5}{3}$, and $d = -\frac{2}{3}$, from which it follows that

$$
Y(s) = \frac{2s}{s^2 + 1} + \frac{5/3}{s^2 + 1} - \frac{2/3}{s^2 + 4}.
$$
 (23)

$$
y = \phi(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t.
$$
 (24)

$$
y^{\text{iv}} - y = 0,\tag{25}
$$

$$
y(0) = 0,
$$
 $y'(0) = 1,$ $y''(0) = 0,$ $y'''(0) = 0.$ (26)

EXAMPLE $\mathbf{2}$

Chapter 6. The Laplace Transform
In this problem we need to assume that the solution $y = \phi(t)$ satisfies the conditions
Corollary 6.2.2 for $n = 4$. The Laplace transform of the differential equation (25) is
 $s^4Y(s) - s^3y($ **Chapter 6. The Laplace Transform**

In this problem we need to assume that the solution $y = \phi(t)$ satisfies the conditions

of Corollary 6.2.2 for $n = 4$. The Laplace transform of the differential equation (25) is
 $s^4 Y(s)$ **Chapter 6. The Laplace Transform**

In this problem we need to assume that the solution $y = \phi(t)$ satisfies the conditions

of Corollary 6.2.2 for $n = 4$. The Laplace transform of the differential equation (25) is
 $s^4Y(s)$ **Chapter 6. The Laplace Transform**

In this problem we need to assume that the solution $y = \phi(t)$ satisfies the conditions

of Corollary 6.2.2 for $n = 4$. The Laplace transform of the differential equation (25) is
 $s^4 Y(s)$ **Chapter 6. The Laplace Transform**

In this problem we need to assume that the solution $y = \phi(t)$ satisfies the conditions

of Corollary 6.2.2 for $n = 4$. The Laplace transform of the differential equation (25) is
 $s^4 Y(s)$

$$
s^{4}Y(s) - s^{3}y(0) - s^{2}y'(0) - sy''(0) - y'''(0) - Y(s) = 0.
$$

$$
Y(s) = \frac{s^2}{s^4 - 1}.
$$
 (27)

$$
Y(s) = \frac{as+b}{s^2-1} + \frac{cs+d}{s^2+1},
$$

$$
(as+b)(s2+1) + (cs+d)(s2-1) = s2
$$
 (28)

equations

$$
2(a+b) = 1, \qquad 2(-a+b) = 1,
$$

Chapter 6. The Laplace Transform

In this problem we need to assume that the solution $y = \phi(t)$ satisfies the conditions

of Corollary 6.2.2 for $n = 4$. The Laplace transform of the differential equation (25) is
 $s^4 Y(s)$ **Chapter 6. The Laplace Transform**

In this problem we need to assume that the solution $y = \phi(t)$ satisfies the conditions

of Corollary 6.2.2 for $n = 4$. The Laplace transform of the differential equation (25) is
 $s^4Y(s)$ $\frac{1}{2}$. If we set $s = 0$ in Eq. (28), then $b - a$ **Chapter 6. The Laplace Transform**

ssume that the solution $y = \phi(t)$ satisfies the conditions

he Laplace transform of the differential equation (25) is
 $-s^2y'(0) - sy''(0) - y''(0) - Y(s) = 0$.

In (26) and solving for $Y(s)$, we ha 1 2^{\cdot} **Chapter 6. The Laplace Transform**

of Corollary 6.2.2 for $n = 4$. The Laplace transform of the differential equation (25) is
 $s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y''(0) - Y(s) = 0$.

Then, using the initial conditions (26) and sol In this problem we need to assume that the solution $y = \phi(t)$ satisfies the conditions
of Corollary 6.2.2 for $n = 4$. The Laplace transform of the differential equation (25) is
 $s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y''(0) - Y(s) = 0$.
 Then, using the initial conditions (26) and solving for $Y(s)$, we have
 $Y(s) = \frac{s^2}{s^4 - 1}$. (27)

A partial fraction expansion of $Y(s)$ is
 $Y(s) = \frac{s^2}{s^2 - 1} + \frac{cs + d}{s^2 + 1}$.

and it follows that
 $(as + b)(s^2 + 1) + (cs + d)(s^2 -$ Then, using the initial conditions (26) and solving for $Y(s)$, we have
 $Y(s) = \frac{s^2}{s^4 - 1}$. (27)

A partial fraction expansion of $Y(s)$ is
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and it follows that
 $(as + b)(s^2 + 1) + (cs + d)(s^2 -$

$$
Y(s) = \frac{1/2}{s^2 - 1} + \frac{1/2}{s^2 + 1},\tag{29}
$$

$$
y = \phi(t) = \frac{\sinh t + \sin t}{2}.
$$
 (30)

 $Y(s) = \frac{as+b}{s^2-1} + \frac{cs+d}{s^2+1}$.

(d it follows that
 $(as + b)(s^2 + 1) + (cs + d)(s^2 - 1) = s^2$ (28)

real s. By setting $s = 1$ and $s = -1$, respectively, in Eq. (28) we obtain the pair of
 $2(a + b) = 1$, $2(-a + b) = 1$.

d therefore $a = 0$ study of mechanical vibrations and in the analysis of electric circuits; the governing

study of mechanical vibrations and the analysis of electric circuits; the analysis of electric conditions

and therefore $a = 0$ and and it follows that
 $(as + b)(s^2 + 1) + (cs + d)(s^2 - 1) = s^2$ (28)

for all s. By setting $s = 1$ and $s = -1$, respectively, in Eq. (28) we obtain the pair of

equations
 $2(a + b) = 1$, $2(-a + b) = 1$,

and therefore $a = 0$ and $b = \frac{1}{2}$ and it follows that
 $(as + b)(s^2 + 1) + (cs + d)(s^2 - 1) = s^2$

(2)

for all s. By setting $s = 1$ and $s = -1$, respectively, in Eq. (28) we obtain the pair

equations
 $2(a + b) = 1$, $2(-a + b) = 1$,

and therefore $a = 0$ and $b = \frac{1}{2}$. equations
 $2(a + b) = 1$, $2(-a + b) = 1$,

and therefore $a = 0$ and $b = \frac{1}{2}$. If we set $s = 0$ in Eq. (28), then $b - d = 0$, so $d = \frac{1}{2}$.

Finally, equating the coefficients of the cubic terms on each side of Eq. (28), we f 2(a + b) = 1, 2(-a + b) = 1,
and therefore $a = 0$ and $b = \frac{1}{2}$. If we set $s = 0$ in Eq. (28), then $b - d = 0$, so $d = \frac{1}{2}$.
Finally, equating the coefficients of the cubic terms on each side of Eq. (28), we find
that and therefore $a = 0$ and $b = \frac{1}{2}$. If we set $s = 0$ in Eq. (28), then $b - d = 0$, so $d = \frac{1}{2}$.

Finally, equating the coefficients of the cubic terms on each side of Eq. (28), we find

that $a + c = 0$, so $c = 0$. Thus
 $\frac{1}{2}$
 $\frac{1}{2}$ $Y(s) = \frac{1/2}{s^2 - 1} + \frac{1/2}{s^2 + 1}$. (29)

and from lines 7 and 5 of Table 6.2.1 the solution of the initial value problem (25),

(26) is
 $y = \phi(t) = \frac{\sinh t + \sin t}{2}$. (30)

The most important elementary applications of the La $Y(s) = \frac{1}{s^2 - 1} + \frac{1}{s^2 + 1}$. (29)

and from lines 7 and 5 of Table 6.2.1 the solution of the initial value problem (25),

(26) is
 $y = \phi(t) = \frac{\sinh t + \sin t}{2}$. (30)

The most important elementary applications of the Laplac the solution of the initial value problem (25),
 $=\frac{\sinh t + \sin t}{2}$. (30)

plications of the Laplace transform are in the

the analysis of electric circuits; the governing

A vibrating spring-mass system has the equation
 $\$ of the initial value problem (25),
 $\frac{\sin t}{t}$. (30)

the Laplace transform are in the

of electric circuits; the governing

ring—mass system has the equation
 $F(t)$, (31)

k the spring constant, and $F(t)$

g an electric $y = \phi(t) = \frac{\sinh t + \sin t}{2}$. (30)

The most important elementary applications of the Laplace transform are in the

study of mechanical vibrations and in the analysis of electric circuits; the governing

equations were derived

$$
m\frac{d^2u}{dt^2} + \gamma \frac{du}{dt} + ku = F(t),\tag{31}
$$

$$
L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t),
$$
\n(32)

$$
L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}(t).
$$
 (33)

Solution 2.1 Value Problems
We have noted previously in Section 3.8 that Eq. (31) for the spring–mass system
d Eq. (32) or (33) for the electric circuit are identical mathematically, differing only
the interpretation of **307**

We have noted previously in Section 3.8 that Eq. (31) for the spring-mass system

and Eq. (32) or (33) for the electric circuit are identical mathematically, differing only

in the interpretation of the constants an **307**
 ideal Value Problems
 38 We have noted previously in Section 3.8 that Eq. (31) for the spring-mass system

and Eq. (32) or (33) for the electric circuit are identical mathematically, differing only

in the inter **First Value Problems 307**
We have noted previously in Section 3.8 that Eq. (31) for the spring-mass system
and Eq. (32) or (33) for the electric circuit are identical mathematically, differing only
in the interpretation **307**
We have noted previously in Section 3.8 that Eq. (31) for the spring–mass system
and Eq. (32) or (33) for the electric circuit are identical mathematically, differing only
in the interpretation of the constants and **307**
 **Corresponding to the Conference Corresponding in Section 3.8 that Eq. (31) for the spring-mass system

and Eq. (32) or (33) for the electric circuit are identical mathematically, differing only

in the interpretati In the problem s**
 In the problem s

We have noted previously in Section 3.8 that Eq. (31) for the spring-mass system

Id Eq. (32) or (33) for the electric circuit are identical mathematically, differing only

the inte **307**
We have noted previously in Section 3.8 that Eq. (31) for the spring-mass system
and Eq. (32) or the elective circuit are identical mathematically, differential
in the interpretation of the constants and variables a **Solution Solution Complemes**
 Complementation Coefficients Coefficients Coefficients Coefficients Coefficients and Eq. (32) or (33) for the electric circuit are identical mathematically, differing only in **307**
 Usual Value Problems
 We have noted previously in Section 3.8 that Eq. (31) for the spring-mass system

and Eq. (32) or (33) for the electric circuit are identical mathematically, differing only

in the interpr **Itial Value Problems 307**
We have noted previously in Section 3.8 that Eq. (31) for the spring-mass system
and Eq. (32) or (33) for the clearted circuit are identical mathematically, differing only
in the interpretation

PROBLEMS

conresponding physical problem is of immediate interest.

\nIn the problem lists following this and other sections in this chapter are numerous initial value problems for second order linear differential equations with constant coefficients. Many can be interpreted as models of particular physical systems, but usually we do not point this out explicitly.

\nIn each of Problems 1 through 10 find the inverse Laplace transform of the given function.

\n1.
$$
\frac{3}{s^2 + 4}
$$

\n2.
$$
\frac{4}{(s-1)^3}
$$

\n3.
$$
\frac{2}{s^2 + 3s - 4}
$$

\n4.
$$
\frac{3s}{s^2 - s - 6}
$$

\n5.
$$
\frac{2s + 2}{s^2 + 2s + 5}
$$

\n6.
$$
\frac{2s - 3}{s^2 - 4}
$$

\n7.
$$
\frac{2s + 1}{s^2 - 2s + 2}
$$

\n8.
$$
\frac{8s^2 - 4s + 12}{s(s^2 + 4)}
$$

\n9.
$$
\frac{1 - 2s}{s^2 + 4s + 5}
$$

\n10.
$$
\frac{2s - 3}{s^2 + 2s + 10}
$$

\nIn each of Problems 11 through 23 use the Laplace transform to solve the given initial value problem.

\n11.
$$
y'' - y' - 6y = 0; \quad y(0) = 1, \quad y'(0) = -1
$$

\n12.
$$
y'' + 3y' + 2y = 0; \quad y(0) = 1, \quad y'(0) = 0
$$

\n13.
$$
y'' - 2y' + 2y = 0; \quad y(0) = 1, \quad y'(0) = 0
$$

\n14.
$$
y'' - 4y' + 4y = 0; \quad y(0) = 1, \quad y'(0) = 1
$$

problem.

3.
$$
\frac{2}{s^2 + 3s - 4}
$$

\n4. $\frac{3s}{s^2 - s - 6}$
\n5. $\frac{2s + 2}{s^2 + 2s + 5}$
\n6. $\frac{2s - 3}{s^2 - 4}$
\n7. $\frac{2s + 1}{s^2 - 2s + 2}$
\n8. $\frac{8s^2 - 4s + 12}{s(s^2 + 4)}$
\n9. $\frac{1 - 2s}{s^2 + 4s + 5}$
\n10. $\frac{2s - 3}{s^2 + 2s + 10}$
\nIn each of Problems 11 through 23 use the Laplace transform to solve the given initial value problem.
\n11. $y'' - y' - 6y = 0$; $y(0) = 1$, $y'(0) = -1$
\n12. $y'' + 3y' + 2y = 0$; $y(0) = 1$, $y'(0) = 0$
\n13. $y'' - 2y' + 2y = 0$; $y(0) = 1$, $y'(0) = 0$
\n14. $y'' - 2y' + 2y = 0$; $y(0) = 1$, $y'(0) = 1$
\n15. $y'' - 2y' - 2y = 0$; $y(0) = 2$, $y'(0) = 1$
\n16. $y'' + 2y' + 5y = 0$; $y(0) = 2$, $y'(0) = 0$
\n17. $y^{10} - 4y' + 4y' + y = 0$; $y(0) = 2$, $y'(0) = 0$
\n18. $y^{10} - y = 0$; $y(0) = 1$, $y'(0) = 0$, $y''(0) = 1$, $y''(0) = 0$, $y'''(0) = 0$
\n19. $y^{10} - 4y = 0$; $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$, $y'''(0) = 0$
\n11. y^{10

24.
$$
y'' + 4y = \begin{cases} 1, & 0 \le t < \pi, \\ 0, & \pi \le t < \infty; \end{cases}
$$
 $y(0) = 1, y'(0) = 0$
25. $y'' + y = \begin{cases} t, & 0 \le t < 1, \\ 0, & 1 \le t < \infty; \end{cases}$ $y(0) = 0, y'(0) = 0$

- 26. $y'' + 4y = \begin{cases} t, & 0 \le t < 1, \\ 1, & 1 \le t < \infty; \end{cases}$ t, $0 \le t < 1$, $y(0) = 0$, $y'(0) = 0$
- **Chapter 6. The Laplace Transform**

26. $y'' + 4y = \begin{cases} t, & 0 \le t < 1, \\ 1, & 1 \le t < \infty; \end{cases}$ $y(0) = 0, y'(0) = 0$

27. The Laplace transforms of certain functions can be found conveniently from their Taylor

series consinues, $\sin t$ **Chapter 6. The Laplace Transform**

y'' + 4y = $\begin{cases} t, & 0 \le t < 1, \\ 1, & 1 \le t < \infty; \end{cases}$ y(0) = 0, y'(0) = 0

The Laplace transforms of certain functions can be found conveniently from their Taylor

series expansions.

(a) **Chapter 6. The Laplace Transform**
 $y'' + 4y = \begin{cases} t, & 0 \le t < 1, \\ 1, & 1 \le t < \infty; \end{cases}$
 $y(0) = 0, y'(0) = 0$

The Laplace transforms of certain functions can be found conveniently from their Taylor

series expansions.

(a) Using **Chapter 6. The Laplace Transform**
 $y'' + 4y = \begin{cases} 1, & 0 \le t < 1, \\ 1, & 1 \le t < \infty; \end{cases}$ $y(0) = 0, & y'(0) = 0$

The Laplace transforms of certain functions can be found conveniently from their Taylor

(i) Using the Taylor series **Chapter 6. The Laplace Transform**
 $y'' + 4y = \begin{cases} t, & 0 \le t < 1, \\ 1, & 1 \le t < \infty; \end{cases}$ $y(0) = 0, y'(0) = 0$

The Laplace transforms of certain functions can be found conveniently from their Taylor

series expansions.

(a) Using **Chapter 6. The Laplace Transform**
 $y'' + 4y = \begin{cases} t, & 0 \le t < 1, \\ 1, & 1 \le t < \infty; \end{cases}$ $y(0) = 0, y'(0) = 0$

The Laplace transforms of certain functions can be found conveniently from their Taylor

series expansions.

(a) Using **Chapter 6. The Laplace Transform**
 $y'' + 4y = \begin{cases} t_1 & 0 \le t < 1, \\ 1 \le t < \infty; \end{cases}$ $y(0) = 0, \quad y'(0) = 0$

The Laplace transforms of certain functions can be found conveniently from their Taylor

series expansions.

(a) Using t **Chapter 6. The Laplace Transform**
 $y'' + 4y = \begin{cases} t, & 0 \le t < 1, \\ 1, & 1 \le t < \infty; \end{cases}$ $y(0) = 0, y'(0) = 0$

The Laplace transforms of certain functions can be found conveniently from their Taylor

series expansions.

(a) Using **Chapter 6. The Laplace Transform**
 $y'' + 4y = \begin{cases} t, & 0 \le t < 1, \\ 1, & 1 \le t < \infty; \end{cases}$ $y(0) = 0, y'(0) = 0$

The Laplace transforms of certain functions can be found conveniently from their Taylor

series expansions.

(a) Using The Laplace Transform

0

weniently from their Taylor

e computed term by term,

Laplace transform of this

has the Taylor series (see

ed term by term, verify that $y'' + 4y = \begin{cases} t, & 0 \le t < 1, \\ 1, & 1 \le t < \infty; \end{cases}$ $y(0) = 0, y'(0) = 0$

The Laplace transforms of certain functions can be found conveniently from their Taylor

scribes expansions.

(a) Using the Taylor series for sin *t*,

(a)
	-

$$
\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!},
$$

$$
\mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1}, \qquad s > 1.
$$

$$
f(t) = \begin{cases} (\sin t)/t, & t \neq 0, \\ 1, & t = 0. \end{cases}
$$

$$
\mathcal{L}\{f(t)\} = \arctan(1/s), \qquad s > 1.
$$

series expansions,

(a) Using the Taylor series for sin t,
 $\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n-1}}{(2n+1)!}$,

and assuming that the Laplace transform of this series can be computed term by term,

verify that
 $\mathcal{L}[\sin t] = \frac{1}{s^2$ werify that
 $\mathcal{L}|\sin t| = \frac{1}{s^2 + 1}$, $s > 1$.

(b) Let
 $f(t) = \begin{cases} \sin t/2, & \text{if } t \neq 0, \\ 1, & \text{if } t = 0. \end{cases}$

Find the Taylor series for f about $t = 0$. Assuming that the Laplace transform of this

function can be compute

$$
J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n} (n!)^2}.
$$

$$
\mathcal{L}{J_0(t)} = (s^2 + 1)^{-1/2}, \qquad s > 1,
$$

and

$$
\mathcal{L}\{J_0(\sqrt{t})\} = s^{-1}e^{-1/4s}, \qquad s > 0.
$$

$$
F(s) = \int_0^\infty e^{-st} f(t) \, dt.
$$

 $\mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1}, \quad s > 1.$ (b) Let
 $f(t) = \begin{cases} (\sin t)/t, & t \neq 0, \\ 1, & t = 0. \end{cases}$

Find the Taylor series for f about $t = 0$. Assuming that the Laplace transform of this

function can be computed term by term, verify th $f(t) = \begin{cases} \sin t/t, & t \neq 0, \\ 1, & t = 0. \end{cases}$

Find the Taylor series for f about $t = 0$. Assuming that the Laplace transform of this function can be computed term by term, verify that
 $\mathcal{L}[f(t)] = \arctan(1/s), s > 1.$

Section 5.8) $f(t) =\begin{cases} \sin(t)/t, & t \neq 0, \\ 1, & t \neq 0, \end{cases}$

Find the Taylor series for f about $t = 0$. Assuming that the Laplace transform of this function can be computed term) term, verify that
 $\mathcal{L}(f(t)) = \arctan(1/s)$.

(c) The Bessel Find the Taylor series for f about $t = 0$. Assuming that the Laplace transform of this

function can be computed term by term, verify that
 $L(f(t)) = \arctan(1/s)$.

(c) The Bessel function of the first kind of order zero J_0 h term, verify that
 $= \arctan(1/s)$, $s > 1$.

inst kind of order zero J_0 has the Taylor series (see
 $J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n} (n!)^2}$.
 $= \arctan 50$ computed term by term, verify that
 $= (s^2 + 1)^{-1/2}$, $s > 1$,
 C(r) The Bessel function of the first kind of order zero J_0 has the Taylor series (see
Section 5.8)
 $J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n} (n!)^2}$.

Assuming that the following Laplace transforms can be computed term by (e) In Bessel tunction of the Inst kind of order zero J_0 has the Taylor series (see
Section 5.8)
 $J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^n (t!)^2}$.
Assuming that the following Laplace transforms and be computed term by term, v Assuming that the following Laplace transforms can be computed term by term, verify that
 $\mathcal{L}(J_0(t)) = (s^2 + 1)^{-1/2}, \qquad s > 1,$

and
 $\mathcal{L}(J_0(\sqrt{r})) = s^{-1}e^{-1/4s}, \qquad s > 0,$

Problems 28 through 36 are concerned with different e transforms can be computed term by term, verify that
 $=(s^2 + 1)^{-1/2},$ $s > 1,$
 $(s^2 + 1)^{-1/2},$ $s > 0.$

with differentiation of the Laplace transform.
 $s) = \int_0^\infty e^{-st} f(t) dt$.

as f satisfies the conditions of Theorem 6.

 $\mathcal{L}(s) = \mathcal{L}\{(-t)^n f(t)\}\;$ hence differentiating the Laplace tra

- 29. te^{at} 30. $t^2 \sin bt$
- 31. t^n
33. te^{at} sin bt 32. $t^n e^{at}$ e^{at} 33. $te^{at} \sin bt$ 34. $te^{at} \cos bt$
35. Consider Bessel's equation of order zero
-

Value Problems
 Recall from Section 5.4 that $t = 0$ **is a regular singular point for this equation, and therefore solutions may become unbounded as** $t \rightarrow 0$ **. However, let us try to determine whether there are any solutio Value Problems**
Solutions may become unbounded as $t \to 0$ is a regular singular point for this equation, and therefore
solutions may become unbounded as $t \to 0$. However, let us try to determine whether there
there is su **Value Problems** 3009

Recall from Section 5.4 that $t = 0$ is a regular singular point for this equation, and therefore

solutions may become unbounded as $t \to 0$. However, let us try to determine whether there

are any s **Value Problems**
 Solution Solution S.4 that $t = 0$ **is a regular singular point for this equation, and therefore solutions may become unbounded as** $t \to 0$ **. However, let us try to determine whether there there there is su Value Problems**
 Solution S.4 that $t = 0$ is a regular singular point for this equation, and therefore

solutions may become unbounded as $t \to 0$. However, let us try to determine whether there

are any solution by $= \$ **309**
 SUP
 $t \to 0$. However, let us try to determine whether there
 $t = 0$ and have finite derivatives there. Assuming that
 $Y(s) = L[\phi(t)]$.
 y^2) $Y'(s) + sY(s) = 0$.

, where *c* is an arbitrary constant.

suming that it is **Value Problems**
 Solution S4 that $t = 0$ is a regular singular point for this equation, and therefore

solutions may become unbounded as $t \to 0$. However, let us try to determine whether there

there is such a solution **309**

= 0 is a regular singular point for this equation, and therefore

ded as $t \to 0$. However, let us try to determine whether there

initie at $t = 0$ and have finite derivatives there. Assuming that
 $i(t)$, let $Y(s) = E$ **Value Problems 309**
 Recall from Section 5.4 that $t = 0$ is a regular singular point for this equation, and therefore

solutions may become unbounded as $t \to 0$. However, let us try to determine whether there

are an **309**
 309
 c point for this equation, and therefore
 Γ , let us try to determine whether there

therefore direct derivatives there. Assuming that
 Γ
 Γ
 Γ and for $s > 1$ and assuming that it is
 Γ , show **Sollems**
 Sollems
 Sollems
 Sollem 5.4 that $t = 0$ is a regular singular point for this equation, and therefore

may become unbounded as $t \rightarrow 0$. However, let us try to determine whether there

the a solution $y = \phi$ **309**

on, and therefore

the whether there

the start of the start of the solution of the

component of the solution of the **Has Problems**
 All or Forestion 5.4 that $t = 0$ is a regular singular point for this equation, and therefore

rations may become unbounded as $t \to 0$. However, let us try to determine whether there

ration show that Y **Solute Problems**
 Solution Society A.4 that $t = 0$ **is a regular singular point for this equation, and therefore

Recoll from Section 5.4 that** $t = 0$ **and have finite derivatives there. Assuming that

there is such a solu 369**
 Start Value Problems

Recall from Section 5.4 that $t = 0$ is a regular singular point for this equation, and therefore

solutions have become unbounded as $t \to 0$. However, let us from the following the factor of **Value Problems**
 Solution Section 5.4 that $t = 0$ is a regular singular point for this equation, and therefore

selutions may become unbounded as $t \to 0$. However, let us try to determine whether there

are any solutio **Solution**
 Solution
 Solutions
 Solutions
 Solutions
 Solutions may become unbounded as $t \to 0$. However, let us try to determine whether there
 once the youtions that remain finite at $t = 0$ and have initie

$$
(1 + s2)Y'(s) + sY(s) = 0.
$$

$$
y = c \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n} (n!)^2} = c J_0(t),
$$

where J_0 is the Bessel function of the first kind of order zero. Note that $J_0(0) = 1$, and that J_0 has finite derivatives of all orders at $t = 0$. It was shown in Section 5.8 that the second solution of this equation becomes unbounded as $t \to 0$.

(a) $y'' - ty = 0$; $y(0) = 1$, $y'(0) = 0$ (Airy's equation)

(b) $(1 - t^2)y'' - 2ty' + \alpha(\alpha + 1)y = 0$; $y(0) = 0$, $y'(0) = 1$ (Legendre's equation)

Value Problems
 Solution Section 5.4 that $t = 0$ is a regular singular point for this equation, and therefore

solutions may become unbounded as $t \to 0$. However, let us try to determine whether there

are any solutio **Source the CO**

Section 5.4 that $t = 0$ is a regular singular point for this equation, and therefore

y become unbounded as $t \to 0$. However, let us try to determine whether there

ions that remain finite at $t = 0$ and h Recall from Section 5.4 that $t = 0$ is a regular singular point for this equation, and therefore
solutions may become unbounded as $t \to 0$. However, let us try to determine whether there
are ay solutions that remain finit Recall from Section 5.4 that $t = 0$ is a regular singular point for this equation, and therefore
solutions may become unbounded as $t \to 0$. However, let us try to determine whether there
are any solutions that termain fin solutions may become unbounded as $t = 0$. However, let us try to determine whether there are a solutions in the remain finite at $t = 0$ and have finite derivatives there. Assuming that the set second as $t = 0$ and the sec solven the Happen transform $t = 0$ and the contribution $\int_0^{\infty} f(x) dx = 0$ and the laplace transform is not of $(1 + s^2)Y'(s) + sY(s) = 0$.

(a) Show that $Y(s) = c(t)$, let $Y(s) = L(\phi(t))$.

(b) Show that $Y(s) = c(t + s^2)^{-1/2}$ in a binom chere is such a solution $y = \phi(t)$, let $Y(s) = \mathcal{L}(\phi(t))$.

(a) Show that $Y(s) = c(1 + s^2)^{1/2}(s) + sY(s) = 0$.

(b) Show that $Y(s) = c(1 + s^2)^{-1/2}$, where c is an arbitrary constant.

(c) Expanding (1 + s²)^{-1/2} i an biominal se variable. (a) Enow and $Y(s) = c(1 + s^2)Y'(s) + sY(s) = 0$.

(b) Show that $Y(s) = c(1 + s^2)^{-1/2}$, where c is an arbitrary constant.

(c) Expanding $(1 + s^2)^{-1/2}$ in a binomial series valid for $s > 1$ and assuming that it is permissible to tak (c) Expanding $(1 + s^2)^{-1/2}$ in a binomial series valid for $s > 1$ and assuming that it is
permissible to take the inverse transform term by term, show that
permissible to take the inverse transform term by term, show tha $y = c \sum_{n=0}^{\infty} \frac{2\pi}{2n(1)^2} = cJ_0(t)$.

Where J_q is the Bessel function of the first kind of order zero. Note that J_q (0) = 1, and that J_q has finite derivatives of all orders are $t = 0$. It was shown in Section J where J_n is the Bessel function $\pi_1 a^2$ (m)
 $\pi_2 a^2$ (m) is the Hessel function of the first kind of order zero. Note that $J_n(0) = 1$, and that
 J_0 has finite derivatives of all orders at $t = 0$. It was shown solution of this equation becomes unbounded as $t \rightarrow 0$.

Solution of the following initial value problems use the results of Problem 28 to find the differential equation satisfied by $Y(s) = L[\phi(t)]$, where $y = \phi(t)$ is the sol 1 28 to find the
solution of the
dre's equation)
f second order
in the equation
illustrates that
with variable
independent
show that
show that
ed to calculate
and $P(s)$ is a
and $P(s)$ is a
(s) $Q(s)$ has a For each of the following initial value problems use the results of Problem 28 to find the given initial capacity $Y(s) = L\{\phi(t)\}\$, where $y = \phi(t)$ is the solution of the given initial value problem.

(a) $y'' - (y = 0; y'(0) = 1, y'($ differential equation satisfied by $Y(s) = L(\phi(t))$, where $y = \phi(t)$ is the solution of the

given initial value problem.

(a) $y'' - (y) = 1$, $y(0) = 0$, $(y(0) = 0)$, $y'(0) = 1$ (Legendre's equation)

(b) $(1 - t^2)y'' - 2ty' + \alpha(t + 1)y = 0$; Note that the differential equators of $Y(s)$ is of this order in part (a), but of second order
for part (b). This is due to the fact that *t* appears at most to the first power in the equation
of part (a), whereas it appe in part (b). This is due to the fact that r appears at most to the first power in the equation

of part (a), whereas it appears to the second power in that of part (b). This illustrates that

the Laplace transform is not

$$
g(t) = \int_0^t f(\tau) d\tau.
$$

$$
G(s) = F(s)/s.
$$

$$
F(s) = P(s)/Q(s),
$$

where $Q(s)$ is a polynomial of degree *n* with distinct zeros r_1, \ldots, r_n and $P(s)$ is a polynomial of degree less than *n*. In this case it is possible to show that $P(s)/Q(s)$ has a variable.

Suppose that
 $g(t) = \int_0^t f(\tau) d\tau$.

If $G(s)$ and $F(s)$ are the Laplace transforms of $g(t)$ and $f(t)$, respectively, show that
 $G(s) = F(s)/s$.

In this problem we show how a general partial fraction expansion can b and the limit of the material state that

d $f(t)$, respectively, show that

expansion can be used to calculate

d,

then the limit as r_1, \ldots, r_n and $P(s)$ is a

sible to show that $P(s)/Q(s)$ has a

d,
 $\frac{A_n}{1-r_n}$,

(i)

$$
\frac{P(s)}{Q(s)} = \frac{A_1}{s - r_1} + \dots + \frac{A_n}{s - r_n},
$$
 (i)

$$
A_k = P(r_k)/Q'(r_k), \qquad k = 1, \dots, n. \tag{ii}
$$

Hint: One way to do this is to multiply Eq. (i) by $s - r_k$ and then to take the limit as $s \to r_k$.