(b) Show that

$$\mathcal{L}^{-1}\{F(s)\} = \sum_{k=1}^{n} \frac{P(r_k)}{Q'(r_k)} e^{r_k t}. \qquad \text{(iii)}$$

6.3 Step Functions

In Section 6.2 we outlined the general procedure involved in solving initial value problems by means of the Laplace transform. Some of the most interesting elementary applications of the transform method occur in the solution of linear differential equations with discontinuous or impulsive forcing functions. Equations of this type frequently arise in the analysis of the flow of current in electric circuits or the vibrations of mechanical systems. In this section and the following ones we develop some additional properties of the Laplace transform that are useful in the solution of such problems. Unless a specific statement is made to the contrary, all functions appearing below will be assumed to be piecewise continuous and of exponential order, so that their Laplace transforms exist, at least for s sufficiently large.

To deal effectively with functions having jump discontinuities, it is very helpful to introduce a function known as the **unit step function**, or **Heaviside function**. This function will be denoted by u_c, and is defined by

$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c, \end{cases} \qquad c \geq 0. \qquad (1)$$

The graph of $y = u_c(t)$ is shown in Figure 6.3.1. The step can also be negative. For instance, Figure 6.3.2 shows the graph $y = 1 - u_c(t)$.

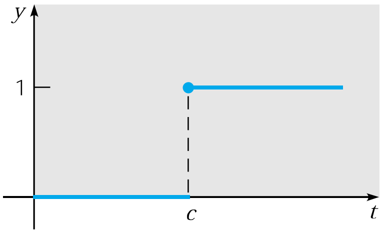

FIGURE 6.3.1 Graph of $y = u_c(t)$.

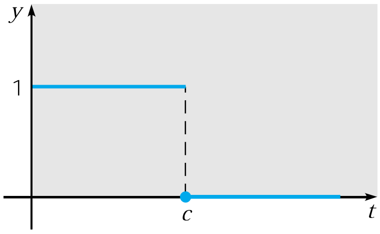

FIGURE 6.3.2 Graph of $y = 1 - u_c(t)$.

EXAMPLE 1

Sketch the graph of $y = h(t)$, where

$$h(t) = u_\pi(t) - u_{2\pi}(t), \qquad t \geq 0.$$

From the definition of $u_c(t)$ in Eq. (1) we have

$$h(t) = \begin{cases} 0 - 0 = 0, & 0 \leq t < \pi, \\ 1 - 0 = 1, & \pi \leq t < 2\pi, \\ 1 - 1 = 0, & 2\pi \leq t < \infty. \end{cases}$$

Thus the equation $y = h(t)$ has the graph shown in Figure 6.3.3. This function can be thought of as a rectangular pulse.

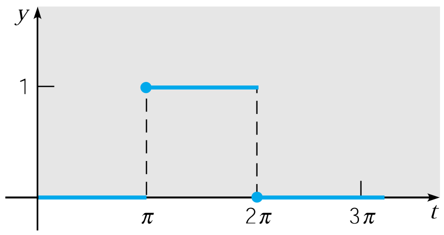

FIGURE 6.3.3 Graph of $y = u_\pi(t) - u_{2\pi}(t)$.

The Laplace transform of u_c is easily determined:

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\,dt = \int_c^\infty e^{-st}\,dt$$
$$= \frac{e^{-cs}}{s}, \qquad s > 0. \tag{2}$$

For a given function f, defined for $t \geq 0$, we will often want to consider the related function g defined by

$$y = g(t) = \begin{cases} 0, & t < c, \\ f(t-c), & t \geq c, \end{cases}$$

which represents a translation of f a distance c in the positive t direction; see Figure 6.3.4. In terms of the unit step function we can write $g(t)$ in the convenient form

$$g(t) = u_c(t) f(t-c).$$

The unit step function is particularly important in transform use because of the following relation between the transform of $f(t)$ and that of its translation $u_c(t)f(t-c)$.

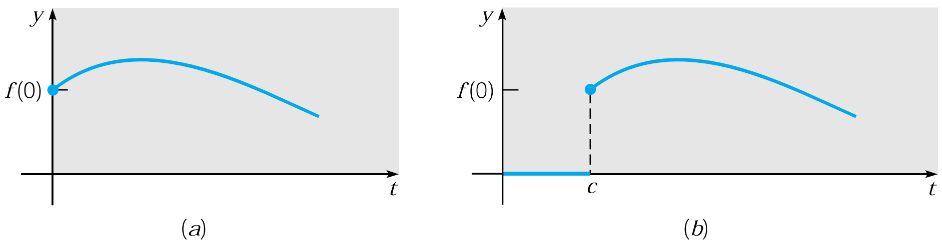

FIGURE 6.3.4 A translation of the given function. (a) $y = f(t)$; (b) $y = u_c(t)f(t-c)$.

Theorem 6.3.1 If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if c is a positive constant, then

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s), \qquad s > a. \tag{3}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$u_c(t)f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}. \tag{4}$$

Theorem 6.3.1 simply states that the translation of $f(t)$ a distance c in the positive t direction corresponds to the multiplication of $F(s)$ by e^{-cs}. To prove Theorem 6.3.1 it is sufficient to compute the transform of $u_c(t)f(t-c)$:

$$\begin{aligned}\mathcal{L}\{u_c(t)f(t-c)\} &= \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt\\ &= \int_c^\infty e^{-st}f(t-c)\,dt.\end{aligned}$$

Introducing a new integration variable $\xi = t - c$, we have

$$\begin{aligned}\mathcal{L}\{u_c(t)f(t-c)\} &= \int_0^\infty e^{-(\xi+c)s}f(\xi)\,d\xi = e^{-cs}\int_0^\infty e^{-s\xi}f(\xi)\,d\xi\\ &= e^{-cs}F(s).\end{aligned}$$

Thus Eq. (3) is established; Eq. (4) follows by taking the inverse transform of both sides of Eq. (3).

A simple example of this theorem occurs if we take $f(t) = 1$. Recalling that $\mathcal{L}\{1\} = 1/s$, we immediately have from Eq. (3) that $\mathcal{L}\{u_c(t)\} = e^{-cs}/s$. This result agrees with that of Eq. (2). Examples 2 and 3 illustrate further how Theorem 6.3.1 can be used in the calculation of transforms and inverse transforms.

EXAMPLE 2

If the function f is defined by

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi/4,\\ \sin t + \cos(t - \pi/4), & t \ge \pi/4,\end{cases}$$

find $\mathcal{L}\{f(t)\}$. The graph of $y = f(t)$ is shown in Figure 6.3.5.

Note that $f(t) = \sin t + g(t)$, where

$$g(t) = \begin{cases} 0, & t < \pi/4,\\ \cos(t - \pi/4), & t \ge \pi/4.\end{cases}$$

Thus

$$g(t) = u_{\pi/4}(t)\cos(t - \pi/4),$$

and

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\{\sin t\} + \mathcal{L}\{u_{\pi/4}(t)\cos(t - \pi/4)\}\\ &= \mathcal{L}\{\sin t\} + e^{-\pi s/4}\mathcal{L}\{\cos t\}.\end{aligned}$$

Introducing the transforms of $\sin t$ and $\cos t$, we obtain

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2+1} + e^{-\pi s/4}\frac{s}{s^2+1} = \frac{1 + se^{-\pi s/4}}{s^2+1}.$$

You should compare this method with the calculation of $\mathcal{L}\{f(t)\}$ directly from the definition.

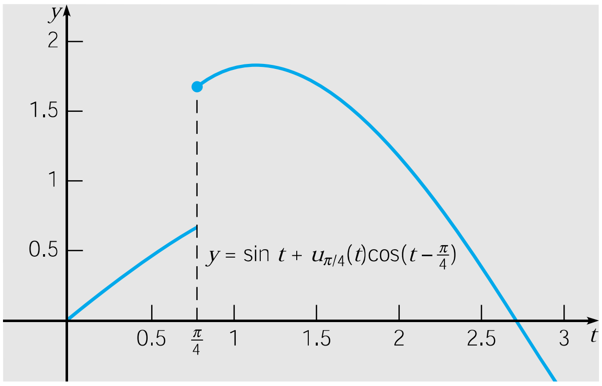

FIGURE 6.3.5 Graph of the function in Example 2.

EXAMPLE 3

Find the inverse transform of

$$F(s) = \frac{1 - e^{-2s}}{s^2}.$$

From the linearity of the inverse transform we have

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$$
$$= t - u_2(t)(t-2).$$

The function f may also be written as

$$f(t) = \begin{cases} t, & 0 \le t < 2, \\ 2, & t \ge 2. \end{cases}$$

The following theorem contains another very useful property of Laplace transforms that is somewhat analogous to that given in Theorem 6.3.1.

Theorem 6.3.2 If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \ge 0$, and if c is a constant, then

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c), \qquad s > a + c. \tag{5}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$e^{ct}f(t) = \mathcal{L}^{-1}\{F(s-c)\}. \tag{6}$$

According to Theorem 6.3.2, multiplication of $f(t)$ by e^{ct} results in a translation of the transform $F(s)$ a distance c in the positive s direction, and conversely. The proof of this theorem requires merely the evaluation of $\mathcal{L}\{e^{ct}f(t)\}$. Thus

$$\mathcal{L}\{e^{ct}f(t)\} = \int_0^\infty e^{-st}e^{ct}f(t)\,dt = \int_0^\infty e^{-(s-c)t}f(t)\,dt$$
$$= F(s-c),$$

which is Eq. (5). The restriction $s > a + c$ follows from the observation that, according to hypothesis (ii) of Theorem 6.1.2, $|f(t)| \le Ke^{at}$; hence $|e^{ct}f(t)| \le Ke^{(a+c)t}$. Equation (6) follows by taking the inverse transform of Eq. (5), and the proof is complete.

The principal application of Theorem 6.3.2 is in the evaluation of certain inverse transforms, as illustrated by Example 4.

EXAMPLE 4

Find the inverse transform of

$$G(s) = \frac{1}{s^2 - 4s + 5}.$$

By completing the square in the denominator we can write

$$G(s) = \frac{1}{(s-2)^2 + 1} = F(s-2),$$

where $F(s) = (s^2 + 1)^{-1}$. Since $\mathcal{L}^{-1}\{F(s)\} = \sin t$, it follows from Theorem 6.3.2 that

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = e^{2t}\sin t.$$

The results of this section are often useful in solving differential equations, particularly those having discontinuous forcing functions. The next section is devoted to examples illustrating this fact.

PROBLEMS

In each of Problems 1 through 6 sketch the graph of the given function on the interval $t \ge 0$.

1. $u_1(t) + 2u_3(t) - 6u_4(t)$
2. $(t-3)u_2(t) - (t-2)u_3(t)$
3. $f(t-\pi)u_\pi(t)$, where $f(t) = t^2$
4. $f(t-3)u_3(t)$, where $f(t) = \sin t$
5. $f(t-1)u_2(t)$, where $f(t) = 2t$
6. $f(t) = (t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t)$

In each of Problems 7 through 12 find the Laplace transform of the given function.

7. $f(t) = \begin{cases} 0, & t < 2 \\ (t-2)^2, & t \ge 2 \end{cases}$
8. $f(t) = \begin{cases} 0, & t < 1 \\ t^2 - 2t + 2, & t \ge 1 \end{cases}$
9. $f(t) = \begin{cases} 0, & t < \pi \\ t - \pi, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$
10. $f(t) = u_1(t) + 2u_3(t) - 6u_4(t)$
11. $f(t) = (t-3)u_2(t) - (t-2)u_3(t)$
12. $f(t) = t - u_1(t)(t-1)$, $t \ge 0$

In each of Problems 13 through 18 find the inverse Laplace transform of the given function.

13. $F(s) = \dfrac{3!}{(s-2)^4}$
14. $F(s) = \dfrac{e^{-2s}}{s^2 + s - 2}$
15. $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2 - 2s + 2}$
16. $F(s) = \dfrac{2e^{-2s}}{s^2 - 4}$
17. $F(s) = \dfrac{(s-2)e^{-s}}{s^2 - 4s + 3}$
18. $F(s) = \dfrac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$