1

TRIGONOMETRIC FOURIER SERIES

1. Periodic Functions

A function $f(x)$ is called *periodic* if there exists a constant $T > 0$ for which

$$f(x + T) = f(x), \tag{1.1}$$

for any x in the domain of definition of $f(x)$. (It is understood that both x and $x + T$ lie in this domain.) Such a constant T is called a *period* of the function $f(x)$. The most familiar periodic functions are $\sin x$, $\cos x$, $\tan x$, etc. Periodic functions arise in many applications of mathematics to problems of physics and engineering. It is clear that the sum, difference, product, or quotient of two functions of period T is again a function of period T.

If we plot a periodic function $y = f(x)$ on any closed interval $a \leqslant x \leqslant a + T$, we can obtain the entire graph of $f(x)$ by periodic repetition of the portion of the graph corresponding to $a \leqslant x \leqslant a + T$ (see Fig. 1).

If T is a period of the function $f(x)$, then the numbers $2T, 3T, 4T, \ldots$ are also periods. This follows immediately by inspecting the graph of a periodic function or from the series of equalities[1]

$$f(x) = f(x + T) = f(x + 2T) = f(x + 3T) = \cdots$$

[1] We suggest that the reader prove the validity not only of these equalities but also of the following equalities:

$$f(x) = f(x - T) = f(x - 2T) = f(x - 3T) = \cdots$$

which are obtained by repeated use of the condition (1.1). Thus, if T is a period, so is kT, where k is any positive integer, i.e., if a period exists, it is *not unique.*

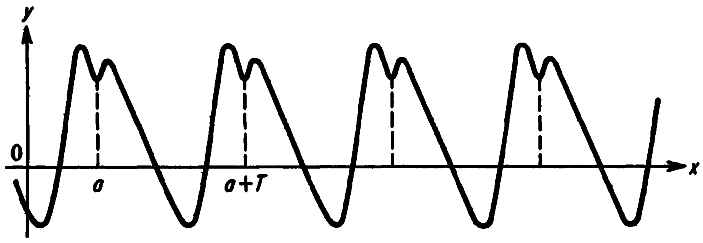

FIGURE 1

Next, we note the following property of any function $f(x)$ of period T:

If $f(x)$ is integrable on any interval of length T, then it is integrable on any other interval of the same length, and the value of the integral is the same, i.e.,

$$\int_a^{a+T} f(x)\,dx = \int_b^{b+T} f(x)\,dx, \tag{1.2}$$

for any a and b.

This property is an immediate consequence of the interpretation of an integral as an area. In fact, each integral (1.2) equals the area included between the curve $y = f(x)$, the x-axis and the ordinates drawn at the end points of the interval, where areas lying above the x-axis are regarded as positive and areas lying below the x-axis are regarded as negative. In the present case, the areas represented by the two integrals are the same, because of the periodicity of $f(x)$ (see Fig. 2).

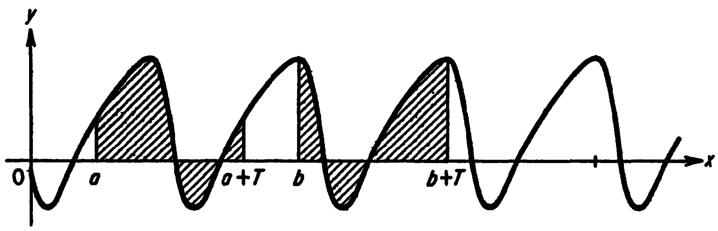

FIGURE 2

Hereafter, when we say that a function $f(x)$ of period T is *integrable*, we shall mean that it is integrable on an interval of length T. It follows from the property just proved that $f(x)$ is also integrable on any interval of finite length.

2. Harmonics

The simplest periodic function, and the one of greatest importance for the applications, is

$$y = A \sin(\omega x + \varphi),$$

where A, ω, and φ are constants. This function is called a *harmonic of amplitude* $|A|$, *(angular) frequency* ω, *and initial phase* φ. The period of such a harmonic is $T = 2\pi/\omega$, since for any x

$$A \sin\left[\omega\left(x + \frac{2\pi}{\omega}\right) + \varphi\right] = A \sin[(\omega x + \varphi) + 2\pi] = A \sin(\omega x + \varphi).$$

The terms "amplitude," "frequency," and "initial phase" stem from the following mechanical problem involving the simplest kind of oscillatory motion, i.e., *simple harmonic motion*: Suppose that a point mass M, of mass m, moves along a straight line under the action of a *restoring force* F which is proportional to the distance of M from a fixed origin O and which is directed towards O (see Fig. 3). Regarding s as positive if M lies to the right of O and

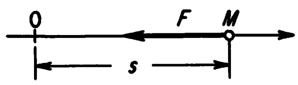

FIGURE 3

negative if M lies to the left of O, i.e., assigning the usual positive direction to the line, we find that $F = -ks$, where $k > 0$ is a constant of proportionality. Therefore

$$m\frac{d^2s}{dt^2} = -ks$$

or

$$\frac{d^2s}{dt^2} + \omega^2 s = 0,$$

where we have written $\omega^2 = k/m$, so that $\omega = \sqrt{k/m}$.

It is easily verified that the solution of this differential equation is the function $s = A\sin(\omega t + \varphi)$, where A and φ are constants, which can be calculated from a knowledge of the position and velocity of the point M at the initial time $t = 0$. This function s is a harmonic, and in fact is a periodic function of time with period $T = 2\pi/\omega$. Thus, under the action of the restoring force F, the point M undergoes oscillatory motion. The amplitude $|A|$ is the maximum deviation of the point M from O, and the quantity $1/T$ is the number of oscillations in an interval containing 2π units of time (e.g., seconds). This explains the term "frequency". The quantity φ is the initial phase and characterizes the initial position of the point, since for $t = 0$ we have $s_0 = \sin \varphi$.

We now examine the appearance of the curve $y = A \sin(\omega x + \varphi)$. We assume that $\omega > 0$, since otherwise $\sin(-\omega x + \varphi)$ is merely replaced by $-\sin(\omega x - \varphi)$. The simplest case is obtained when $A = 1$, $\omega = 1$, $\varphi = 0$; this gives the familiar *sine curve* $y = \sin x$ [see Fig. 4(a)]. For $A = 1$, $\omega = 1$, $\varphi = \pi/2$, we obtain the *cosine curve* $y = \cos x$, whose graph is the same as that of $y = \sin x$ shifted to the left by an amount $\pi/2$.

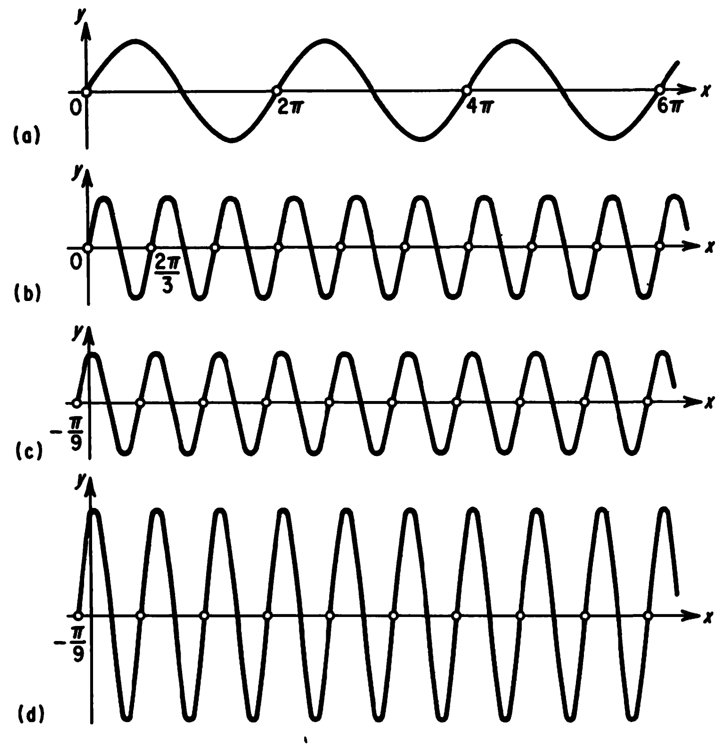

FIGURE 4

Next, consider the harmonic $y = \sin \omega x$, and set $\omega x = z$, thereby obtaining $y = \sin x$, an ordinary sine curve. Thus, the graph of $y = \sin \omega x$ is obtained by deforming the graph of a sine curve: This deformation reduces to a uniform compression along the x-axis by a factor ω if $\omega > 1$, and to a uniform expansion along the x-axis by a factor $1/\omega$ if $\omega < 1$. Figure 4(b) shows the harmonic $y = \sin 3x$, of period $T = 2\pi/3$.

Now, consider the harmonic $y = \sin(\omega x + \varphi)$, and set $\omega x + \varphi = \omega z$, so that $x = z - \varphi/\omega$. We already know the graph of $\sin \omega z$. Therefore, the graph of $y = \sin(\omega x + \varphi)$ is obtained by shifting the graph of $y = \sin \omega x$ along the x-axis by the amount $-\varphi/\omega$. Figure 4(c) represents the harmonic

$$y = \sin\left(3x + \frac{\pi}{3}\right)$$

with period $2\pi/3$ and initial phase $\pi/3$.

Finally, the graph of the harmonic $y = A \sin(\omega x + \varphi)$ is obtained from that of the harmonic $y = \sin(\omega x + \varphi)$ by multiplying all ordinates by the number A. Figure 4(d) shows the harmonic

$$y = 2\sin\left(3x + \frac{\pi}{3}\right).$$

These results may be summarized as follows:

The graph of the harmonic $y = A \sin(\omega x + \varphi)$ is obtained from the graph of the familiar sine curve by uniform compression (or expansion) along the coordinate axes plus a shift along the x-axis.

Using a well-known formula from trigonometry, we write

$$A \sin(\omega x + \varphi) = A(\cos \omega x \sin \varphi + \sin \omega x \cos \varphi).$$

Then, setting

$$a = A \sin \varphi, \qquad b = A \cos \varphi, \tag{2.1}$$

we convince ourselves that every harmonic can be represented in the form

$$a \cos \omega x + b \sin \omega x. \tag{2.2}$$

Conversely, every function of the form (2.2) is a harmonic. To prove this, it is sufficient to solve (2.1) for A and B. The result is

$$A = \sqrt{a^2 + b^2}, \quad \sin \varphi = \frac{a}{A} = \frac{a}{\sqrt{a^2 + b^2}}, \quad \cos \varphi = \frac{b}{A} = \frac{b}{\sqrt{a^2 + b^2}},$$

from which φ is easily found.

From now on, we shall write harmonics in the form (2.2). For example, for the harmonic shown in Fig. 4(d), this form is

$$2 \sin\left(3x + \frac{\pi}{3}\right) = \sqrt{3} \cos 3x + \sin 3x$$

It will also be convenient to explicitly introduce the period T in (2.2). If we set $T = 2l$, then, since $T = 2\pi/\omega$, we have

$$\omega = \frac{2\pi}{T} = \frac{\pi}{l},$$

and therefore, the harmonic with period $T = 2l$ can be written as

$$a \cos \frac{\pi x}{l} + b \sin \frac{\pi x}{l}. \tag{2.3}$$

3. Trigonometric Polynomials and Series

Given the period $T = 2l$, consider the harmonics

$$a_k \cos \frac{\pi k x}{l} + b_k \sin \frac{\pi k x}{l} \qquad (k = 1, 2, \ldots) \tag{3.1}$$

with frequencies $\omega_k = \pi k/l$ and periods $T_k = 2\pi/\omega_k = 2l/k$. Since

$$T = 2l = kT_k,$$

the number $T = 2l$ is simultaneously a period of all the harmonics (3.1), for an integral multiple of a period is again a period (see Sec. 1). Therefore, every sum of the form

$$s_n(x) = A + \sum_{k=1}^{n} \left(a_k \cos \frac{\pi k x}{l} + b_k \sin \frac{\pi k x}{l} \right),$$

where A is a constant, is a function of period $2l$, since it is a sum of functions of period $2l$. (The addition of a constant obviously does not destroy periodicity; in fact, a constant can be regarded as a function for which *any* number is a period.) The function $s_n(x)$ is called a *trigonometric polynomial of order n* (and period $2l$).

Even though it is a sum of various harmonics, a trigonometric polynomial in general represents a function of a much more complicated nature than a simple harmonic. By suitably choosing the constants $A, a_1, b_1, a_2, b_2, \ldots$ we can form functions $y = s_n(x)$ with graphs quite unlike the smooth and symmetric graph of a simple harmonic. For example, Fig. 5 shows the trigonometric polynomial

$$y = \sin x + \tfrac{1}{2} \sin 2x + \tfrac{1}{4} \sin 3x.$$

The *infinite trigonometric series*

$$A + \sum_{k=1}^{\infty} \left(a_k \cos \frac{\pi k x}{l} + b_k \sin \frac{\pi k x}{l} \right)$$

(if it converges) also represents a function of period $2l$. The nature of functions which are sums of such infinite trigonometric series is even more diverse. Thus, the following question arises naturally: Can any given

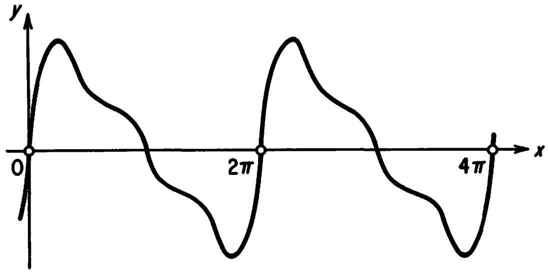

FIGURE 5

function of period $T = 2l$ be represented as the sum of a trigonometric series? We shall see later that such a representation is in fact possible for a very wide class of functions.

For the time being, suppose that $f(x)$ belongs to this class. This means that $f(x)$ can be expanded as a sum of harmonics, i.e., as a sum of functions with a very simple structure. The graph of the function $y = f(x)$ is obtained as a "superposition" of the graphs of these harmonics. Thus, to give a mechanical interpretation, we can represent a complicated oscillatory motion $f(x)$ as a sum of individual oscillations which are particularly simple. However, one must not imagine that trigonometric series are applicable only to oscillation phenomena. This is far from being the case. In fact, the concept of a trigonometric series is also very useful in studying many phenomena of a quite different nature.

If

$$f(x) = A + \sum_{k=1}^{\infty} \left(a_k \cos \frac{\pi k x}{l} + b_k \sin \frac{\pi k x}{l} \right), \tag{3.2}$$

then, setting $\pi x/l = t$ or $x = tl/\pi$, we find that

$$\varphi(t) = f\left(\frac{tl}{\pi}\right) = A + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt), \tag{3.3}$$

where the harmonics in this series all have period 2π. This means that if a function $f(x)$ of period $2l$ has the expansion (3.2), then the function $\varphi(t) = f(tl/\pi)$ is of period 2π and has the expansion (3.3). Obviously, the converse is also true, i.e., if a function $\varphi(t)$ of period 2π has the expansion (3.3), then the function $f(x) = \varphi(\pi x/l)$ is of period $2l$ and has the expansion (3.2).

Thus, it is enough to know how to solve the problem of expansion in trigonometric series for functions of the "standard" period 2π. Moreover, in this case, the series has a simpler appearance. Therefore, we shall develop the theory for series of the form (3.3), and only the final results will be converted to the "language" of the general series (3.2).

4. A More Precise Terminology. Integrability. Series of Functions

We now introduce a more precise terminology and recall some facts from differential and integral calculus. When we say that $f(x)$ is integrable on the interval $[a, b]$, we mean that the integral

$$\int_a^b f(x)\,dx \tag{4.1}$$

(which may be improper) exists in the elementary sense. Thus, our integrable functions $f(x)$ will always be either continuous or have a finite number of points of discontinuity in the interval $[a, b]$, at which the function can be either bounded or unbounded.

In courses on integral calculus, it is proved that if a function has a finite number of discontinuities, then if the integral

$$\int_a^b |f(x)|\,dx$$

exists, so does the integral (4.1). (The converse is not always true.) In this case, the function $f(x)$ is said to be *absolutely integrable*. If $f(x)$ is absolutely integrable and $\varphi(x)$ is a bounded integrable function, then the product $f(x)\varphi(x)$ is absolutely integrable. The following rule for integration by parts holds:

Let $f(x)$ and $\varphi(x)$ be continuous on $[a, b]$, but perhaps non-differentiable at a finite number of points. Then, if $f'(x)$ and $\varphi'(x)$ are absolutely integrable,[2] we have

$$\int_a^b f(x)\varphi'(x)\,dx = \Big[f(x)\varphi(x)\Big]_{x=a}^{x=b} - \int_a^b f'(x)\varphi(x)\,dx. \tag{4.2}$$

Another familiar result is the fact that if the functions $f_1(x), f_2(x), \ldots, f_n(x)$ are integrable on $[a, b]$, then their sum is also integrable, and

$$\int_a^b \left[\sum_{k=1}^n f_k(x)\right] dx = \sum_{k=1}^n \int_a^b f_k(x)\,dx. \tag{4.3}$$

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[2] Instead of absolute integrability of both derivatives, we can weaken this requirement to absolute integrability of just one of the derivatives. However, the stronger form of the requirement is sufficient for what follows.

We now consider an *infinite* series of functions

$$f_1(x) + f_2(x) + \cdots + f_k(x) + \cdots = \sum_{k=1}^{\infty} f_k(x). \tag{4.4}$$

Such a series is said to be *convergent* for a given value of x if its *partial sums*

$$s_n(x) = \sum_{k=1}^{n} f_k(x) \qquad (n = 1, 2, \ldots)$$

have a finite limit

$$s(x) = \lim_{n\to\infty} s_n(x).$$

The quantity $s(x)$ is said to be the *sum* of the series, and is obviously a function of x. If the series converges for all x in the interval $[a, b]$, then its sum $s(x)$ is defined on the whole interval $[a, b]$.

We now ask whether the formula (4.3) can be extended to the case of a convergent series of functions which are integrable on the interval $[a, b]$, i.e., is the formula

$$\int_a^b \left[\sum_{k=1}^{\infty} f_k(x) \right] dx = \int_a^b s(x)\, dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)\, dx \tag{4.5}$$

valid? In other words, can the series be integrated *term by term*? It turns out that (4.5) is not always valid, if for no other reason than that a series of integrable or even continuous functions may not even have an integrable sum. A similar problem arises in connection with the possibility of term by term differentiation of series. We now single out an important class of series of functions to which these operations can be applied.

The series (4.4) is said to be *uniformly convergent* on the interval $[a, b]$ if for any positive number ε, there exists a number N such that the inequality

$$|s(x) - s_n(x)| \leqslant \varepsilon \tag{4.6}$$

holds for all $n \geqslant N$ and *for all x in the interval* $[a, b]$. Thus, if we examine the graph of the sum of the series $s(x)$ and of the partial sum $s_n(x)$, uniform convergence means that for all sufficiently large indices n and for all x, the curve representing $s(x)$ and the curve representing $s_n(x)$ are less than ε apart, where ε is any preassigned number, so that the two curves are *uniformly*[3] close (see Fig. 6).

Not every series which converges on an interval $[a, b]$ converges uniformly there. The following is a very useful and simple test for the uniform convergence of a series of functions (Weierstrass' M-test):

[3] I.e., for all x in $[a, b]$.

If the series of positive numbers

$$M_1 + M_2 + \cdots + M_k + \cdots$$

converges and if for any x in the interval $[a, b]$ *we have* $|f_k(x)| \leqslant M_k$ *from a certain k on, then the series* (4.3) *converges uniformly* (*and absolutely*) *on* $[a, b]$.

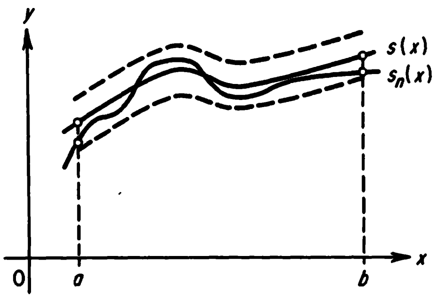


FIGURE 6

The following important theorems are valid:

THEOREM 1. *If the terms of the series* (4.4) *are continuous on* $[a, b]$ *and if the series is uniformly convergent on* $[a, b]$, *then*

a) *The sum of the series is continuous;*

b) *The sum can be integrated term by term, i.e.,* (4.5) *holds.*

THEOREM 2. *If the series* (4.4) *converges, if its terms are differentiable and if the series*

$$f_1'(x) + f_2'(x) + \cdots + f_k'(x) + \cdots = \sum_{k=1}^{\infty} f_k'(x)$$

is uniformly convergent on $[a, b]$, *then*

$$\left(\sum_{k=1}^{\infty} f_k(x)\right)' = s'(x) = \sum_{k=1}^{\infty} f_k'(x),$$

i.e., the series (4.4) *can be differentiated term by term.*[4]

5. The Basic Trigonometric System. The Orthogonality of Sines and Cosines

By the *basic trigonometric system* we mean the system of functions

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, \ldots \qquad (5.1)$$

[4] In courses on analysis, it is usually assumed also that the derivatives are continuous, in order to simplify the proof.

All these functions have the common period 2π (although $\cos nx$ and $\sin nx$ also have the smaller period $2\pi/n$). We now prove some auxiliary formulas.

For any integer $n \neq 0$, we have

$$\begin{aligned} \int_{-\pi}^{\pi} \cos nx\, dx &= \left[\frac{\sin nx}{n}\right]_{x=-\pi}^{x=\pi} = 0, \\ \int_{-\pi}^{\pi} \sin nx\, dx &= \left[-\frac{\cos nx}{n}\right]_{x=-\pi}^{x=\pi} = 0, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} \int_{-\pi}^{\pi} \cos^2 nx\, dx &= \int_{-\pi}^{\pi} \frac{1+\cos 2nx}{2}\, dx = \pi, \\ \int_{-\pi}^{\pi} \sin^2 nx\, dx &= \int_{-\pi}^{\pi} \frac{1-\cos 2nx}{2}\, dx = \pi. \end{aligned} \tag{5.3}$$

Using the familiar trigonometric formulas

$$\begin{aligned} \cos\alpha\cos\beta &= \tfrac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)], \\ \sin\alpha\sin\beta &= \tfrac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)] \end{aligned}$$

we find that

$$\begin{aligned} &\int_{-\pi}^{\pi} \cos nx \cos mx\, dx \\ &\qquad = \tfrac{1}{2}\int_{-\pi}^{\pi} [\cos(n+m)x + \cos(n-m)x]\, dx = 0, \\ &\int_{-\pi}^{\pi} \sin nx \sin mx\, dx \\ &\qquad = \tfrac{1}{2}\int_{-\pi}^{\pi} [\cos(n-m)x - \cos(n+m)x]\, dx = 0 \end{aligned} \tag{5.4}$$

for any integers n and m ($n \neq m$). Finally, using the formula

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)],$$

we find that

$$\begin{aligned} &\int_{-\pi}^{\pi} \sin nx \cos mx\, dx \\ &\qquad = \tfrac{1}{2}\int_{-\pi}^{\pi} [\sin(n+m)x + \sin(n-m)x]\, dx = 0 \end{aligned} \tag{5.5}$$

for any n and m. The formulas (5.2), (5.4), and (5.5) show that the integral over the interval $[-\pi, \pi]$ of the product of any two *different* functions of the system (5.1) vanishes.

We shall agree to call two functions $\varphi(x)$ and $\psi(x)$ *orthogonal*[5] on the interval $[a, b]$ if

$$\int_a^b \varphi(x)\psi(x)\,dx = 0.$$

With this definition, we can say that the functions of the system (5.1) are pairwise orthogonal on the interval $[-\pi, \pi]$, or more briefly, that *the system* (5.1) *is orthogonal on* $[-\pi, \pi]$.

As we know, the integral of a periodic function is the same over any interval whose length equals the period (see Sec. 1). Therefore, the formulas (5.2) through (5.5) are valid not only for the interval $[-\pi, \pi]$ but also for any interval $[a, a + 2\pi]$, i.e., the system (5.1) is orthogonal on every such interval.

6. Fourier Series for Functions of Period 2π

Suppose the function $f(x)$ of period 2π has the expansion

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx), \tag{6.1}$$

where, to simplify the subsequent formulas, we denote the constant term by $a_0/2$. We now pose the problem of determining the coefficients a_0, a_k and b_k $(k = 1, 2, \ldots)$ from a knowledge of $f(x)$. To do this, we make the following *assumption*: It is assumed that the series (6.1), and the series to be written presently, can be integrated term by term, i.e., it is assumed that for all these series the integral of the sum equals the sum of the integrals. [It is thereby also assumed that the function $f(x)$ is integrable.] Then, integrating (6.1) from $-\pi$ to π, we obtain

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx + \sum_{k=1}^{\infty}\left(a_k \int_{-\pi}^{\pi} \cos kx\,dx + b_k \int_{-\pi}^{\pi} \sin kx\,dx\right).$$

By (5.2), all the integrals in the sum vanish, so that

$$\int_{-\pi}^{\pi} f(x)\,dx = \pi a_0. \tag{6.2}$$

Next, we multiply both sides of (6.1) by $\cos nx$ and integrate the result from $-\pi$ to π, as before, obtaining

[5] In geometry, the word *orthogonality* connotes perpendicularity. One must not think that the concept of orthogonality of two functions corresponds to anything like perpendicularity of their graphs, despite the fact that this concept is related to a suitably generalized notion of perpendicularity. In this regard, see Ch. 2, Sec. 10.

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos nx\,dx$$
$$+\sum_{k=1}^{\infty}\left(a_k\int_{-\pi}^{\pi}\cos kx\cos nx\,dx\right.$$
$$\left.+\,b_k\int_{-\pi}^{\pi}\sin kx\cos nx\,dx\right).$$

By (5.2), the first integral on the right vanishes. Since the functions of the system (5.1) are pairwise orthogonal, all the integrals in the sum also vanish, except one. The only integral that remains is the coefficient of a_n:

$$\int_{-\pi}^{\pi}\cos^2 nx\,dx = \pi$$

[see (5.3)]. Thus we have

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = a_n\pi. \qquad (6.3)$$

Similarly, we find that

$$\int_{-\pi}^{\pi} f(x)\sin nx\,dx = b_n\pi. \qquad (6.4)$$

It follows from (6.2) to (6.4) that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad (n = 0, 1, 2, \ldots),$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad (n = 1, 2, \ldots). \qquad (6.5)$$

Thus, finally, if $f(x)$ is integrable and can be expanded in a trigonometric series, and if this series and the series obtained from it by multiplying by $\cos nx$ and $\sin nx$ $(n = 1, 2, \ldots)$ can be integrated term by term, then the coefficients a_n and b_n are given by the formulas (6.5).

Now, suppose we are given an integrable function $f(x)$ of period 2π, and we wish to represent $f(x)$ as the sum of a trigonometric series. If such a representation is possible at all (and if the requirement of term by term integrability is satisfied), then by what has been said, the coefficients a_n and b_n must be given by (6.5). Therefore, in looking for a trigonometric series whose sum is a given function $f(x)$, it is natural to examine first the series whose coefficients are given by (6.5), and to see whether this series has the required properties. As we shall see later, this will be the case for a large class of functions.

The coefficients a_n and b_n calculated by the formulas (6.5) are called the *Fourier coefficients* of the function $f(x)$, and the trigonometric series with these coefficients is called the *Fourier series* of $f(x)$. Incidentally, we note that the formulas (6.5) involve integrating a function of period 2π. Therefore, the interval of integration $[-\pi, \pi]$ can be replaced by any other interval of length 2π (see Sec. 1), so that together with the formulas (6.5), we have

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_a^{a+2\pi} f(x)\cos nx\,dx \qquad (n = 0, 1, 2, \ldots), \\ b_n &= \frac{1}{\pi}\int_a^{a+2\pi} f(x)\sin nx\,dx \qquad (n = 1, 2, \ldots). \end{aligned} \tag{6.6}$$

The above considerations make it natural to devote special attention to Fourier series. It we form the Fourier series of a function $f(x)$ without deciding in advance whether it converges to $f(x)$, we write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

This notation means only that the Fourier series written on the right *corresponds* to the function $f(x)$. The sign $\sim$ can be replaced by the sign $=$ only if we succeed in proving that the series converges and that its sum equals $f(x)$. A simple consequence of these considerations is the following theorem, which is quite useful:

THEOREM 1. *If a function $f(x)$ of period 2π can be expanded in a trigonometric series which converges uniformly on the whole real axis,*[6] *then this series is the Fourier series of $f(x)$.*

Proof. Suppose that $f(x)$ satisfies (6.1), where the series is uniformly convergent. By Theorem 1 of Sec. 4, $f(x)$ is continuous and term by term integration of the series is possible. This gives the formula (6.2). Next, we consider the equality

$$f(x)\cos nx = \frac{a_0}{2}\cos nx + \sum_{k=1}^{\infty} (a_k \cos kx \cos nx + b_k \sin kx \cos nx), \tag{6.7}$$

and show that the series on the right is uniformly convergent. Set

$$s_m(x) = \frac{a_0}{2} + \sum_{k=1}^{m} (a_k \cos kx + b_k \sin kx),$$

and let ε be an arbitrary positive number. If the series (6.1) converges uniformly, then there exists a number N such that

$$|f(x) - s_m(x)| \leqslant \varepsilon$$

[6] By the periodicity of $f(x)$ we can require uniform convergence on $[-\pi, \pi]$, rather than on the whole real axis.

for all $m \geqslant N$. The product $s_m(x) \cos nx$ is obviously the mth partial sum of the series (6.7). Then, the inequality

$$|f(x) \cos nx - s_m(x) \cos nx| = |f(x) - s_m(x)| \, |\cos nx| \leqslant \varepsilon,$$

which holds for all $m \geqslant N$, implies the uniform convergence of the series (6.7). It follows that this series can be integrated term by term, and the result of the integration is the formula (6.3). Similarly, we prove the formula (6.4). Thus, finally, the formulas (6.5) hold for the coefficients a_n and b_n, which means that (6.1) is the Fourier series of $f(x)$.

The modern theory of Fourier series allows us to prove the following more general result, whose proof we cannot give because of its complexity:

THEOREM 2. *If an absolutely integrable function $f(x)$ of period 2π can be expanded in a trigonometric series which converges to $f(x)$ everywhere, except possibly at a finite number of points (within one period), then this series is the Fourier series of $f(x)$.*

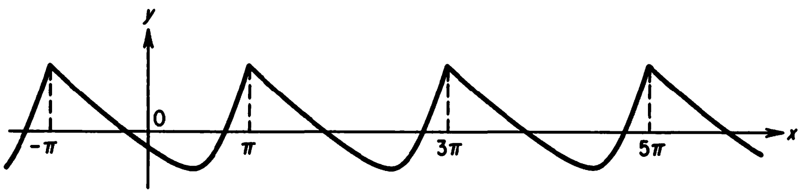
This theorem confirms the assertion made above, that in looking for a trigonometric series which has a given function $f(x)$ as its sum, we should first consider the Fourier series of $f(x)$.

7. Fourier Series for Functions Defined on an Interval of Length 2π

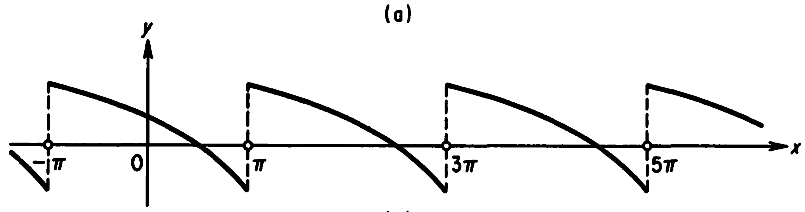
A problem which arises quite often in the applications is that of expanding a function $f(x)$ in trigonometric series, when $f(x)$ is defined only on the interval $[-\pi, \pi]$. In this case, nothing at all is said about the periodicity of $f(x)$. Nevertheless, this does not prevent us from writing the Fourier series of $f(x)$, since the formulas (6.5) involve only the interval $[-\pi, \pi]$. Moreover, $f(x)$ can be extended by periodicity from $[-\pi, \pi]$ onto the whole x-axis. This leads to a periodic function which coincides with $f(x)$ on $[-\pi, \pi]$ and which has a Fourier series identical with that of $f(x)$. In fact, if the Fourier series of $f(x)$ turns out to converge to $f(x)$, then, since it is a periodic function, the sum of this Fourier series automatically gives us the required periodic extension of $f(x)$ from $[-\pi, \pi]$ onto the whole x-axis.

Thus, it does not matter whether we talk about the Fourier series of a function defined on $[-\pi, \pi]$, or whether we talk about the Fourier series of the function obtained from $f(x)$ by *periodic extension* along the x-axis. This implies that it is sufficient to formulate the tests for convergence of Fourier series for the case of periodic functions.

In connection with the problem of extending $f(x)$ by periodicity from the interval $[-\pi, \pi]$ onto the whole x-axis, the following remarks are in order: If $f(-\pi) = f(\pi)$, there is no difficulty in making the extension, since in this case, if $f(x)$ is continuous on $[-\pi, \pi]$, its extension will be continuous on the whole x-axis [see Fig. 7(a)]. However, if $f(-\pi) \neq f(\pi)$, we cannot accomplish the required extension without changing the values of $f(-\pi)$ and $f(\pi)$, since the periodicity requires that $f(-\pi)$ and $f(\pi)$ coincide. This difficulty can be avoided in two ways: (1) We can completely avoid considering the values of $f(x)$ at $x = -\pi$ and $x = \pi$, thereby making the function undefined at these points and hence making the periodic extension of $f(x)$ undefined at the points $x = (2k + 1)\pi$, $k = 0, \pm 1, \pm 2, \ldots$; (2) We can suitably modify the values of the function $f(x)$ at $x = -\pi$ and $x = \pi$ by making these values equal. It is important to note that in both cases, the Fourier coefficients will have the same values as before, since changing the values of a function at a finite number of points, or even failing to define it at a finite number of points, cannot affect the value of an integral, in particular, the values of the integrals (6.5) defining the Fourier coefficients. Thus, whether or not we carry out the indicated modification of the function $f(x)$, its Fourier series remains unchanged.



(a)



(b)

FIGURE 7

It should be observed that if $f(-\pi) \neq f(\pi)$ and if $f(x)$ is continuous on the interval $[-\pi, \pi]$, then the periodic extension of $f(x)$ onto the whole x-axis will have discontinuities at all the points $x = (2k + 1)\pi$, $k = 0, \pm 1, \pm 2, \ldots$, no matter how we change the values of the function at $x = -\pi$ and $x = \pi$ [see Fig. 7(b)]. The problem of finding the values to which the Fourier series of $f(x)$ may be expected to converge at $x = \pm\pi$, when $f(-\pi) \neq f(\pi)$, is a special one, and will be solved later.

Finally, suppose that $f(x)$ is defined on an arbitrary interval $[a, a + 2\pi]$ of length 2π, and that it is required to expand $f(x)$ in a trigonometric series.

As before, we arrive at the conclusion that it does not matter whether we talk about the Fourier series of $f(x)$ or about the Fourier series of the function obtained from $f(x)$ by extending it periodically onto the whole x-axis. If $f(x)$ is continuous on the interval $[a, a + 2\pi]$ but $f(a) \neq f(a + 2\pi)$, we obtain an extension which is discontinuous at the points $x = a + 2k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$.

8. Right-Hand and Left-Hand Limits. Jump Discontinuities

We introduce the notation

$$\lim_{\substack{x \to x_0 \\ x < x_0}} f(x) = f(x_0 - 0), \quad \lim_{\substack{x \to x_0 \\ x > x_0}} f(x) = f(x_0 + 0),$$

provided these limits exist and are finite.[7] The first of these limits is called the *left-hand limit* of $f(x)$ at the point x_0, and the second is called the *right-hand limit* of $f(x)$ at x_0. These limits both exist at points of continuity (by the very definition of continuity), and we have

$$f(x_0 - 0) = f(x_0) = f(x_0 + 0) \tag{8.1}$$

at continuity points.

If x_0 is a point of discontinuity of the function $f(x)$, then the right-hand and left-hand limits (either or both of them) may exist in some cases and fail to exist in others. If both limits exist, we say that the point x_0 is a point of discontinuity of the first kind, or simply, a point of *jump discontinuity*. If at least one of these limits does not exist, then the point x_0 is called a *point of discontinuity of the second kind*. We shall be particularly interested in jump discontinuities. If x_0 is such a point, then the quantity

$$\delta = f(x_0 + 0) - f(x_0 - 0) \tag{8.2}$$

is called the *jump* of the function $f(x)$ at x_0.

The following example illustrates this situation. Suppose that

$$f(x) = \begin{cases} -x^3 & \text{for } x < 1, \\ 0 & \text{for } x = 1, \\ \sqrt{x} & \text{for } x > 1, \end{cases} \tag{8.3}$$

with the graph shown in Fig. 8. The value of the function at $x = 1$ is indicated by the little circle. At $x = 1$, the left-hand and right-hand limits are obviously

$$f(1 - 0) = -1, \qquad f(1 + 0) = 1.$$

[7] If $x_0 = 0$, we do not write $f(0 + 0)$ and $f(0 - 0)$, but simply $f(+0)$ and $f(-0)$.

Therefore, the jump of the function at $x = 1$ is

$$\delta = f(1 + 0) - f(1 - 0) = 2,$$

which is in complete agreement with the intuitive idea of a jump (see Fig. 8).

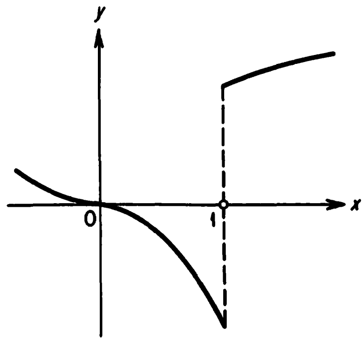


FIGURE 8

If $f(x)$ is a function which is continuous on the interval $[-\pi, \pi]$, then if $f(-\pi) \neq f(\pi)$, jump discontinuities appear in making the periodic extension of $f(x)$ from $[-\pi, \pi]$ onto the whole x-axis [see Fig. 7(b)], and all the jump discontinuities are equal to the number

$$\delta = f(-\pi) - f(\pi).$$

9. Smooth and Piecewise Smooth Functions

The function $f(x)$ is said to be *smooth* on the interval $[a, b]$ if it has a *continuous* derivative on $[a, b]$. In geometrical language, this means that the direction of the tangent changes *continuously*, without jumps, as it moves along the curve $y = f(x)$ [see Fig. 9(a)]. Thus, the graph of a smooth function is a smooth curve without any "corners."[8]

The function $f(x)$ is said to be *piecewise smooth* on the interval $[a, b]$ if either $f(x)$ and its derivative are both continuous on $[a, b]$, or they have only a finite number of jump discontinuities on $[a, b]$. It is easy to see that the graph of a piecewise smooth function is either a continuous curve or a discontinuous curve which can have a finite number of *corners* (at which the derivative has jumps). As we approach any discontinuity or corner (from one side or the other), the direction of the tangent approaches a definite limiting position, since the derivative can have only jump discontinuities.

[8] "Corner" = Russian "угловая точка," a point at which the curve has two distinct tangents. (*Translator*)

Figures 9(b) and 9(c) illustrate the graphs of continuous and discontinuous piecewise smooth functions. From now on, we shall regard smooth functions as a special case of piecewise smooth functions.

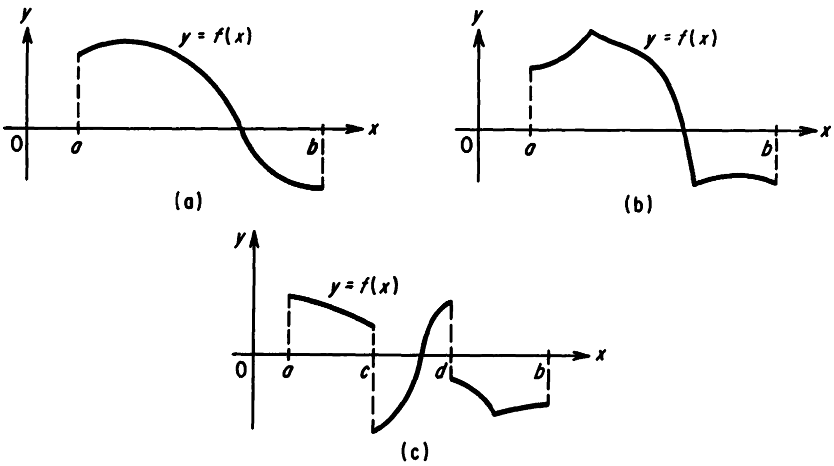


FIGURE 9

A continuous or discontinuous function $f(x)$ which is defined on the whole x-axis, is said to be piecewise smooth if it is piecewise smooth on every interval of finite length. In particular, this concept applies to periodic functions. Every piecewise smooth function $f(x)$ [whether continuous or discontinuous] is bounded and has a bounded derivative everywhere, except at its corners and points of discontinuity [at all these points, $f'(x)$ does not exist].

10. A Criterion for the Convergence of Fourier Series

We now give a more useful criterion for the convergence of a Fourier series, deferring the proof of this criterion until Ch. 3:

The Fourier series of a piecewise smooth (continuous or discontinuous) function $f(x)$ of period 2π converges for all values of x. The sum of the series equals $f(x)$ at every point of continuity and equals the number

$$\tfrac{1}{2}[f(x+0)+f(x-0)],$$

the arithmetic mean of the right-hand and left-hand limits, at every point of discontinuity (see Fig. 10). *If $f(x)$ is continuous everywhere, then the series converges absolutely and uniformly.*