Thus, the Fourier series of a function $f(x)$ defined on the interval $[-\pi, \pi]$ and continuous at $x = \pm\pi$ behaves at the points $x = \pm\pi$ just as it does at the other points of continuity, provided that $f(-\pi) = f(\pi)$. However, if

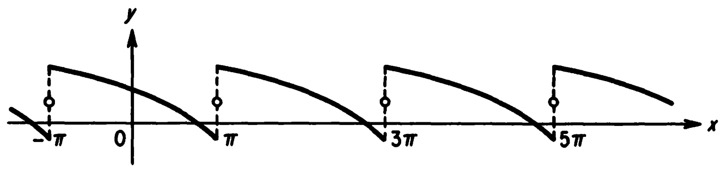

FIGURE 11

$f(-\pi) \neq f(\pi)$, the series obviously cannot converge to $f(x)$ at $x = \pi$, and in this case, it is meaningful to pose the problem of expanding $f(x)$ in Fourier series only for $-\pi < x < \pi$ and not for $-\pi \leqslant x \leqslant \pi$. A similar remark can be made concerning the Fourier series of a function specified in an interval of the type $[a, a + 2\pi]$, where a is any number.

In solving any concrete problem, if the reader draws a graph of the periodic extension of the function (this is always recommended!) and bears in mind the criterion just formulated, then the nature of the behavior of the Fourier series at the end points of the interval will be immediately apparent.

11. Even and Odd Functions

Let the function $f(x)$, defined either on the whole x-axis or on some interval, be symmetric with respect to the origin of coordinates. We say that $f(x)$ is an *even* function if

$$f(-x) = f(x)$$

for every x. This definition implies that the graph of any even function $y = f(x)$ is symmetric with respect to the y-axis [see Fig. 12(a)]. It follows

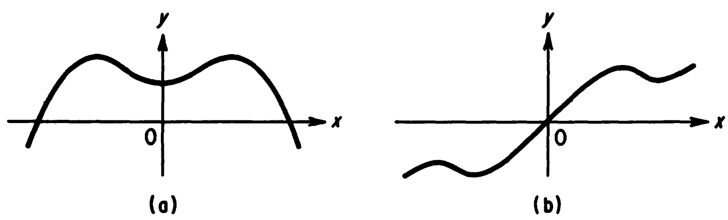

FIGURE 12

from the interpretation of the integral as an area that for even functions we have

$$\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx \tag{11.1}$$

for any l, provided that $f(x)$ is defined and integrable on the interval $[-l, l]$.

We say that the function $f(x)$ is *odd* if

$$f(-x) = -f(x)$$

for every x. In particular, for an odd function we have

$$f(-0) = -f(0),$$

so that $f(0) = 0$. The graph of any odd function $y = f(x)$ is symmetric with respect to the point O [see Fig. 12(b)]. For odd functions

$$\int_{-l}^{l} f(x)\,dx = 0 \tag{11.2}$$

for any l, provided that $f(x)$ is defined and integrable on the interval $[-l, l]$.

The following properties are simple consequences of the definition of even and odd functions:

(a) The product of two even or odd functions is an even function;
(b) The production of an even and an odd function is an odd function.

In fact, if $\varphi(x)$ and $\psi(x)$ are even functions, then for $f(x) = \varphi(x)\psi(x)$, we have

$$f(-x) = \varphi(-x)\psi(-x) = \varphi(x)\psi(x) = f(x),$$

while if $\varphi(x)$ and $\psi(x)$ are odd, we have

$$f(-x) = \varphi(-x)\psi(-x) = [-\varphi(x)][-\psi(x)] = \varphi(x)\psi(x) = f(x).$$

This proves Property (a). On the other hand, if $\varphi(x)$ is even and $\psi(x)$ is odd, then

$$f(-x) = \varphi(-x)\psi(-x) = \varphi(x)[-\psi(x)] = -\varphi(x)\psi(x) = -f(x),$$

which proves Property (b).

12. Cosine and Sine Series

Let $f(x)$ be an *even* function defined on the interval $[-\pi, \pi]$, or else an even periodic function. Since $\cos nx$ $(n = 0, 1, 2, \ldots)$ is obviously an even function, then by Property (a) of Sec. 11 the function $f(x) \cos nx$ is also even. On the other hand, the function $\sin nx$ $(n = 1, 2, \ldots)$ is odd, so that the function $f(x) \sin nx$ is also odd, by Property (b) of Sec. 11. Then, using

(6.5), (11.1) and (11.2), we find that the Fourier coefficients of the even function $f(x)$ are

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \qquad (n = 0, 1, 2, \ldots), \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 \qquad (n = 1, 2, \ldots). \end{aligned} \tag{12.1}$$

Therefore, the Fourier series of an even function contains only *cosines*, i.e.,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

where the coefficients a_n are given by the formula (12.1).

Now, let $f(x)$ be an *odd* function, defined on the interval $[-\pi, \pi]$, or else an odd periodic function. Since $\cos nx$ $(n = 0, 1, 2, \ldots)$ is an even function, the function $f(x) \cos nx$ is odd, by Property (b) of Sec. 11, and since $\sin nx$ $(n = 1, 2, \ldots)$ is odd, the function $f(x) \sin nx$ is even, by Property (a) of Sec. 11. Then, using (6.5), (11.1), and (11.2), we find that the Fourier coefficients of the odd function $f(x)$ are

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 \qquad (n = 0, 1, 2, \ldots), \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \qquad (n = 1, 2, \ldots). \end{aligned} \tag{12.2}$$

Therefore, the Fourier series of an odd function contains only *sines*, i.e.,

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx,$$

where the coefficients b_n are given by the formula (12.2). Since the Fourier series of an odd function contains only sines, it obviously vanishes for $x = -\pi$, $x = 0$, and $x = \pi$ (and in general for $x = k\pi$), regardless of the values of $f(x)$ at these points.

A problem which often arises is that of making an expansion in cosine series or sine series of an absolutely integrable function $f(x)$ defined on the interval $[0, \pi]$. To expand $f(x)$ in *cosine* series, we can reason as follows: Make the *even* extension of $f(x)$ from the interval $[0, \pi]$ onto the interval $[-\pi, 0]$ [see Fig. 13(a)]. Then all the previous considerations apply to the even extension of $f(x)$, so that its Fourier coefficients can be calculated by the formulas

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \qquad (n = 0, 1, 2, \ldots), \\ b_n &= 0 \qquad (n = 1, 2, \ldots), \end{aligned} \tag{12.3}$$

which involve only the values of $f(x)$ in the interval $[0, \pi]$. Therefore, for computational purposes, there is no need to actually make the even extension of $f(x)$ from $[0, \pi]$ onto $[-\pi, 0]$.

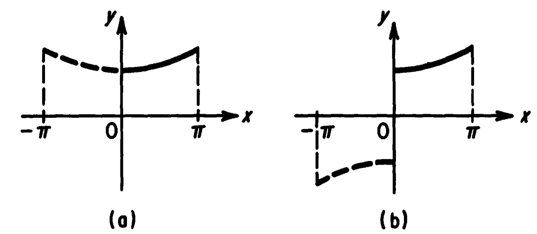

FIGURE 13

To expand $f(x)$ in *sine* series, we first make the *odd* extension of $f(x)$ from the interval $[0, \pi]$ onto the interval $[-\pi, 0]$ [see Fig. 13(b)]. In doing so, the oddness requires that we set $f(0) = 0$. Then, the previous considerations again apply to the odd extension of $f(x)$, so that its Fourier coefficients are given by the formulas

$$
\begin{aligned}
a_n &= 0 \qquad (n = 0, 1, 2, \ldots), \\
b_n &= \frac{2}{\pi}\int_0^\pi f(x) \sin nx \, dx \qquad (n = 1, 2, \ldots),
\end{aligned}
\tag{12.4}
$$

which involve only the values of $f(x)$ in the interval $[0, \pi]$. Therefore, as in the case of cosine series, there is no need to actually make the odd extension of $f(x)$ from $[0, \pi]$ onto $[-\pi, 0]$.

However, in order to avoid mistakes in using the convergence criterion of Sec. 10, it is still recommended that a sketch be made of the function $f(x)$ and its even (or odd) extension onto the interval $[-\pi, 0]$, as well as of its periodic extension (with period 2π) onto the whole x-axis. This sketch will help in investigating the behavior of the "extended" function, which is the function to which the convergence criterion has to be applied.

13. Examples of Expansions in Fourier Series

Example 1. *Expand* $f(x) = x^2$ $(-\pi \leqslant x \leqslant \pi)$ *in Fourier series.* The function $f(x)$ is even; the graph of $f(x)$ together with its periodic extension is shown in Fig. 14. The extended function is continuous and piecewise smooth. Therefore, by the criterion of Sec. 10, its Fourier series converges to $f(x) = x^2$ everywhere in $[-\pi, \pi]$, and converges to the periodic extension of $f(x)$ outside $[-\pi, \pi]$. Moreover, the convergence is absolute and uniform.

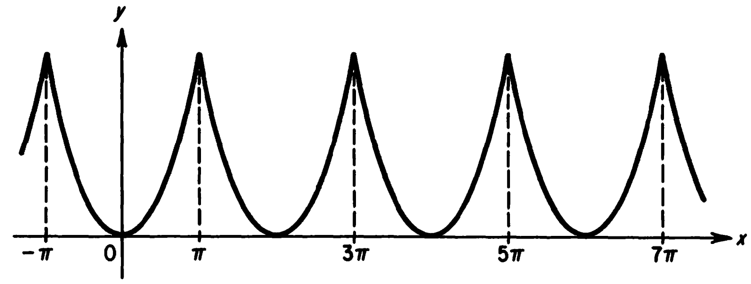

FIGURE 14

A calculation shows that

$$a_0 = \frac{2}{\pi}\int_0^\pi x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_{x=0}^{x=\pi} = \frac{2\pi^2}{3}.$$

Furthermore, integrating by parts, we find that

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi x^2\cos nx\,dx = -\frac{4}{\pi n}\int_0^\pi x\sin nx\,dx \\ &= \frac{4}{\pi n^2}[x\cos nx]_{x=0}^{x=\pi} - \frac{4}{\pi n^2}\int_0^\pi \cos nx\,dx \\ &= \frac{4}{n^2}\cos n\pi = (-1)^n\frac{4}{n^2}, \end{aligned}$$

while $b_n = 0$ ($n = 1, 2, \ldots$), since $f(x)$ is even. Therefore, for $-\pi \leqslant x \leqslant \pi$, we have

$$x^2 = \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right). \tag{13.1}$$

Example 2. *Expand* $f(x) = |x|$ $(-\pi \leqslant x \leqslant \pi)$ *in Fourier series.* The function $f(x)$ is even; Fig. 15 shows the graph of $f(x)$ together with its periodic extension. The extended function is continuous and piecewise smooth, so

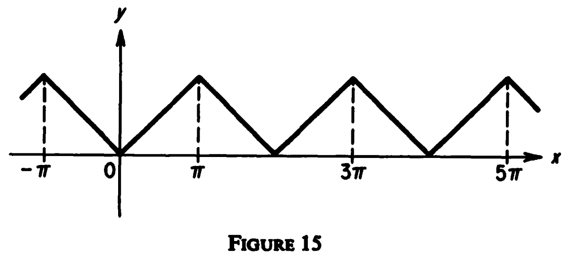

FIGURE 15

that the criterion of Sec. 10 is applicable. Therefore, its Fourier series converges to $f(x) = |x|$ everywhere in $[-\pi, \pi]$ and converges to the periodic extension of $f(x)$ outside $[-\pi, \pi]$. Moreover, the convergence is absolute and uniform.

Since $|x| = x$ for $x \geqslant 0$, we have

$$a_0 = \frac{2}{\pi}\int_0^\pi x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_{x=0}^{x=\pi} = \pi,$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi x\cos nx\,dx = -\frac{2}{\pi n}\int_0^\pi \sin nx\,dx \\ &= \frac{2}{\pi n^2}\left[\cos nx\right]_{x=0}^{x=\pi} = \frac{2}{\pi n^2}\left[\cos n\pi - 1\right] \\ &= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]. \end{aligned}$$

It follows that $a_n = 0$ for even n, and that $a_n = -4/\pi n^2$ for odd n. Finally, $b_n = 0$ $(n = 1, 2, \ldots)$, since $f(x)$ is even. Thus, for $-\pi \leqslant x \leqslant \pi$, we have

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right). \tag{13.2}$$

Example 3. *Expand $f(x) = |\sin x|$ in Fourier series.* This function is defined for all x, and represents a continuous, piecewise smooth, even function. Its graph is shown in Fig. 16. The criterion of Sec. 10 is applicable, and hence $f(x) = |\sin x|$ is everywhere equal to its Fourier series, which is absolutely and uniformly convergent.

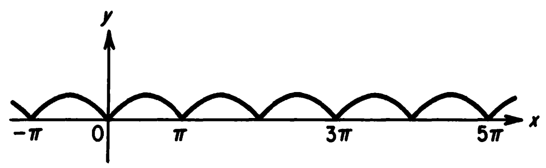

FIGURE 16

Since $|\sin x| = \sin x$ for $0 \leqslant x \leqslant \pi$, we have

$$a_0 = \frac{2}{\pi}\int_0^\pi \sin x\,dx = \frac{4}{\pi},$$

and

$$a_n = \frac{2}{\pi}\int_0^\pi \sin x \cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^\pi [\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= -\frac{1}{\pi}\left[\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_{x=0}^{x=\pi}$$

$$= -\frac{1}{\pi}\left[\frac{(-1)^{n+1}-1}{n+1} - \frac{(-1)^{n-1}-1}{n-1}\right] = -2\frac{(-1)^n+1}{\pi(n^2-1)},$$

for $n \neq 1$, while for $n = 1$

$$a_1 = \frac{2}{\pi}\int_0^\pi \sin x\cos x\,dx = \frac{1}{\pi}\int_0^\pi \sin 2x\,dx = 0.$$

Moreover, $b_n = 0$ $(n = 1, 2, \ldots)$, since $f(x)$ is even. Therefore, for all x we have

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \cdots\right).$$

Example 4. *Expand* $f(x) = x$ $(-\pi < x < \pi)$ *in Fourier series.* The function $f(x)$ is odd; Fig. 17 shows the graph of $f(x)$ together with its periodic extension. The extended function is piecewise smooth and discontinuous at the points $x = (2k+1)\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$. The test of Sec. 10 is applicable, and the Fourier series of $f(x)$ converges to zero at the points of discontinuity.

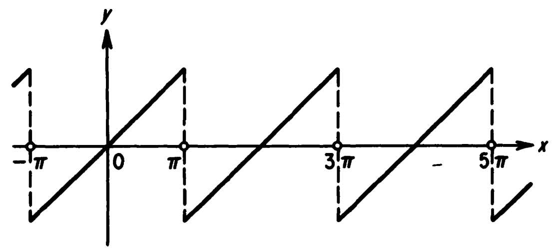

FIGURE 17

Since $f(x)$ is odd

$$a_n = 0 \qquad (n = 0, 1, 2, \ldots),$$

$$b_n = \frac{2}{\pi}\int_0^\pi x\sin nx\,dx$$

$$= -\frac{2}{\pi n}[x\cos nx]_{x=0}^{x=\pi} + \frac{2}{\pi n}\int_0^\pi \cos nx\,dx$$

$$= -\frac{2}{n}\cos n\pi = \frac{2}{n}(-1)^{n+1}.$$

Therefore, for $-\pi < x < \pi$, we have

$$x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right). \tag{13.3}$$

Example 5. *Expand* $f(x) = 1$ $(0 < x < \pi)$ *in sine series.* Making the odd extension of $f(x)$ onto the interval $[-\pi, 0]$ produces a discontinuity at $x = 0$. Figure 18 shows the graph of $f(x)$ and its odd extension, together with its subsequent periodic extension (with period 2π) over the whole x-axis. The convergence criterion of Sec. 10 is applicable to this "extended" function. Therefore, its Fourier series converges to $f(x) = 1$ for $0 < x < \pi$. Outside the interval $0 < x < \pi$, it converges to the function shown in Fig. 18, with the sum of the series being equal to zero at the points $x = k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$.

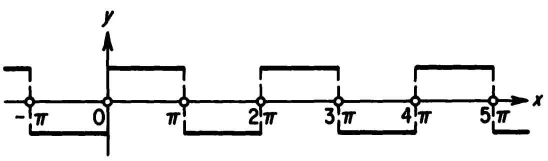

FIGURE 18

Since

$$a_n = 0 \qquad (n = 0, 1, 2, \ldots),$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin nx\,dx$$

$$= \frac{2}{\pi n}[-\cos nx]_{x=0}^{x=\pi} = \frac{2}{\pi n}[1 - (-1)^n],$$

we have

$$1 = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right). \tag{13.4}$$

for $0 < x < \pi$.

Example 6. *Expand* $f(x) = x$ $(0 < x < 2\pi)$ *in Fourier series.* This example bears a superficial resemblance to Example 4, but the difference is immediately apparent if we construct the periodic extension of $f(x)$ (see Fig. 19). The criterion of Sec. 10 is applicable to this extended function. At the points of discontinuity, the Fourier series converges to the arithmetic mean of the right-hand and left-hand limits, i.e., to the value π. The function $f(x)$ is neither even nor odd.

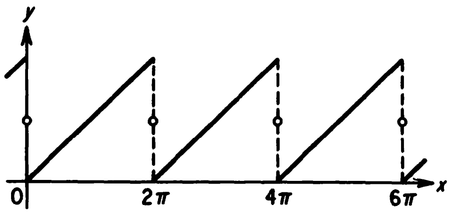

FIGURE 19

Since

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{x=0}^{x=2\pi} = 2\pi,$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi n}[x\sin nx]_{x=0}^{x=2\pi} - \frac{1}{\pi n}\int_0^{2\pi}\sin nx\,dx = 0 \qquad (n = 1, 2, \ldots),$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= -\frac{1}{\pi n}[x\cos nx]_{x=0}^{x=2\pi} + \frac{1}{\pi n}\int_0^{2\pi}\cos nx\,dx = -\frac{2}{n},$$

we have

$$x = \pi - 2\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right), \tag{13.5}$$

for $0 < x < 2\pi$.

Example 7. *Expand* $f(x) = x^2$ $(0 < x < 2\pi)$ *in Fourier series.* This example resembles Example 1, but the graph of the periodic extension of $f(x)$ immediately shows the difference (see Fig. 20). The criterion of Sec. 10 is

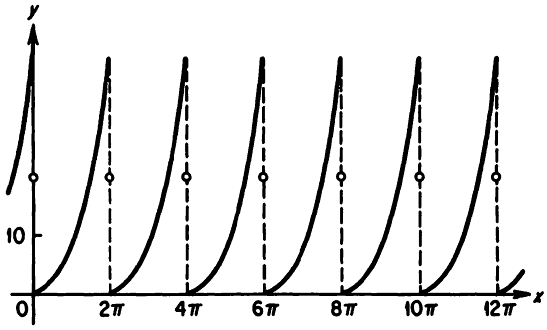

FIGURE 20

applicable, and at the points of discontinuity the series converges to the arithmetic mean of the right-hand and left-hand limits, i.e., to the value $2\pi^2$. The function $f(x)$ is neither even nor odd.

Since

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{x=0}^{x=2\pi} = \frac{8\pi^2}{3},$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx = -\frac{2}{\pi n}\int_0^{2\pi} x \sin nx\,dx$$

$$= \frac{2}{\pi n^2}[x\cos nx]_{x=0}^{x=2\pi} - \frac{2}{\pi n^2}\int_0^{2\pi}\cos nx\,dx = \frac{4}{n^2},$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$

$$= -\frac{1}{\pi n}[x^2\cos nx]_{x=0}^{x=2\pi} + \frac{2}{\pi n}\int_0^{2\pi} x\cos nx\,dx$$

$$= -\frac{4\pi}{n} - \frac{2}{\pi n^2}\int_0^{2\pi}\sin nx\,dx = -\frac{4\pi}{n},$$

we have

$$x^2 = \frac{4\pi^2}{3} + 4\left(\cos x - \pi\sin x + \frac{\cos 2x}{2^2} - \frac{\pi\sin 2x}{2} + \cdots + \frac{\cos nx}{n^2} - \frac{\pi\sin nx}{n} + \cdots\right) \tag{13.6}$$

$$= \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2} - \frac{\pi\sin nx}{n}\right)$$

$$= \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\sin nx}{n},$$

for $0 < x < 2\pi$.

Example 8. *Expand $f(x) = Ax^2 + Bx + C$ $(-\pi < x < \pi)$, where A, B, and C are constants, in Fourier series.* The graph of $f(x)$ is a parabola. By periodic extension, we can obtain a continuous or a discontinuous function, depending on the choice of the constants A, B, and C. Figure 21 shows a possible extension for certain values of A, B, and C.

We could calculate the Fourier coefficients from the appropriate formulas, but there is no need to do so, since we can use the expansions for the functions x^2 and x $(-\pi < x < \pi)$, given in Examples 1 and 4. The result is

$$Ax^2 + Bx + C = \frac{A\pi^2}{3} + C + 4A\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2} - 2B\sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n}.$$

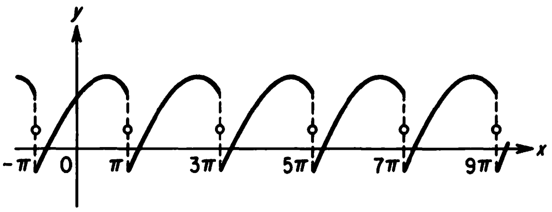

FIGURE 21

Example 9. *Expand* $f(x) = Ax^2 + Bx + C$ $(0 < x < 2\pi)$ *in Fourier series.* Figure 22 shows the periodic extension of $f(x)$ for a certain choice of the constants A, B, and C. Using the expansions of the functions x^2 and x $(0 < x < 2\pi)$, given in Examples 6 and 7, we find that

$$Ax^2 + Bx + C = \frac{4A\pi^2}{3} + B\pi + C + 4A\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} - (4\pi A - 2B)\sum_{n=1}^{\infty}\frac{\sin nx}{n},$$

for $0 < x < 2\pi$.

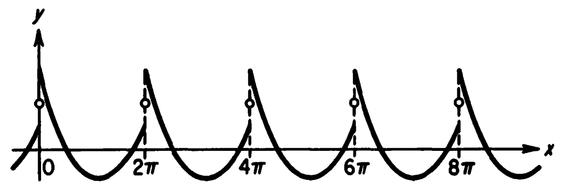

FIGURE 22

We can use these examples to calculate the sums of some important trigonometric series. For example, (13.5) immediately gives

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n} = \frac{\pi - x}{2} \qquad (0 < x < 2\pi), \tag{13.7}$$

and from (13.5) and (13.6), we infer that

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} = \frac{3x^2 - 6\pi x + 2\pi^2}{12} \qquad (0 < x < 2\pi). \tag{13.8}$$

Since the terms of the series on the left do not exceed $1/n^2$ in absolute value, the series is uniformly convergent, which means that its sum is continuous for all x (see Sec. 4). Therefore, (13.8) is valid for $0 \leqslant x \leqslant 2\pi$, and not just for $0 < x < 2\pi$.

Similarly, (13.3) gives

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n} = \frac{x}{2} \qquad (-\pi < x < \pi), \tag{13.9}$$

(13.1) gives

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos nx}{n^2} = \frac{\pi^2 - 3x^2}{12} \qquad (-\pi \leqslant x \leqslant \pi), \tag{13.10}$$

(13.4) gives

$$\sum_{n=0}^{\infty}\frac{\sin (2n+1)x}{2n+1} = \frac{\pi}{4} \qquad (0 < x < \pi), \tag{13.11}$$

and (13.2) gives

$$\sum_{n=0}^{\infty}\frac{\cos (2n+1)x}{(2n+1)^2} = \frac{\pi^2 - 2\pi x}{8} \qquad (0 \leqslant x \leqslant \pi). \tag{13.12}$$

Moreover, subtracting (13.11) from (13.7), we obtain

$$\sum_{n=1}^{\infty}\frac{\sin 2nx}{2n} = \frac{\pi - 2x}{4} \qquad (0 < x < \pi), \tag{13.13}$$

and subtracting (13.2) from (13.8), we obtain

$$\sum_{n=1}^{\infty}\frac{\cos 2nx}{(2n)^2} = \frac{6x^2 - 6\pi x + \pi^2}{24} \qquad (0 \leqslant x \leqslant \pi). \tag{13.14}$$

These formulas also allow us to calculate the sums of some *numerical* series. For example, if we set $x = 0$, (13.8) and (13.10) become

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots, \quad \frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

while if we set $x = \pi/2$, (13.11) becomes

$$\frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots$$

14. The Complex Form of a Fourier Series

Let the function $f(x)$ be integrable on the interval $[-\pi, \pi]$, and form its Fourier series