The coefficients c_n given by (14.4) are called the *complex Fourier coefficients* of the function $f(x)$. They satisfy the relations

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{14.7}$$

In fact, by Euler's formula and (14.4), we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} f(x)\cos nx\,dx - i\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right]$$
$$= \tfrac{1}{2}(a_n - ib_n) = c_n$$

for positive indices and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} f(x)\cos nx\,dx + i\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right]$$
$$= \tfrac{1}{2}(a_n + ib_n) = c_{-n}$$

for negative indices. It is useful to bear in mind that if $f(x)$ is *real*, then the coefficients c_n and c_{-n} are *complex conjugates*. This is an immediate consequence of (14.4).

Incidentally, we note that the formula (14.7) can also be obtained directly, just as the formulas (14.2) were (see Sec. 6), if we assume that the sign $=$ appears in (14.6) instead of the sign $\sim$ and that term by term integration is legitimate. In fact, multiplying both sides of the equality

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

by e^{-inx} and integrating term by term over the interval $[-\pi, \pi]$, we obtain

$$\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = 2\pi c_n, \tag{14.8}$$

since for $k \neq n$ (see Sec. 5) we have

$$c_k \int_{-\pi}^{\pi} e^{i(k-n)x}\,dx$$
$$= c_k \int_{-\pi}^{\pi} [\cos(k-n)x + i\sin(k-n)x]\,dx = 0,$$

i.e., all the integrals on the right vanish except the one corresponding to the index $k = n$, while for $k = n$, we obtain the number $2\pi c_n$. The formula (14.7) is an immediate consequence of (14.8).

15. Functions of Period $2l$

If it is required to expand a function $f(x)$ of period $2l$ in Fourier series, we set $x = lt/\pi$, thereby obtaining the function $\varphi(t) = f(lt/\pi)$ of period 2π (see Sec. 3). For $\varphi(t)$ we can form the Fourier series

$$\varphi(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt), \tag{15.1}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(t) \cos nt \, dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lt}{\pi}\right) \cos nt \, dt \qquad (n = 0, 1, 2, \ldots),$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(t) \sin nt \, dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lt}{\pi}\right) \sin nt \, dt \qquad (n = 1, 2, \ldots).$$

Returning to the original variable x by setting $t = \pi x/l$, we obtain

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{\pi n x}{l} + b_n \sin\frac{\pi n x}{l}\right), \tag{15.2}$$

where

$$\begin{aligned} a_n &= \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{\pi n x}{l}\, dx \qquad (n = 0, 1, 2, \ldots), \\ b_n &= \frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{\pi n x}{l}\, dx \qquad (n = 1, 2, \ldots). \end{aligned} \tag{15.3}$$

The coefficients (15.3) are still called the *Fourier coefficients* of $f(x)$, and the series (15.2) is still called the *Fourier series* of $f(x)$. If the equality holds in (15.1), then the equality holds in (15.2), and conversely.

We could have constructed a theory of series of the form (15.2) directly, by starting from a trigonometric system of the form

$$1, \cos\frac{\pi x}{l}, \sin\frac{\pi x}{l}, \ldots, \cos\frac{\pi n x}{l}, \sin\frac{\pi n x}{l}, \ldots, \tag{15.4}$$

just as we did in the case of the basic trigonometric system (5.1). The system (15.4) consists of functions with the common period $2l$, and it is easily verified that these functions are orthogonal on every interval of length $2l$. The considerations of Secs. 6, 7, 10, 12, and 14 can be repeated as applied to the system (15.4), and the result is a formulation analogous to that given in these sections, except that π is replaced by l. In particular, instead of a function $f(x)$ of period $2l$, we can consider a function defined only on the interval $[-l, l]$ [or on any other interval of length $2l$, provided we appropriately change the limits of integration in (15.3)]. The Fourier series of such a function is identical with that of its periodic extension onto the whole

x-axis. The convergence criterion of Sec. 10 continues to "work," if we replace the period 2π by the period $2l$.

If $f(x)$ is even, the formulas (15.3) become

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{\pi n x}{l}\,dx \qquad (n = 0, 1, 2, \ldots),$$
$$b_n = 0 \qquad (n = 1, 2, \ldots), \tag{15.5}$$

while if $f(x)$ is odd, they become

$$a_n = 0 \qquad (n = 0, 1, 2, \ldots)$$
$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{\pi n x}{l}\,dx \qquad (n = 1, 2, \ldots). \tag{15.6}$$

As in Sec. 12, we can use this fact to expand a function $f(x)$ defined only on the interval $[0, l]$ in cosine series or in sine series (making the even or the odd extension of $f(x)$ onto the interval $[-l, 0]$).

The complex form of the series (15.2) is

$$f(x) \sim \sum_{n=-\infty}^{+\infty} c_n e^{i\pi n x/l},$$

where

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-i\pi n x/l}\,dx \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

or

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2} \qquad (n = 1, 2, \ldots).$$

Example 1. *Expand the function $f(x)$, defined by*

$$f(x) = \begin{cases} \cos\dfrac{\pi x}{l} & \text{for } 0 \leqslant x \leqslant \dfrac{l}{2}, \\ 0 & \text{for } \dfrac{l}{2} < x \leqslant l \end{cases}$$

in cosine series. Figure 23 shows the graph of $f(x)$ and its even extension onto the interval $[-l, 0]$, together with its subsequent periodic extension

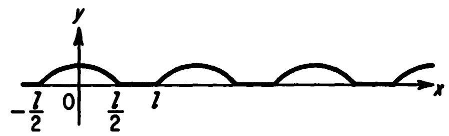

FIGURE 23

(with period $2l$) onto the whole x-axis. The convergence criterion can obviously be applied everywhere.

For $l/2 < x \leqslant l$, we have $f(x) = 0$, so that

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^{l/2} \cos\frac{\pi x}{l}\,dx = \frac{2}{\pi},$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{\pi n x}{l}\,dx = \frac{2}{l}\int_0^{l/2} \cos\frac{\pi x}{l}\cos\frac{\pi n x}{l}\,dx.$$

Making the substitution $\pi x/l = t$, we obtain

$$a_n = \frac{2}{\pi}\int_0^{\pi/2} \cos t \cos nt\,dt = \frac{1}{\pi}\int_0^{\pi/2} [\cos (n+1)t + \cos (n-1)t]\,dt,$$

whence

$$a_1 = \frac{1}{\pi}\int_0^{\pi/2} (\cos 2t + 1)\,dt = \frac{1}{\pi}\left[\frac{\sin 2t}{2} + t\right]_{t=0}^{t=\pi/2} = \frac{1}{2},$$

$$a_n = \frac{1}{\pi}\left[\frac{\sin (n+1)t}{n+1} + \frac{\sin (n-1)t}{n-1}\right]_{t=0}^{t=\pi/2} \qquad (n > 1).$$

Therefore, for odd $n > 1$

$$a_n = 0,$$

while, for even n

$$a_n = -\frac{2(-1)^{n/2}}{\pi(n^2-1)}, \quad b_n = 0 \qquad (n = 1, 2, \ldots).$$

Thus we have

$$\frac{1}{\pi} + \frac{1}{2}\cos\frac{\pi x}{l} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos\frac{2\pi n x}{l} = \begin{cases} \cos\frac{\pi x}{l} & \text{for } 0 \leqslant x \leqslant \frac{l}{2}, \\ 0 & \text{for } \frac{l}{2} < x \leqslant l. \end{cases}$$

This series converges on the whole x-axis to the function shown in Fig. 23.

Example 2. *Expand the function $f(x)$, defined by*

$$f(x) = \begin{cases} x & \text{for } 0 \leqslant x \leqslant \frac{l}{2}, \\ l - x & \text{for } \frac{l}{2} < x \leqslant l, \end{cases}$$

in sine series. Figure 24 shows the graph of $f(x)$ and its odd extension onto the interval $[-l, 0]$, together with its subsequent periodic extension (with period $2l$) onto the whole x-axis. The convergence criterion can be applied everywhere.

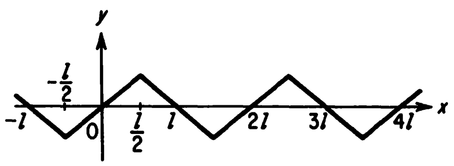

FIGURE 24

In this case, we have

$$a_n = 0 \qquad (n = 0, 1, 2, \ldots),$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{\pi n x}{l}\,dx$$

$$= \frac{2}{l}\int_0^{l/2} x\sin\frac{\pi n x}{l}\,dx + \frac{2}{l}\int_{l/2}^{l}(l-x)\sin\frac{\pi n x}{l}\,dx \qquad (n = 1, 2, \ldots).$$

Setting $\pi x/l = t$, we obtain

$$b_n = \frac{2l}{\pi^2}\int_0^{\pi/2} t\sin nt\,dt + \frac{2l}{\pi^2}\int_{\pi/2}^{\pi}(\pi - t)\sin nt\,dt$$

$$= \frac{2l}{\pi^2}\left[-\frac{t\cos nt}{n}\right]_{t=0}^{t=\pi/2} + \frac{2l}{\pi^2 n}\int_0^{\pi/2}\cos nt\,dt$$

$$+ \frac{2l}{\pi^2}\left[-\frac{(\pi - t)\cos nt}{n}\right]_{t=\pi/2}^{t=\pi} - \frac{2l}{\pi^2 n}\int_{\pi/2}^{\pi}\cos nt\,dt$$

$$= \frac{4l}{\pi^2 n^2}\sin\frac{\pi n}{2}.$$

Therefore

$$\frac{4l}{\pi^2}\left(\sin\frac{\pi x}{l} - \frac{1}{3^2}\sin\frac{3\pi x}{l} + \frac{1}{5^2}\sin\frac{5\pi x}{l} - \cdots\right) = \begin{cases} x & \text{for } 0 \leqslant x \leqslant \dfrac{l}{2}, \\ l - x & \text{for } \dfrac{l}{2} < x \leqslant l. \end{cases}$$

This series converges on the whole x-axis to the function shown in Fig. 24.

PROBLEMS

1. Expand the following functions in Fourier series:

a) $f(x) = e^{ax}$ $(-\pi < x < \pi)$, where $a \neq 0$ is a constant;
b) $f(x) = \cos ax$ $(-\pi \leqslant x \leqslant \pi)$, where a is not an integer;

c) $f(x) = \sin ax\ (-\pi < x < \pi)$, where a is not an integer;

d) $f(x) = \begin{cases} 0 & \text{for} \quad -\pi < x < 0, \\ x & \text{for} \quad 0 \leqslant x < \pi. \end{cases}$

2. Using the expansion of Prob. 1b, show that

$$\frac{1}{\sin z} = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{z - n\pi} + \frac{1}{z + n\pi}\right],$$

$$\cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \left[\frac{1}{z - n\pi} + \frac{1}{z + n\pi}\right],$$

where z is any number which is not a multiple of π.

3. Using the expansion of Prob. 1a, expand the following functions in Fourier series:

a) The hyperbolic cosine

$$\cosh x = \frac{e^{ax} + e^{-ax}}{2} \qquad (-\pi \leqslant x \leqslant \pi);$$

b) The hyperbolic sine

$$\sinh x = \frac{e^{ax} - e^{-ax}}{2} \qquad (-\pi < x < \pi).$$

4. Expand the following functions in Fourier cosine series:

a) $f(x) = \sin ax\ (0 \leqslant x \leqslant \pi)$, where a is not an integer;

b) $f(x) = \begin{cases} 1 & \text{for} \quad 0 \leqslant x \leqslant h, \\ 0 & \text{for} \quad h < x \leqslant \pi; \end{cases}$

c) $f(x) = \begin{cases} 1 - \dfrac{x}{2h} & \text{for} \quad 0 \leqslant x \leqslant 2h, \\ 0 & \text{for} \quad 2h < x \leqslant \pi. \end{cases}$

5. Expand the following functions in Fourier sine series:

a) $f(x) = \begin{cases} \sin \dfrac{\pi x}{l} & \text{for} \quad 0 \leqslant x < \dfrac{l}{2}, \\ 0 & \text{for} \quad \dfrac{l}{2} < x \leqslant l; \end{cases}$

b) $f(x) = \begin{cases} \sin \dfrac{\pi x}{l} & \text{for} \quad 0 \leqslant x < \dfrac{l}{2}, \\ -\sin \dfrac{\pi x}{l} & \text{for} \quad \dfrac{l}{2} < x \leqslant l. \end{cases}$

6. Expand the periodic function

$$f(x) = \left|\cos \frac{\pi x}{l}\right|, \quad l = \text{const}, \ l > 0$$

in Fourier series.

7. Let $f(x)$ have period 2π and let $|f(x) - f(y)| \leqslant c|x - y|^{\alpha}$, for some constants