1.1

# Chapter One : Matrices and Systems of Linear Equations.

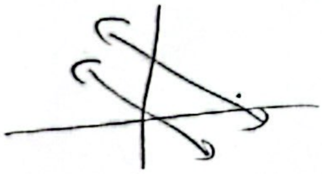
## sec 1.1 System of Linear equations.

Linear equation $\quad ax = b \quad$ one variable

$ax + by = c \quad$ two variables.

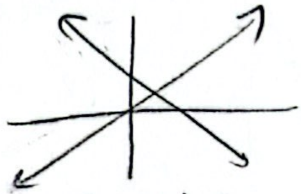
$a_1x_1 + a_2x_2 + \dots + a_nx_n \quad$ n-variables (unknowns.)

Each equation represents a Line



no solution

(a)



one solution
unique solution

(b)

infinite number of solutions

(c)

If the system has no solution then it is called <u>Inconsistent</u>

If the system has at least one solution it is <u>Consistent</u>

1) example: $\qquad x_1 + x_2 + \dots =$

Consider the following systems:

(a) $\begin{aligned} x + 2y &= 5 \\ 2x + 3y &= 8 \end{aligned}$ $\qquad$ (b) $\begin{aligned} x - y + z &= 2 \\ 2x + y - z &= 4 \end{aligned}$ $\qquad$ (c) $\begin{aligned} x + y &= 2 \\ x - y &= 1 \\ x &= 4 \end{aligned}$

find the (if any) solution sets for each system.

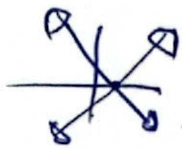
<u>Solution</u> التعويض

(a) $\{(1,2)\}$

(b) $\{(2,a,a) : a \in \mathbb{R}\}$

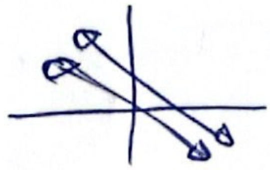
(c) no solution : Inconsistent system.

<u>ex</u> (i) $\begin{aligned} x + y &= 2 \\ x - y &= 2 \end{aligned}$ $\qquad \{(2,0)\}$



intersect at one point

(ii) $\begin{aligned} x + y &= 2 \\ x + y &= 1 \end{aligned}$



parallel

(iii) $\begin{aligned} x + y &= 2 \\ -x - y &= -2 \end{aligned}$

same Line

Homogeneous

(c) Eno 15

Definition: Two systems are said to be equivalent if the have the same variables and the same solution set.

ex Consider the systems:

(1)
$$3x + 2y - z = -2$$
$$y = 3$$
$$2z = 4$$

and

(2)
$$3x + 2y - z = -2 \text{ ——— } ①$$
$$-3x - y + z = 5 \text{ ——— } ②$$
$$3x + 2y + z = 2 \text{ ——— } ③$$

Are the systems equivalent.

Solution:

Consistent system

(1) $2z = 4 \Rightarrow z = 2$

$y = 3$

thus: $3x + 6 - 2 = -2 \Rightarrow 3x = -6 \Rightarrow \boxed{x = -2}$

$\boxed{(-2, 3, 2)}$

Consistent system

(2) $① + ② \Rightarrow y = 3$

$① + ③ \Rightarrow 6x + 12 = 0 \Rightarrow 6x = -12 \Rightarrow \boxed{x = -2}$

$\boxed{(-2, 3, 2)}$

The two systems has the same variables: $x, y, z$ and the same solution set $\{(-2, 3, 2)\}$.

Then are equivalent.

# Notes

(1) The System of Linear equations:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m = b_2$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m = b_n$$

$a_{ij} \in \mathbb{R} \quad \forall i,j$

(1) $n \times m$ system where $n$ is the number of equations and $m$ is the number of variables.

(2) If the system has solution it is called Consistent.
If not it is called Inconsistent

(3) If $b_1 = b_2 = \cdots = b_n =$ Zero
then the system is called Homogeneous
If not then it is called nonhomogeneous

(4) To obtain equivalent system:
(a) We can change the Order of the equations.
(b) Both sides of an equation can be multiplied by any constant $\neq 0$
(c) We can add (subtract) a multiple of one equation to another.

(5) If the solution set contains only one point then the system has a unique solution.

(3) The Matrix form of the system is

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2m} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} & b_n \end{array}\right]$$

## Back substitution Method for nxn systems

For any nxn system in the strict triangular form we can use the back substitution method.

**Ex**

$$\left.\begin{aligned} 3x + 2y + z &= 1 \\ y + 2z &= 2 \\ 2z &= 4 \end{aligned}\right\} \Rightarrow \begin{matrix} 2, 3, 1 \\ 0, 1, 2 \\ 0, 0, -3 \end{matrix} \neq 0$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 3 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -3 & 2 \end{array}\right]$$

**Ex**

The solution is $\{(-3, 4, 2)\}$.

**Ex** Solve the System

$$x_1 + 2x_2 + x_3 = 3$$
$$3x_1 - x_2 - 3x_3 = -1$$
$$2x_1 + 3x_2 + x_3 = 4$$

**Solution** We will try to write the strict triangular form:

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$$x_1 + 2x_2 + x_3 = 3 \quad \text{——— } R_1$$
$$-3x_1 + x_2 + 3x_3 = 1 \quad \text{——— } R_2$$
$$2x_1 + 3x_2 + x_3 = 4 \quad \text{——— } R_3$$

$$x_1 + 2x_2 + x_3 = 3$$

$(3R_1 + R_2)$ :
$$7x_2 + 6x_3 = 10$$
$$2x_1 + 3x_2 + x_3 = 4$$

$(-3R_1 + R_2)$

$$x_1 + 2x_2 + x_3 = 3$$
$$7x_2 + 6x_3 = 10$$

$-2R_1 + R_3$
$$-x_2 - x_3 = -2$$

$$x_1 + 2x_2 + x_3 = 3$$
$$7x_2 + 6x_3 = 10$$

$(7R_3)$
$$-7x_2 - 7x_3 = -14$$

$$x_1 + 2x_2 + x_3 = 3$$
$$7x_2 + 6x_3 = 10$$
$$-x_3 = -4$$

$$x_1 + 2x_2 + x_3 = 3$$
$$7x_2 + 6x_3 = 10$$
$$x_3 = 4$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 7 & 6 & 10 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

In the strict triangular form.
We can use back substitution:

From $R_3$ : $\boxed{x_3 = 4}$

from $R_2$ : $7x_2 + 6(4) = 10 \Rightarrow x_2 = -2$

from $R_1$ : $x_1 + 2(-2) + 4 = 3 \Rightarrow \boxed{x_1 = 3}$

The solution set is $\{(3, -2, 4)\}$

It is unique solution.

# Matrix form of a system:

**example**

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 3x_1 - x_2 - 3x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 4 \end{aligned}$$

The matrix form (representation) of this system is:

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix}}_{3 \times 3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

$3 \times 3$: number of rows ($\boxed{\text{equ.}}$) $\times$ number of columns ($\boxed{\text{variables}}$)

The Coefficient matrix: $\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix}$

The augmented matrix

pivot

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right] \longrightarrow \text{pivotal row}$$

To convert this system to a strict triangular system we can employ the following on the augmented matrix

1. Interchange two rows
2. Multiply a row by a non zero real number.
3. Replace a row by its sum with a multiple of another row.

$$\begin{array}{c} \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ \boxed{3} & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right]$$

$$\begin{array}{c} R_1 \\ 3R_1-R_2 \\ \ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 7 & 6 & 10 \\ 0 & 1 & 1 & 2 \end{array}\right]$$

$$\begin{array}{c} \\ \\ -7R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 7 & 6 & 10 \\ 0 & -7 & -7 & -14 \end{array}\right]$$

$$\begin{array}{c} \\ R_3+R_2 \\ \ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 7 & 6 & 10 \\ 0 & 0 & -1 & -4 \end{array}\right]$$

$-x_3 = -4 \Rightarrow \boxed{x_3 = 4}$

$7x_2 + 6(4) = 10 \Rightarrow x_2 = -2$

$x_1 + 2(-2) + 1(4) = 3 \Rightarrow x_1 = 3$

The solution set $= \{(3,-2,4)\}$.

H.W example 4 page 8 :

**Ex** Use the Elementary Row operations to Solve the system:

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$$
\begin{aligned}
-x_2 - x_3 + x_4 &= 0\\
x_1 + x_2 + x_3 + x_4 &= 6\\
2x_1 + 4x_2 + x_3 - 2x_4 &= -1\\
3x_1 + x_2 - 2x_3 + 2x_4 &= 3
\end{aligned}
$$

Solution

$$
\begin{array}{c}
R_1\\ R_2\\ R_3\\ R_4
\end{array}
\left[\begin{array}{cccc|c}
0 & -1 & -1 & 1 & 0\\
1 & 1 & 1 & 1 & 6\\
2 & 4 & 1 & -2 & -1\\
3 & 1 & -2 & 2 & 3
\end{array}\right]
$$

Interchange $R_1$ and $R_2$

$$
\left[\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 6\\
0 & -1 & -1 & 1 & 0\\
2 & 4 & 1 & -2 & -1\\
3 & 1 & -2 & 2 & 3
\end{array}\right]
$$

pivotal row

$$
\begin{array}{c}
\\ \\ -2R_1+R_3\\ 3R_1+R_4
\end{array}
\left[\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 6\\
0 & -1 & -1 & 1 & 0\\
0 & 2 & -1 & -4 & -13\\
0 & -2 & -5 & -1 & -15
\end{array}\right]
$$

$$
\begin{array}{c}
\\ \\ 2R_2+R_3\\ -2R_2+R_4
\end{array}
\left[\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 6\\
0 & -1 & -1 & 1 & 0\\
0 & 0 & -3 & -2 & -13\\
0 & 0 & -3 & -3 & -15
\end{array}\right]
$$

$$
\left[\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 6\\
0 & 1 & 1 & -1 & 0\\
0 & 0 & 3 & 2 & 13\\
0 & 0 & 0 & -1 & -2
\end{array}\right]
$$

The current augmented matrix represent strictly triangular System.

$\boxed{x_4 = 2}$

$3x_3 + 2(x_4) = 13$

$3x_3 + 4 = 13$

$\Rightarrow \boxed{x_3 = 3}$

$x_2 + x_3 - x_4 = 0$

$x_2 + 3 - 2 = 0 \Rightarrow \boxed{x_2 = -1}$ $\{(2,-1,3,2)\}$

$x_1 + x_2 + x_3 + x_4 = 6$

$x_1 + 4 = 6$

$\boxed{x_1 = 2}$

**Note** ① If an $n \times n$ system can be reduced to strictly triangular form, then it will have a unique solution i.e the solution set contains just one point.

② The method to obtain strictly triangular form may breaks down

if

(a) If all possible choices for a pivot element are equal to zero.

or

(b) If the given system is $n \times m$ when $n \neq m$.

Alternativly: we can reduce the system ~~for~~ to **echelon form**.

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## Elementary row operations:

1. Interchange two rows
2. Multiply a row by a nonzero constant
3. Replace a row by its sum with a multiple of another row.

To solve $n \times m$ systems of Linear equations one can use the Gaussian elimination method that is to convert the system to row echelon form r.e.f and get equivalent system.

Matrix is in the row echelon form if:

1. The first nonzero entry in each nonzero row is 1
2. The row ~~which~~ all whose all entries are zero is below
3. k row number of Leading zero is $n$
   k+1 row " " " " is $> n$

ex

The matrix is in the row echelon form if

① The first nonzero entry in each non zero row is 1

② Rows with all entries zeros are below.

③ row : k  number of Leading zeros n
row : k+1  number of Leading zeros > n

<u>example</u>

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \checkmark \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \checkmark \quad \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \checkmark , \quad \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & 6 \end{bmatrix}$$

$$\underset{\text{No}}{\begin{bmatrix} 2 & 4 & 6 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{bmatrix}} \quad \underset{\text{No}}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} , \quad \underset{\text{No}}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} , \quad \underset{\text{No}}{\begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}}$$

<u>Ex</u> Use $\boxed{\text{Gaussian elimination}}$ to obtain coefficient matrix in row echelon form.

$\underline{\underline{x}}$ Use Gaussian elimination to solve !

$$x_1 + 2x_2 + x_3 = 1$$
$$2x_1 - x_2 + x_3 = 2$$
$$4x_1 + 3x_2 + 3x_3 = 4$$
$$2x_1 - x_2 + 3x_3 = 5$$

Over

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 2 & -1 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 \end{array}\right]$$

$$\left[\begin{array}{cccc} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1/5 & 0 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_3 = \frac{3}{2}$$

$$x_2 = -\frac{1}{5}\left(\frac{3}{2}\right) = -\frac{3}{10}$$

$$x_1 + 2\left(-\frac{3}{10}\right) + \left(\frac{3}{2}\right) = 1$$

$$x_1 = 0.1$$

$$\left(\frac{3}{2}, -\frac{3}{10}, 0.1\right)$$

unique solution

$$x_1 + x_2 = 1$$
$$x_1 - x_2 = 3$$
$$-x_1 + 2x_2 = -2$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 2 & -2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 3 & -1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array}\right]$$

since $0x_1 + 0x_2 = 2$
$0 \neq 2$ $\rightarrow\leftarrow$

No solution

---

<u>ex</u>

$$x_1 + 2x_2 + x_3 = 1$$
$$2x_1 - x_2 + x_3 = 2$$
$$4x_1 + 3x_2 + 3x_3 = 4$$
$$3x_1 + x_2 + 2x_3 = 3$$

$(4\times 3)$

Infinite number of solut.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1/5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_3 = a$, $a \in \mathbb{R}$ is free variable

$x_2 + \frac{1}{5}(a) = 0 \Rightarrow x_2 = -\frac{a}{5}$

$x_1 + 2x_2 + x_3 = 1 \rightarrow x_1 = 1 - 3 a$

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Ex solve

$$x_1 + 2x_2 + x_3 = 1$$
$$2x_1 + 4x_2 + 2x_3 = 3$$

Underdetermined

no sol

[illegible Arabic]

infinit..

sol unique.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad \text{no solutn}$$

ex

$$x_1 + x_2 + x_3 + x_4 + x_5 = 2$$
$$x_1 + x_2 + x_3 + 2x_4 + 2x_5 = 3$$
$$x_1 + x_2 + x_3 + 2x_4 + 3x_5 = 2$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

$$\begin{array}{cccccc} x_1 & & & x_4 & x_5 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

let $x_2 = a$

$x_3 = b$

$\boxed{x_5 = -1}$

$x_4 + x_5 = 1 \Rightarrow \boxed{x_4 = 2}$

$x_1 = 2 - [1 + a + b]$

$\boxed{x_1 = 1 - a - b}$

# Row Echelon form

If the reduction to strict triangular form breaks down or if the system is $n \times m$ and $n \neq m$, we will use the elementary row operations to get the Row echelon form (r.e.f)

**Example (1)** Consider the system represented by the augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ R_1+R_2 \\ 2R_1+R_3 \\ \\ -R_1+R_5 \end{array} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{r} \rightarrow \\ \rightarrow \\ \\ \\ \\ \end{array} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right]$$

This system is Inconsistent since by $R_4$ : $0 = -4$
and by $R_5$ : $0 = -3$

<u>Now</u>

If have
$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 1 & 1 & 2 & 2 & 4 & 4 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Leading variables are $x_1, x_3, x_5$
free variables are $x_2, x_4$
$\beta \quad \alpha$

$\boxed{x_5 = 3}$

$x_3 + x_4 + 2x_5 = 0 \Rightarrow x_3 + x_4 = -6$

$\boxed{x_3 = -6 - \alpha}$

$x_1 + x_2 + x_3 + x_4 + x_5 = 1$

$x_1 = 4 - \beta$

the solution set is

$$\{ (4-\beta, \beta, -6-\alpha, \alpha, 3) : \alpha, \beta \in \mathbb{R} \}$$

<u>ex</u> $(4, 0, -6, 0, 3)$
$(3, 1, -6, 0, 3)$

This form is called row echelon form and the method is called <u>Gaussian elimination</u>.

## Notes

If number of equations $= m$
and number of variables $= n$

the system will be
$m \times n$ system.

- case (1) $n = m$
  - no solution
  - one solution (unique solution)
  - infinit
- $m > n$
  - no solution
  - unique solution
  - infinite number of sol
  - Overdetermined Systems
- $m < n$
  - no solution
  - infinite number of solut.
  - Underdetermined systems

1) <u>No solution</u> $\left[\begin{array}{cccc|c} 0 & 0 & 0 \cdots & 0 & b \end{array}\right]$, $b \neq 0$

2) <u>Infinite number of solution</u> : free variable(s)

3) one solution $\equiv$ unique solution $\left[\begin{array}{c|c} \text{strict triangular form} & \\ 0 \; 0 \; 0 \; 0 \; 0 & 0 \end{array}\right]$ !!

For a Homogeneous system
- unique solution
- infinite number of solutions

# Reduced Row Echelon form: rref

A matrix is in rref if:

(1) The matrix is in row echelon form.
(2) The first nonzero entry in each row is the only nonzero in its column.

**ex** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

**Example (1)** Use Gauss-Jordan reduction to solve

**Note**: Gauss-Jordan reduction is the process of using the elementary row operations to convert the matrix to reduced ref $\equiv$ (rref)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right] \sim_{(1)} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & -10 \\ 0 & -1 & -1 & -2 \end{array}\right] \sim_{(2)} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -7 & -6 & -10 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

we get the rref: $x_3 = 4$, $x_2 = -2$, $x_1 = 3$ $\quad (3, -2, 4)$ unique solution.

**Ex②** Use Gauss – Jordan reduction to solve (example page 16)

⋮

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right] \; \frac{1}{4}$$

$$\cancel{\left[\begin{array}{cccc|} 1 & -1 & 1 & -3 \\ 0 & 4 & -4 & 8 \\ 0 & & & \end{array}\right.} \sim \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & -3 & 3 & 0 \end{array}\right] \; \div 3$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

$$\sim \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \end{array}$$

$x_4$ is free variable, Let $\boxed{x_4 = a}$, $a \in \mathbb{R}$

by $R_3$: $x_3 - x_4 = 0 \Rightarrow x_3 - a = 0 \Rightarrow \boxed{x_3 = a}$

by $R_2$: $x_2 + x_4 = 0 \Rightarrow x_2 + a = 0 \Rightarrow \boxed{x_2 = -a}$

by $R_1$: $x_1 - x_4 = 0 \Rightarrow x_1 - a = 0 \Rightarrow \boxed{x_1 = a}$

The solution is $\{(a, -a, a, a) ; a \in \mathbb{R}\}$

example $(0,0,0,0)$
$(1,-1,1,1)$
$(0.5, -0.5, 0.5, 0.5)$ ....... infinite number of solutions

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Consider a Linear System whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right]$$

find the values of $a, b$

(1) the system is inconsistent.

(2) the system has infinitely many solutions

(3) the system has unique (one) solution.

Sol write the ref

$$\begin{array}{c} \\ -R_1+R_2 \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & a-3 & b-2 \end{array}\right] \underset{-2R_2+R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a-5 & b-4 \end{array}\right]$$

[1] $0 \; 0 \; 0 \; 0 =$ non zero

so: $a-5=0 \Rightarrow \boxed{a=5}$

$b-4 \neq 0 \Rightarrow b \neq 4$

$\left.\right\}$ $a=5$, $b \in \mathbb{R} \setminus \{4\}$

The result $[0 \; 0 \; 0 \mid \text{nonzero}]$

(2) $0 \; 0 \cdots 0 = 0$ to get free variables

$\left.\begin{array}{l} a-5=0 \\ b-4=0 \end{array}\right\}$ $\begin{array}{l} a=5 \\ b=4 \end{array}$

[3] $a-5 \neq 0$ i.e $a \neq 5$

$b-4$ any real number so $b \in \mathbb{R}$

i.e $a \in \mathbb{R} \setminus \{5\}$

$b \in \mathbb{R}$

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