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1.1 Propositional Logic

Our discussion begins with an introduction to the basic building blocks of logic-propositions.

Definition: A proposition

A proposition(or a statement) is a declarative sentence that is either true or false, but not both.

Examples of propositions :

1. 5 + 2 = 8. (F)
2. Beijing is the capital of China (T)
3. 1 + 1 = 2. (T)
4. 2 is a prime number (T)

But, the following are NOT propositions:

1. It is raining today. (either T or F)
2. How are you? (a question is not a proposition)
3. x + 5 = 3 (since x is not specified, neither true nor false)
4. She is very talented. (since she is not specified, neither true nor false)
5. There are other life forms on other planets in the universe. (either T or F)
6. Just do it ! (imperative command)

We use letters to denote **propositional variables** (or **statement variables**), that is, variables that represent propositions, just as letters are used to denote numerical variables.

Definition: Negation

Let p be a proposition. The statement "It is not the case that p." is another proposition, called the **negation of p**. The negation of p is denoted by ¬ p and read as "not p."

Example 1: Find the negation of the proposition
p : "I have brown hair."
Solution: The negation is
$\neg p$: "It is not the case that I have brown hair"

This negation can be more simply expressed as

$\neg p$: "I do **not** have brown hair."

Other examples:

– 5 + 2 ≠ 8.
– 10 is not a prime number.
– It is not the case that buses stop running at 9:00pm.

Example 2 : Negate the following propositions :

1. Today is Sunday.
2. 5 is a prime number.
3. "Ahmad's smartphone has at least 32GB of memory"

Solution:

1. Today is not Sunday.
2. It 5 is not a prime number
3. It is not the case that Ahmad's smartphone has at least 32GB of memory
   or This negation can also be expressed as
   - "Ahmad's smartphone does not have at least 32GB of memory"

   or even more simply as
   - "Ahmad's smartphone has less than 32GB of memory."

• A **truth table** displays **the relationships between truth values** (T or F) of different propositions.

Truth table for NOT:

p	$\neg p$
T	F
F	T

Definition: Conjunction

Let p and q be propositions. The proposition **"p and q"** denoted by $p \wedge q$, is true when both p and q are true and is false otherwise. The proposition $p \wedge q$ is called the **conjunction** of p and q.

Example 3 : Find the conjunction of the propositions *p* and *q* where *p* is the proposition “Khalid’s PC has more than 16 GB free hard disk space” and *q* is the proposition “The processor in Khalid’s PC runs faster than 1 GHz.”

Solution:

$p \wedge q$: “Khalid’s PC has more than 16 GB free hard disk space, and its processor runs faster than 1 GHz.”

Other examples:

- Ptuk is located in Ramallah and 5 + 2 = 8
- It is raining today and 2 is a prime number.
- 2 is a prime number and $5 + 2 \neq 8$.
- 13 is a perfect square and 9 is a prime.

Definition: Disjunction

Let p and q be propositions. The proposition "**p or q**" denoted by $p \vee q$, is false when both p and q are false and is true otherwise. The proposition $p \vee q$ is called the **disjunction** of p and q.

Example 4 : What is the disjunction of the propositions *p* and *q* where *p* and *q* are the same propositions as in Example 3?

Solution:

$p \vee q$: “Khalid’s PC has at least 16 GB free hard disk space, or the processor in Khalid’s PC runs faster than 1 GHz.”

Other examples:

- Ptuk is located in Ramallah or 5 + 2 = 8
- Today is Sunday or 2 is a prime number.
- 2 is a prime number or $5 + 2 \neq 8$.
- 13 is a perfect square or 9 is a prime.

Truth tables for Conjunction and disjunction

• Four different combinations of values for p and q

p	q	$p \wedge q$	$p \vee q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Note: (the or is used inclusively, i.e., p $\vee$ q is true when either p or q or both are true).

Definition : Exclusive or

Let p and q be propositions. The proposition "p exclusive or q" denoted by p $\oplus$ q, is true when exactly one of p and q is true and it is false otherwise

Truth table for Exclusive or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Definition: Implication (Conditional Statement)

Let *p* and *q* be propositions. The *conditional statement* $p \rightarrow q$ is the proposition "if *p*, then *q*."
In the conditional statement $p \rightarrow q$, *p* is called the ***hypothesis*** (or ***antecedent*** or ***premise***) and *q* is called the ***conclusion*** (or ***consequence***).

Notes :

1. The conditional statement $p \rightarrow q$ is false when *p* is true and *q* is false, and true otherwise.
2. p $\rightarrow$ q is read in a variety of equivalent ways:
   - if p then q
   - p only if q
   - p is sufficient for q

- q whenever p

Truth table for Conditional Statement

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 5 : Let *p* be the statement "Maria learns discrete mathematics" and *q* the statement "Maria will find a good job." Express the statement $p \to q$ as a statement in English.

Solution:
$p \to q$ represents the statement

- "If Maria learns discrete mathematics, then she will find a good job."
  or
- "Maria will find a good job when she learns discrete mathematics."
  or
- "For Maria to get a good job, it is sufficient for Maria to learn discrete mathematics."

or

- "Maria will find a good job unless she does not learn discrete mathematics."

Example 6 : What is the truth value of these conditional statements?

1. if Steelers win the Super Bowl in 2013 then 2 is a prime.
2. if today is Tuesday then 2 * 3 = 8.

Solution:

1. If F then T ? **T**
2. If T then F ? **F**

Remark 1 :

The mathematical concept of a conditional statement is independent of a cause-and effect relationship between hypothesis and conclusion.

For Example let the following two statements

1. “If Juan has a smartphone, then 2 + 3 = 5”.
2. “If Juan has a smartphone, then 2 + 3 = 6”

- the first statement is true, because its conclusion is true. (*The truth value of the hypothesis does not matter then*.)
- but the second statement is true if Juan does not have a smartphone, even though 2 + 3 = 6 is false.

Remark 2 :

The if-then construction used in many programming languages is different from that used in logic.

Example 7 : What is the value of the variable x after the statement

if 2 + 2 = 4 **then** $x := x + 1$

if $x = 0$ before this statement is encountered?

Solution:

Because 2 + 2 = 4 is true, the assignment statement $x := x + 1$ is executed. Hence, x has the value 0 + 1 = 1 after this statement is encountered.

Converse , Contrapositive, and Inverse

- The **converse** of $p \rightarrow q$ is $q \rightarrow p$.
- The **contrapositive** of $p \rightarrow q$ is $\neg q \rightarrow \neg p$
- The **inverse** of $p \rightarrow q$ is $\neg p \rightarrow \neg q$

Example 8 : What are the contrapositive, the converse, and the inverse of the conditional statement

“The home team wins whenever it is raining?”

Solution:

p: ” It is raining” ***q***: “The home team wins”

$p \rightarrow q$, the original statement can be rewritten as

“If it is raining, then the home team wins.”

Consequently, the **contrapositive** of this conditional statement is

“If the home team does not win, then it is not raining.”

The **converse** is

“If the home team wins, then it is raining.”

The **inverse** is

“If it is not raining, then the home team does not win.”

Only the contrapositive is equivalent to the original statement.

Example 8 : What are the contrapositive, the converse, and the inverse of the conditional statement

" If it snows, the traffic moves slowly"

Solution:

• *p*: "it snows" *q*: "traffic moves slowly".

– **The converse:**

If the traffic moves slowly then it snows.

• **$q \to p$**

– **The contrapositive:**

• If the traffic does not move slowly then it does not snow.

• $\neg q \to \neg p$

– **The inverse:**

• If it does not snow the traffic moves quickly.

• $\neg p \to \neg q$

Definition: **Implication (Conditional Statement)**

Let *p* and *q* be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition "*p* if and only if *q*". Biconditional statements are also called *bi-implications*.

Note: The biconditional statement $p \leftrightarrow q$ is true when *p* and *q* have the same truth values, and is false otherwise.

Truth table for Biconditional Statement

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

• **Notes:**

1. There are some other common ways to express $p \leftrightarrow q$:

- "*p* is necessary and sufficient for *q*"
- "if *p* then *q*, and conversely"

- "p iff q."

2. $p \leftrightarrow q$ means that p and q have the same truth value.
3. this truth table is the exact opposite of ⊕'s!
   - $p \leftrightarrow q$ means $\neg(p \oplus q)$
4. $p \leftrightarrow q$ does not imply p and q are true, or cause each other.

Example 9: Let p be the statement "You can take the flight," and let q be the statement

"You buy a ticket."

Then $p \leftrightarrow q$ is the statement

"You can take the flight if and only if you buy a ticket."

Summary of all connectives

Example 10 :

Let p: 2 is a prime ….. **T**

q: 6 is a prime ….. **F**

• Determine **the truth value** of the following statements:

1. $\neg p$ **F**	2. $p \wedge q$ **F**	3. $p \wedge \neg q$ **T**	4. $p \vee q$ **T**
5. $p \oplus q$ **T**	6. $p \rightarrow q$ **F**	7. $q \rightarrow p$ **T**	8. $q \leftrightarrow p$ **F**

Truth Tables of Compound Propositions

We can use these connectives to build up complicated compound propositions.

The following Table displays the precedence levels of the logical operators, ¬, ∧, ∨, →, and ↔

Precedence of Logical Operators.	
Operator	***Precedence***
¬	1
∧ ∨	2 3
→ ↔	4 5

Example 11: Construct the truth table of the compound proposition

$$(p \vee \neg q) \rightarrow (p \wedge q).$$

Solution:
Simpler if we decompose the sentence to elementary and intermediate propositions

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Example 12 : Construct a truth table for
$(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

Solution:

p	q	¬p	p → q	¬p ↔ q	(p→q)∧(¬p↔q)
T	T	F	T	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

Logic and Bit Operations

- Computers represent information using bits. A **bit** is a symbol with two possible values, namely, 0 (zero) and 1 (one).
- Logical truth values – True and False.
- A bit is sufficient to represent two possible values:
  – 0 (False) or 1(True).

Truth Value	*Bit*
T	1
F	0

- A variable is called a **Boolean variable** if its value is either 1(True) or 0 (False). Consequently, a Boolean variable can be represented using a bit.

- Computer **bit operations** correspond to the logical connectives. By replacing True and False with 1 and 0 in the truth tables for the operators ∧, ∨, and ⊕.
- We will also use the notation *OR*, *AND*, and *XOR* for the operators ∨,∧, and ⊕, respectively as is done in various programming languages.

TABLE 9 **Table for the Bit Operators *OR*, *AND*, and *XOR*.**

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

- Information is often represented using bit strings, which are lists of zeros and ones.

Definition:

A **bit string** is a sequence of zero or more bits. The **length** of this string is the number of bits in the string.

Example 13 : 101010011 is a bit string of length nine.

Bitwise operations

- We can extend bit operations to bit strings.
- The **bitwise *OR*(** ∨) , **bitwise *AND*(** ∧), and **bitwise *XOR*(**⊕) of two strings of the same length are obtained by taking by taking the *OR*, *AND*, and *XOR* of the corresponding bits, respectively.

Example 13: Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of the bit strings 01 1011 0110 and 11 0001 1101.

Solution:

01 1011 0110	01 1011 0110	01 1011 0110
∨ 11 0001 1101	∧ 11 0001 1101	⊕ 11 0001 1101
11 1011 1111	01 0001 0100	10 1010 1011

1.2 Applications of Propositional Logic

- Logic has many important applications to computer science such as:
  - Inference and reasoning
  - specification of software and hardware
  - design computer circuits
  - construct computer programs
  - verify the correctness of programs,
  - build expert systems.

Translation of English sentences

- English sentences are translated into compound statements to remove the ambiguity.
- Once we have translated sentences from English into logical expressions we can
  - analyze these logical expressions to determine their truth values.
  - manipulate them, and use rules of inference to reason (**Inference and reasoning)** about them.

Example 1: How can this English sentence be translated into a logical expression?

"You can access the Internet from campus only if you are a computer science major or you are not a freshman."

Solution:

Parse:

- **If (**You can access the Internet from campus **) then (**you are a computer science major **or** you are not a freshman**)**

Atomic (elementary) propositions:

a: " you can access the Internet from campus"
c: "you are a computer science major "
f: "you are a freshman"

- **Translation:** $a \rightarrow (c \lor \neg f)$.

Example 2: How can this English sentence be translated into a logical expression?

"You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

Solution

Let *q:* "You can ride the roller coaster,"
r : "You are under 4 feet tall,"
and
s: "You are older than 16 years old," .

Then the sentence can be translated to $(r \wedge \neg s) \rightarrow \neg q$.

General rule for translation.
• Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

Example 3: How can this English sentence be translated into a logical expression?

"You can have free coffee if you are senior citizen and it is a Tuesday"

Solution:
Do the following steps:

1. *find logical connectives*

"You can have free coffee if you are senior citizen and it is a Tuesday"

2. *break the sentence into elementary propositions*

"You can have free coffee if you are senior citizen and it is a Tuesday"

a b c

3. *rewrite the sentence in propositional logic*

$$\mathbf{b \wedge c \rightarrow a}$$

Test your self
Assume two elementary statements:
p:" you drive over 65 mph" ; q:" you get a speeding ticket"
• Translate each of these sentences to logic

1. you do not drive over 65 mph. **($\neg p$)**
2. you drive over 65 mph, but you don't get a speeding ticket. **($p \wedge \neg q$)**
3. you will get a speeding ticket if you drive over 65 mph. **($p \rightarrow q$)**
4. if you do not drive over 65 mph then you will not get a speeding ticket. **($\neg p \rightarrow \neg q$)**
5. driving over 65 mph is sufficient for getting a speeding ticket. **($p \rightarrow q$)**

6. you get a speeding ticket, but you do not drive over 65 mph. **(q ∧ ¬p)**

1.3 Propositional Equivalences

Definition:

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a *tautology*.
A compound proposition that is always false is called a *contradiction*.
A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.

Example 1:

- p ∨ ¬p is a **tautology.**

p	¬p	p ∨ ¬p
T	F	T
F	T	T

- p ∧ ¬p is a **contradiction.**

p	¬p	p ∧ ¬p
T	F	F
F	T	F

Logical Equivalences

Definition:

The compound propositions *p* and *q* are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that *p* and *q* are logically equivalent.

- One way to determine whether two compound propositions are equivalent is to use a truth table.
- In particular, the compound propositions *p* and *q* are equivalent if and only if the columns giving their truth values agree.

De Morgan's Laws.
$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Example 2: (*The 2nd De Morgan's Law*) Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Solution:
To convince us that two propositions are logically equivalent use the truth table

p	q	¬p	¬q	¬(p ∨ q)	¬p ∧ ¬q
T	T	F	F	**F**	**F**
T	F	F	T	**F**	**F**
F	T	T	F	**F**	**F**
F	F	T	T	**T**	**T**

Example 3: Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

Solution:

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Because the truth values of $\neg p \vee q$ and $p \rightarrow q$ agree, they are logically equivalent.

Example 4: Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent. This is the *distributive law* of disjunction over conjunction.

Solution:

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Because the truth values of $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ agree, these compound propositions are logically equivalent.

Important logical equivalences

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Some useful equivalences for compound propositions involving conditional statements and biconditional statements.

Logical Equivalences Involving Conditional Statements.
$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to$

Logical Equivalences Involving Biconditional Statements.
$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Constructing New Logical Equivalences

The logical equivalences can be used to construct additional logical equivalences.

Example 5: Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.

Solution:

We have the following equivalences.

$\neg(p \to q) \equiv \neg(\neg p \vee q)$ *by Example 22*

$\equiv \neg(\neg p) \wedge \neg q$ *by the second De Morgan law*

$\equiv p \wedge \neg q$ *by the double negation law*

Example 6: Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution:
(*Note:* we could also easily establish this equivalence using a truth table.)
We have the following equivalences.

$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$ by the second De Morgan law

$\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]$ by the first De Morgan law

$\equiv \neg p \wedge (p \vee \neg q)$ by the double negation law

$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ by the second distributive law

$\equiv \mathbf{F} \vee (\neg p \wedge \neg q)$ because $\neg p \wedge p \equiv \mathbf{F}$

$\equiv (\neg p \wedge \neg q) \vee \mathbf{F}$ by the commutative law for disjunction

$\equiv \neg p \wedge \neg q$ by the identity law for $\mathbf{F}$

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Example 7: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution:
We will use logical equivalences to demonstrate that it is logically equivalent to **T**.

$(p \wedge q) \rightarrow (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$ by Example 2

$\equiv (\neg p \vee \neg q) \vee (p \vee q)$ by the first De Morgan law

$\equiv (\neg p \vee p) \vee (\neg q \vee q)$ by the associative and commutative laws for disjunction

$\equiv \mathbf{T} \vee \mathbf{T}$ by Example 1 and the commutative law for disjunction

$\equiv \mathbf{T}$ by the domination law

1.4 Predicates and Quantifiers

Propositional logic: the world is described in terms of elementary propositions and their logical combinations

Elementary statements:

• Typically refer to objects, their properties and relations. But these are not explicitly represented in the propositional logic

– **For example:**

• "Omar is a Ptuk student."

Omar ⟶ **a Ptuk student**

object ⟶ a property

Objects and properties are hidden in the statement, it is not possible to reason about them.

Limitations of the propositional logic

(1) Statements that must be repeated for many objects

In propositional logic these must be exhaustively enumerated

- **For example:**

– If Omar is an AC Ptuk graduate then Omar has passed Calculus.

Translation:

– Omar is an AC Ptuk graduate → Omar has passed Calculus.

Similar statements can be written for other Ptuk graduates:

– Adnan is an AC Ptuk graduate → Adnan has passed Calculus

– Amal is an AC Ptuk graduate → Amal has passed Calculus

– **…**

• Solution: make statements with **variables**

– If x is an AC Ptuk graduate then x has passed Calculus.

– x is an AC Ptuk graduate → x has passed Calculus.

(2) Statements that define the property of the group of objects

- **For example:**

– "Every computer connected to the university network is functioning properly."

– All new cars must be registered.

– "There is a computer on the university network that is under attack by an intruder."

– Some of the AC graduates graduate with honors.

• **Solution:** make statements with **quantifiers.**

Predicate logic

To understand predicate logic, we first need to introduce the concept of a predicate.

Predicates

Predicates represent properties or relations among objects.

For examples:

Statements involving variables, such as

"$x > 3$," "$x = y + 3$," "$x + y = z$,"

and

"computer *x* is under attack by an intruder,"

and

"computer *x* is functioning properly,"

- The statement "*x* is greater than 3" has two parts.
  - The first part, the variable *x*, is the subject of the statement(**object**).
  - The second part—the **predicate**, "is greater than 3"—refers to a **property** that the subject of the statement can have.
- We can denote the statement "*x* is greater than 3" by *P(x)*.
- The statement *P(x)* is also said to be the value of the **propositional function** *P* at *x*.

  Let the following examples:
  - Student(x) denotes the statement "x is a student"
  - Person(x) denotes the statement "x is a person"
  - University(x) denotes the statement "x is a university"
- Once a value has been assigned to the variable *x*, the statement *P(x)* becomes a proposition and has a truth value.
  - **Student(**John**)** …. T (if John is a student)
  - **Student**(Ann) …. T (if Ann is a student)
  - **Student**(Jane) ….. F (if Jane is not a student)
  - **University**(Ptuk) …. T
  - **Person**(Ahmad) …. T

Example 1: Let *P(x)* denote the statement "$x > 3$." What are the truth values of *P*(4) and *P(2)*?

Solution:

We obtain the statement *P(4)* by setting $x = 4$ in the statement "$x > 3$." Hence, *P*(4), which is the statement "4 > 3," is true. However, *P*(2), which is the statement "2 > 3," is false(F).

Example 2: Assume a predicate P(x) that represents the statement:
" x is a prime number "

1. What are the truth values of , P(x) for x= 3, 4, 5, 6, and 7 .
2. **Is P(x) a proposition?**

Solution:

1.
   - P(3) T
   - P(4) F
   - P(5) T
   - P(6) F
   - P(7) T

   All statements P(2), P(3), P(4), P(5), P(6), P(7) are propositions
2. **No. Many possible substitutions are possible.**

Predicates can have **more arguments (variables)** which represent the **relations between objects.**

- A predicate with two arguments is denoted by *Q(x, y)*, where *x* and *y* are variables is a **predicate**.
- Once a values has been assigned to the variable *x* and *y*, the statement *Q(x, y)* becomes a ***proposition*** and has a truth value.

For example: Let Older(x,y) denotes the statement "x is older than y"

- **Older** (John, Peter) : "John is older than Peter"
  – this is a proposition because it is either true or false
- **Older** (x, y) : " x is older than y"
  – not a proposition, but after the substitution it becomes one.

- **Similarly** a predicate with three arguments is denoted by *R(x, y, z)*, where *x*, *y* and *z* are variables.

- Once a values has been assigned to the variable *x* , *y*, and z the statement *Q(x, y,z)* becomes a ***proposition*** and has a truth value.

For example: Let **StudyAt**(x,y, z) denotes the statement "x study at university y major z "

- **StudyAt**(Amjad,Ptuk, AC): Amjad study at Ptuk major AC "
  – this is a proposition because it is either true or false

- **In general**, a statement involving the n variables $x_1, x_2, \ldots, x_n$ can be denoted by $P(x_1, x_2, \ldots, x_n)$.
  - When values are assigned to the variables $x_1, x_2, \ldots, x_n$ the statement $P(x_1, x_2, \ldots, x_n)$ has a truth value.

Example 3: Let $Q(x, y)$ denote the statement "$x = y + 3$." What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

Solution:

- To obtain $Q(1, 2)$, set $x = 1$ and $y = 2$ in the statement $Q(x, y)$. Hence, $Q(1, 2)$ is the statement "$1 = 2 + 3$," which is false.
- The statement $Q(3, 0)$ is the proposition "$3 = 0 + 3$," which is true.

Example 4: Let Q(x,y) denote "x+5 >y"

1. Is Q(x,y) a proposition? **No!**
2. Is Q(3,7) a proposition? **Yes.** It is true.
3. What is the truth value of:
   a) Q(3,7) T
   b) Q(1,6) F
   c) Q(2,2) T
4. Is Q(3,y) a proposition? **No!** We cannot say if it is true or false.

Solution:

1. **No!**
2. **Yes.** It is true.
3. the truth value of:
   a) Q(3,7) T
   b) Q(1,6) F
   c) Q(2,2) T
4. **No!** We cannot say if it is true or false.

Example 5: let $R(x, y, z)$ denote the statement " $x + y = z$."
What are the truth values of the propositions $R(1, 2, 3)$ and $R(0, 0, 1)$?

Solution:

- The proposition $R(1, 2, 3)$ is obtained by setting $x = 1$, $y = 2$, and $z = 3$ in the statement $R(x, y, z)$.
- $R(1, 2, 3)$ is the statement "$1 + 2 = 3$," which is true.

- Note that *R*(0, 0, 1), which is the statement "0 + 0 = 1," is false.

Compound statements in predicate logic

Compound statements are obtained via logical connectives

For examples:

- Student(Ann) ∧ Student(Jane)
  - **Translation:** "Both Ann and Jane are students"
  - **Proposition:** yes.
- Country(Sienna) ⋁ River(Sienna)
  - **Translation:** "Sienna is a country or a river"
  - **Proposition:** yes.
- AC -major(x) → Student(x)
  - **Translation:** "if x is an AC-major then x is a student"
  - **Proposition:** no.

Quantifiers

Predicate logic lets us to make statements about groups of objects by using special quantified expressions.

First we want to define the **domain of quantification.**

Definition:

The **domain of quantification**; i.e., what the quantifiers (or variables) range over. The domain must be nonempty. (The domain is sometimes also called the **universe of discourse** or the **domain of discourse**.)

The universe of discourse can be people, students, numbers, etc.

Two types of quantified statements:

- **Universal quantifier** –the property is satisfied by all members of the group.
  - **For example:** " all AC Ptuk graduates have to pass Calculus"
    – the statement is true for all graduates.
- **Existential quantifier** – at least one member of the group satisfy the property.
  - **For example:** "Some AC Ptuk students graduate with honor."
    – the statement is true for some people.

Universal quantifier

Definition:

The universal quantification of P(x) is the proposition: "*P(x) is true for all values of x in the domain of discourse*." The notation **∀ x P(x)** denotes the universal quantification of P(x), and is expressed as **for every x, P(x).** An element for which *P(x)* is false is called a **counterexample** of ∀*xP(x)*.

Example 1: Let *P(x)* be the statement "$x + 1 > x$." What is the truth value of the quantification ∀*xP(x)*, where the domain consists of all real numbers?
Solution:
Because *P(x)* is true for all real numbers *x*, the quantification ∀*xP(x)* **is true**.

Remarks:
1. If the domain is empty, then ∀*xP(x)* is true for any propositional function *P(x)* because there are no elements *x* in the domain for which *P(x)* is false.
2. Remember that the truth value of ∀*xP(x)* depends on the domain!
3. Besides "for all" and "for every," universal quantification can be expressed in many other ways, including "**all of**," "**for each**," "**given any**," "**for arbitrary**," "**for each**," and "**for any**."
4. A statement ∀*xP(x)* is false, where*P(x)* is a propositional function, if and only if *P(x)* is not always true when *x* is in the domain.

Example 2: Let *Q(x)* be the statement "$x < 2$." What is the truth value of the quantification ∀*xQ(x)*, where the domain consists of all real numbers?
Solution:
Q(x) is not true for every real number *x*, because, for instance, *Q(*3*)* is false. That is, $x = 3$ is a counterexample for the statement ∀*xQ(x)*. Thus ∀*xQ(x)* is false.

Example 3: Suppose that *P(x)* is "$x^2 > 0$." Show that the statement ∀*xP(x)* is false.
Solution:
To show that the statement ∀*xP(x)* is false where the universe of discourse consists of all integers, we give a counterexample. We see that $x = 0$ is a

counterexample because $x^2 = 0$ when $x = 0$, so that x^2 is not greater than 0 when $x = 0$.

Remark : When all the elements in the domain can be listed say, $x_1, x_2, \ldots, x_n$ it follows that the universal quantification $\forall x P(x)$ is the same as the conjunction $P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$, because this conjunction is true if and only if $P(x_1)$, $P(x_2), \ldots, P(x_n)$ are all true.

Example 4: What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?

Solution:

The statement $\forall x P(x)$ is the same as the conjunction

$$P(1) \wedge P(2) \wedge P(3) \wedge P(4),$$

because the domain consists of the integers 1, 2, 3, and 4. Because $P(4)$, which is the statement "$4^2 < 10$," is false, it follows that $\forall x P(x)$ is false.

Example 5: What does the statement $\forall x N(x)$ mean if $N(x)$ is "Computer x is connected to the network" and the domain consists of all computers on campus?

Solution: The statement $\forall x N(x)$ means that for every computer x on campus, that computer x is connected to the network. This statement can be expressed in English as "Every computer on campus is connected to the network."

Example 6: What is the truth value of $\forall x(x^2 \geq x)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers?

Solution:

The universal quantification $\forall x(x^2 \geq x)$, where the domain consists of all real numbers, is false. For example, $\left(\frac{1}{2}\right)^2 < \frac{1}{2}$.

Existential quantifier

Definition:

The **existential quantification** of P(x) is the proposition "*There exists an element in the domain (universe) of discourse such that P(x) is true*." The notation **$\exists x\, P(x)$** denotes the existential quantification of P(x), and is expressed as **there is an x such that P(x) is true**.

Example 7: Let $P(x)$ denote the statement "$x > 3$." What is the truth value of the quantification $\exists xP(x)$, where the domain consists of all real numbers?

Solution: Because "$x > 3$" is sometimes true—for instance, when $x = 4$, the existential quantification of $P(x)$, which is $\exists xP(x)$, is true.

Remarks :

1. the statement $\exists xP(x)$ is false if and only if there is no element x in the domain for which $P(x)$ is true.
2. we can also express existential quantification in many other ways, such as by using the words "**for some**," "**for at least one**," or "**there is**."
3. The existential quantification $\exists xP(x)$ is read as
   "**There is an *x* such that *P(x)*,**"
   "**There is at least one *x* such that *P(x)*,**"

   or

   "**For some *xP(x)*.**"

Example 8: Let $Q(x)$ denote the statement "$x = x + 1$."What is the truth value of the quantification $\exists xQ(x)$, where the domain consists of all real numbers?

Solution:

Because $Q(x)$ is false for every real number x, the existential quantification of $Q(x)$, which is $\exists xQ(x)$, is false.

Example 9: Let T(x) denote x > 5 and x is from Real numbers. What is the truth value of $\exists x$ T(x)?

Solution:

Since 10 > 5 is true. Therefore, it is **true that $\exists$x T(x).**

Remark : When all elements in the domain can be listed—say, $x_1, x_2, \ldots , x_n$ the existential quantification $\exists xP(x)$ is the same as the disjunction

$$P(x_1) \vee P(x_2) \vee \cdot\cdot\cdot \vee P(x_n),$$

because this disjunction is true if and only if at least one of $P(x_1), P(x_2),\ldots ,P(x_n)$ is true.

Example 10: What is the truth value of $\exists x P(x)$, where $P(x)$ is the statement "$x2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

Solution:

Because the domain is {1, 2, 3, 4}, the proposition $\exists x P(x)$ is the same as the disjunction

$$P(1) \vee P(2) \vee P(3) \vee P(4).$$

Because $P(4)$, which is the statement "$4^2 > 10$," is true, it follows that $\exists x P(x)$ is true.

- **Recall** that **quantification** is another important way to create a proposition from a propositional function.

Example 11: **Determine whether the following** statements proposition or not

1. AC-major(x) → Student(x)
2. ∀x AC -major(x) → Student(x)

Solution

1. **Translation:** "if x is an AC-major then x is a student"
   It is not a proposition.
2. **Translation:** "(For all people it holds that) if a person is an AC-major then she is a student."
   It is a proposition.

Example 12: **Determine whether the following** statements proposition or not

1. AC-Ptuk- graduate (x) ∧ Honor-student(x)
2. ∃*x* AC-Ptuk- graduate (x) ∧ Honor-student(x)

Solution

1. **Translation:** "x is a AC-Ptuk- graduate and x is an honor student"
   It is not a proposition.
2. **Translation:** "There is a person who is a AC-Ptuk- graduate and who is also an honor student."
   It is a proposition.

Summary of quantified statements

- When ∀**x P(x)** and ∃**x P(x)** are true and false?

Statement	When True?	When False?
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Negating Quantified Expressions

❖ Negation of a quantified expression.

- For instance, consider the negation of the statement :
  - “*Every student in your class has taken a course in calculus.*”
- This statement is a universal quantification, namely,
  - $\forall x P(x)$,

where $P(x)$ is the statement “x has taken a course in calculus”
and the domain consists of the students in your class.

- **The negation of this statement is**
  - “*It is not the case that every student in your class has taken a course in calculus.*”
- This is equivalent to
  - “*There is a student in your class who has not taken a course in calculus.*”
- And this is simply the existential quantification of the negation of the original propositional function, namely,
  - $\exists x \ \neg P(x)$.
- This example illustrates the following logical equivalence:
  - $\neg \forall x P(x) \equiv \exists x \ \neg P(x)$.

- To show that $\neg \forall x P(x)$ and $\exists x P(x)$ are logically equivalent
  - first note that $\neg \forall x P(x)$ is true iff $\forall x P(x)$ is false.
  - Next, note that $\forall x P(x)$ is false iff there is an element x in the domain for which $P(x)$ is false.

- This holds iff there is an element x in the domain for which $\neg P(x)$ is true.
- Finally, note that there is an element x in the domain for which $\neg P(x)$ is true iff $\exists x\ \neg P(x)$ is true.
- we can conclude that $\neg \forall x P(x)$ is true iff $\exists x\ \neg P(x)$ is true.
- It follows that $\neg \forall x P(x) \equiv \exists x\ \neg P(x)$.

❖ Negation of an existential expression.

- For instance, consider the negation of the statement :
  - "*There is a student in this class who has taken a course in calculus*."
- This statement is an existential quantification, namely,
  - $\exists x Q(x)$,

where $Q(x)$ is the statement "x has taken a course in calculus" and the domain consists of the students in your class.

- **The negation of this statement is**
  - "*It is not the case that there is student in this who has taken a course in calculus.*"
- This is equivalent to
  - "*Every student in this class has not taken calculus*."
- And this is simply the universal quantification of the negation of the original propositional function, namely,
  - $\forall x\ \neg Q(x)$.
- This example illustrates the following logical equivalence:
  - $\neg \exists x\, Q(x) \equiv \forall x\ \neg Q(x)$.
- To show that $\neg \exists x Q(x)$ and $\forall x Q(x)$ are logically equivalent
  - first note that $\neg \exists x Q(x)$ is true iff $\exists x Q(x)$ is false.
  - This holds iff no x exists in the domain for which $Q(x)$ is true.
  - Next, note that no x exists in the domain for which $Q(x)$ is true if and only if $Q(x)$ is false for every x in the domain.

- Finally, note that $Q(x)$ is false for every x in the domain if and only if $\neg Q(x)$ is true for all x in the domain,
- we can conclude that $\neg\exists x Q(x)$ is true iff $\forall x\ \neg Q(x)$ is true.
- It follows that $\neg\exists x Q(x) \equiv \forall x\ \neg Q(x)$.

The rules for negations for quantifiers are called **De Morgan's laws for quantifiers**. These rules are summarized in following Table.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Example 13: What are the negations of the statements "

1. "There is an honest politician"
2. "All Americans eat cheeseburgers"?

Solution:

1. Let $H(x)$ denote "x is honest."
   Then the statement
   "There is an honest politician"
   is represented by
   $\exists x H(x)$,
   where the domain consists of all politicians. The negation of this statement is
   $\neg\exists x H(x)$,
   which is equivalent to
   $\forall x\ \neg H(x)$.
   This negation can be expressed as
   "*Every politician is dishonest*."

2. Let $C(x)$ denote "x eats cheeseburgers."
   Then the statement
   "*All Americans eat cheeseburgers*"
   is represented by
   $\forall x C(x)$,
   where the domain consists of all Americans.

The negation of this statement is

$$\neg\forall x C(x),$$

which is equivalent to

$$\exists x \neg C(x).$$

This negation can be expressed in several different ways, including

"Some American does not eat cheeseburgers"

and

"There is an American who does not eat cheeseburgers."

Example 14: What are the negations of the statements

1. $\forall x(x^2 > x)$
2. $\exists x(x^2 = 2)$

Solution:

1. The negation of $\forall x(x^2 > x)$ is the statement

   $\neg\forall x(x^2 > x),$

   which is equivalent to

   $\exists x \neg(x^2 > x).$

   This can be rewritten as

   $\exists x(x^2 \leq x).$

2. The negation of $\exists x(x^2 = 2)$ is the statement

   $\neg\exists x(x^2 = 2),$

   which is equivalent to

   $\forall x \neg(x^2 = 2).$

   This can be rewritten as

   $\forall x(x^2 \neq 2).$

The truth values of these statements depend on the domain.

Example 15: Show that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent.

Solution:

By De Morgan's law for universal quantifiers, we know that

$$\neg\forall x(P(x) \rightarrow Q(x)) \equiv \exists x(\neg(P(x) \rightarrow Q(x))).$$

We know that

$$\neg(P(x) \rightarrow Q(x)) \equiv P(x) \wedge \neg Q(x) \text{ for every } x.$$

It follows that

$$\neg\forall x(P(x) \rightarrow Q(x)) \equiv \exists x(P(x) \wedge \neg Q(x))$$

Translating from English into Logical Expressions

Translating from English to logical expressions becomes even more complex when quantifiers are needed.

Example 16: Express the statement
"Every student in this class has studied calculus"
using predicates and quantifiers.

Solution:

- **Assume:** the domain of discourse are all students in this class.
  we rewrite the statement to obtain:

  "For every student in this class, that student has studied calculus."

  Next, we introduce a variable *x* so that our statement becomes

  "For every student x in this class, x has studied calculus."

  Then C(x) is the statement "*x has studied calculus*."

- **Translation:**
  $\forall x C(x)$.

- **Assume:** the domain of discourse consists of all people.
  we will need to express our statement as
  "For every person x, if person x is a student in this class, then x has studied calculus."

  If $S(x)$ represents the statement that person x is in this class.

- **Translation:**
  $\forall x(S(x) \rightarrow C(x))$.

[*Caution!* Our statement *cannot* be expressed as $\forall x(S(x) \wedge C(x))$!]

Assume: we are interested in the background of people in subjects besides calculus.

Then we would replace *C(x)* by *Q(x,* calculus) in both approaches to obtain

$\forall x Q(x, calculus)$

or

$\forall x(S(x) \rightarrow Q(x, calculus)).$

where *Q(x, y) is* two-variable quantifier for the statement "student *x* has studied subject *y*."

Example 17: Express the statements:

1. "Some student in this class has visited Mexico"
2. "Every student in this class has visited either Canada or Mexico"

using predicates and quantifiers.

Solution:

1. The statement "Some student in this class has visited Mexico"

means that

"There is a student in this class with the property that the student has visited Mexico."

When introducing a variable *x* the statement becomes

"There is a student x in this class having the property that x has visited Mexico."

Let *M(x)*: "*x* has visited Mexico."

- **Assume:** the domain for *x* consists of the students in this class.
- **Translation:** we rewrite the statement to obtain:

$\exists x M(x).$

- **Assume:** the domain of discourse consists of all people. The statement can be expressed as:

"There is a person x having the properties that x is a student in this class and x has visited Mexico."

Let *S(x)* be "*x* is a student in this class."

Our solution becomes $\exists x(S(x) \wedge M(x))$

2. the second statement can be expressed as

"For every x in this class, x has the property that x has visited Mexico or x has visited Canada."

Let *C(x)* be "*x* has visited Canada."

- **Assume:** the domain for *x* consists of the students in this class.

Translation: this second statement can be expressed as:

$$\forall x(C(x) \vee M(x)).$$

- **Assume:** the domain of discourse consists of all people. The statement can be expressed as:

  "For every person x, if x is a student in this class, then x has visited Mexico or x has visited Canada."

In this case, the statement can be expressed as

$$\forall x(S(x) \rightarrow (C(x) \vee M(x))).$$

❖ We can use, V *(x*, Mexico) and *V (x*, Canada) as the same meaning as *M(x)* and *C(x)*,

where *V (x, y) is* a two-place predicate represent *"x has visited country y."*

Additional examples

Write the following informal statements in a formal language:

1. "All Ptuk students are smart."
2. "Someone at Ptuk is smart."
3. "All triangles have three sides"
4. "No dogs have wings"
5. "Some programs are structured"
6. "If a real number is an integer, then it is a rational number"
7. "All bytes have eight bits"
8. "No fire trucks are green"
9. "People who like Homos are smart"
10. "If a number is an integer, then it is a rational number"
11. "All Palestinian like Jerusalem "

Solution:

1. **Sentence***:* *"All Ptuk students are smart".*

- **Assume:** the domain of discourse of x are Ptuk students
- **Translation:**
  - $\forall x\ Smart(x)$
- **Assume:** the universe of discourse are students (all students):
- **Translation:**
  - $\forall x\ at(x,Ptuk) \rightarrow Smart(x)$

- **Assume:** the universe of discourse are people:
- **Translation:**

  - $\forall x\ student(x) \wedge at(x,Ptuk) \rightarrow Smart(x)$

2. **Sentence:** " Someone at Ptuk is smart."

- **Assume:** the domain of discourse are all Ptuk students
- **Translation:**

$\exists x\ \ Smart(x)$

- **Assume:** the universe of discourse are people:

$\exists x\ \ at(x,Ptuk) \rightarrow Smart(x)$

3. **Sentence:** **"**All triangles have three sides"
- **Assume:** the domain of discourse are all triangles
- **Translation:**

$\forall x$ ThreeSided(x) *or* $\forall x \in Triangle \cdot$ ThreeSided(x)

4. **Sentence:** "No dogs have wings"
   - **Translation:**

   $\forall d \in Dog \cdot \neg\ HasWings(d)$
5. **Sentence:** **"**Some programs are structured"
   - **Translation:**

   $\exists p \in Program \cdot structured(p)$
6. **Sentence:** "If a real number is an integer, then it is a rational number"
   - **Translation:**

   $\forall n \in RealNumber \cdot Integer(n) \rightarrow Rational(n)$
7. "All bytes have eight bits"

   $\forall b \in$Byte $\cdot$ EightBits(b)
8. "No fire trucks are green"

   $\forall t \in$FireTruck $\cdot \neg$ Green(t)

9. "People who like Homos are smart"

$$\forall x \in \text{Person} \cdot \text{Like}(x, Homos) \rightarrow \text{Smart}(x)$$
$$\forall x \in \text{Person} \cdot \text{LikeHomos}(x) \rightarrow \text{Smart}(x)$$

10. "If a number is an integer, then it is a rational number"

$$\forall n \ \ \text{Integer}(n) \rightarrow \text{Rational}(n)$$

11. "All Palestinian like Jerusalem "
    - **Assume:** the domain of discourse are all Palestinian
    - **Translation:**

      $$\forall p \in Palestinian \ . \ Likes(p, Jerusalem)$$
    - **Assume:** the domain of discourse are all person
    - **Translation:**

      $$\forall p \ . \ Palestinian(p) \land Likes(p, Jerusalem)$$

1.5 Nested quantifiers

More than one quantifier may be necessary to capture the meaning of a statement in the predicate logic.

For Examples: Assume that the domain for the variables x and y consists of all real numbers.

- The statement

$$\forall x \forall y (x + y = y + x)$$

says that $x + y = y + x$ for all real numbers x and y. This is the **commutative law** for addition of real numbers.

- Likewise, the statement

$$\forall x \exists y (x + y = 0)$$

says that for every real number x there is a real number y such that $x + y = 0$. This states that every real number has an **additive inverse**.

- Similarly, the statement

$$\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$$

is **the associative law** for addition of real numbers.

Example 1: Translate this statement into a logical expression

"Every real number has its corresponding negative."

Solution

• **Translation:**

– **Assume:**

• a real number is denoted as x and its negative as y

• A predicate P(x,y) denotes: "x + y =0"

• Then we can write:

$$\forall x \exists y P(x,y)$$

Example 2: Translate this statement into a logical expression

" There is a person who loves everybody. """

• **Translation:**

– **Assume:**

• Variables x and y denote people

• A predicate L(x,y) denotes: "x loves y"

• Then we can write in the predicate logic:

$$\exists x \, \forall y \, L(x,y)$$

Example 3: Translate into English the statement

$$\forall x \forall y((x > 0) \land (y < 0) \rightarrow (xy < 0)),$$

where the domain for both variables consists of all real numbers.

Solution:

- This statement says that for every real number x and for every real number y, if $x > 0$ and $y < 0$, then $xy < 0$.
- That is, this statement says that for real numbers x and y, if x is positive and y is negative, then xy is negative.
- In the summary this statement says that

  "The product of a positive real number and a negative real number is always a negative real number."

<u>Order of quantifiers</u>

- **The order of nested quantifiers matters if quantifiers are of different type**
  - **$\forall x \exists y$ L(x,y)** is not the same as **$\exists y \, \forall x$ L(x,y)**

For Example:

• **Assume:**

- L(x,y) denotes "x loves y"

- Then: ∀x∃y L(x,y)
- Translates to: Everybody loves somebody.
- And: ∃*y* ∀*x* L(x,y)
- Translates to: There is someone who is loved by everyone.

⇒The meaning of the two is different.

- **The order of nested quantifiers does not matter if quantifiers are of the same type**

For Example:

" For all x and y, if x is a parent of y then y is a child of x"

- **Assume:**
  - Parent(x,y) denotes "x is a parent of y"
  - Child(x,y) denotes "x is a child of y"
- Two equivalent ways to represent the statement:
  - – ∀x ∀y Parent(x,y) → Child(y,x)
  - – ∀y ∀x Parent(x,y) → Child(y,x)

Example 4 : Let *P(x, y)* be the statement "*x* + *y* = *y* + *x*." What are the truth values of the quantifications ∀*x*∀*yP(x, y)* and ∀*y*∀*xP(x, y)* where the domain for all variables consists of all real numbers?

Solution:

The quantification

∀x∀yP(x, y)

denotes the proposition

"For all real numbers *x*, for all real numbers *y*, *x* + *y* = *y* + *x*."

Because *P(x, y)* is true for all real numbers *x* and *y* (it is the commutative law for addition, which is an axiom for the real numbers , the proposition

∀*x*∀*yP(x, y)* is **true**.

Note that the statement

∀*y*∀*xP(x, y)* says

"For all real numbers *y*, for all real numbers *x*, *x* + *y* = *y* + *x*."

This has the same meaning as the statement

"For all real numbers *x*, for all real numbers *y*, *x* + *y* = *y* + *x*."

That is, ∀*x*∀*yP(x, y)* and ∀*y*∀*xP(x, y)* have the same meaning, and both are true.

Example 5 : Let $Q(x, y)$ denote "$x + y = 0$." What are the truth values of the quantifications $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$, where the domain for all variables consists of all real numbers?

Solution:
The quantification $\exists y \forall x Q(x, y)$ denotes the proposition
"There is a real number y such that for every real number x, $Q(x, y)$."
I.e., no matter what value of y is chosen, there is only one value of x for which $x + y = 0$.
Because there is no real number y such that $x + y = 0$ for all real numbers x, the statement $\exists y \forall x Q(x, y)$ is **false.**

The quantification $\forall x \exists y Q(x, y)$
denotes the proposition
"For every real number x there is a real number y such that $Q(x, y)$."
Given a real number x, there is a real number y such that $x + y = 0$; namely, $y = -x$.
Hence, the statement $\forall x \exists y Q(x, y)$ is **true**.

Translating Mathematical Statements into Statements Involving Nested Quantifiers

Example 6 : Translate the statement "The sum of two positive integers is always positive" into a logical expression.

Solution:
- **Assume:**
  - the domain for both variables consists of all integers.
  - we first rewrite it so that the implied quantifiers as
    "For every two integers, if these integers are both positive, then the sum of these integers is positive."
  - Next, we introduce the variables x and y to obtain
    "For all positive integers x and y, $x + y$ is positive."

- **Translation:**
  - we can express this statement as

$$\forall x \forall y((x > 0) \wedge (y > 0) \rightarrow (x + y > 0)),$$

- **Assume:**

– the domain for both variables consists of all positive integers.
– Then the statement becomes
  "For every two positive integers, the sum of these integers is positive."
– We can express this as

$$\forall x \forall y(x + y > 0),$$

However, we avoided sentences whose translation into logical expressions required the use of nested quantifiers.

Example 7: Express the statement
  "If a person is female and is a parent, then this person is someone's mother"
as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.

Solution:

- **Assume:**

– the domain consisting of all people.
– variables x and y denote people
– We introduce the propositional functions
  $F(x)$ to represent "x is female,"
  $P(x)$ to represent "x is a parent,"
  and
  $M(x, y)$ to represent "x is the mother of y."
– The statement can be expressed as
  "For every person x, if person x is female and person x is a parent, then there exists a person y such that person x is the mother of person y."
– Then the original statement can represented as

$$\forall x((F(x) \wedge P(x)) \rightarrow \exists y M(x, y)).$$

– We can move $\exists y$ to the left so that it appears just after $\forall x$, because y does not appear in $F(x) \wedge P(x)$.
– We obtain the logically equivalent expression

$$\forall x \exists y((F(x) \wedge P(x)) \rightarrow M(x, y)).$$

Translation exercise

Suppose:

– Variables x,y denote people
– L(x,y) denotes "x loves y".

Translate:

• Everybody loves Raymond. $\forall x\ L(x,\text{Raymond})$

- Everybody loves somebody. $\forall x \exists y\, L(x,y)$
- There is somebody whom everybody loves. $\exists y \forall x\, L(x,y)$
- There is somebody who Raymond doesn't love. $\exists y\, \neg L(\text{Raymond},y)$
- There is somebody whom no one loves. $\exists y\, \forall x\, \neg L(x,y)$

Translating from Nested Quantifiers into English

Example 8: Translate the statement $\forall x(C(x) \vee \exists y(C(y) \wedge F(x, y)))$ into English, where
$C(x)$ is "*x* has a computer,"
$F(x, y)$ is "*x* and *y* are friends,"
and the domain for both *x* and *y* consists of all students in your school.

Solution:
The statement says that for every student *x* in your school, *x* has a computer or there is a student *y* such that *y* has a computer and *x* and *y* are friends.
In other words,

"every student in your school has a computer or has a friend who has a computer."

Negating Nested Quantifiers

Example 9: Express the negation of the statement $\forall x \exists y(xy = 1)$ so that no negation precedes a quantifier.
Solution:
We find that

$$\forall x \exists y(xy = 1)$$

is equivalent to

$$\exists x\, \neg \exists y(xy = 1),$$

which is equivalent to

$$\exists x \forall y\, \neg(xy = 1).$$

Because $\neg(xy = 1)$ can be expressed more simply as $xy \neq 1$, we conclude that our negated statement can be expressed as

$$\exists x \forall y(xy \neq 1).$$

1.6 Rules of Inference

Valid Arguments in Propositional Logic

Consider the following argument involving propositions:
"If you have a current password, then you can log onto the network."
"You have a current password."
Therefore,
"You can log onto the network."

We would like to determine whether the conclusion *"You can log onto the network"* must be true when the premises "If you have a current password, then you can log onto the network" and "You have a current password" are both true.

Use **p** to represent "You have a current password"
and
q to represent "You can log onto the network."
Then, the argument has the form

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

where ∴ is the symbol that denotes "therefore."

We want to show that the statement $((p \to q) \land p) \to q$ is a tautology.

Definition:

An argument in propositional logic is a sequence of propositions. All but the final proposition in the argument are called premises and the final proposition is called the conclusion.

An argument is valid if the truth of all its premises implies that the conclusion is true

Chapter 2 Basic Structures: Sets, Functions, Sequences, and Sums

2.1 Sets

- In Discrete math we study the discrete structures used to represent discrete objects .
- Many discrete structures are built using sets
  - Sets = collection of objects

Examples of discrete structures built with the help of sets:

- Combinations
- Relations
- Graphs

Definition: A set

A set is a (unordered) collection of objects. These objects are sometimes called elements or members of the set. We write $a \in A$ to denote that a is an element of the set A. The notation $a \in A$ denotes that a is not an element of the set A.

For examples:

- Vowels in the English alphabet
  V = { a, e, i, o, u }
- The set O of odd positive integers less than 10
  O = {1, 3, 5, 7, 9}
- First seven prime numbers.
  X = { 2, 3, 5, 7, 11, 13, 17 }

Representing a set by:

1) Listing (enumerating) the members of the set.
2) Definition by property, using the set builder notation
   {x| x has property P}.

For Example:

- Even integers between 50 and 63.
  1) E = {50, 52, 54, 56, 58, 60, 62}

2) E = {x| 50 <= x < 63, x is an even integer}

If enumeration of the members is hard we often use ellipses.
For example: a set of integers between 1 and 100

- A= {1,2,3 …, 100}

Important sets in discrete math

- Natural numbers:
  N = {0,1,2,3, …}
- Integers
  Z = {…, -2,-1,0,1,2, …}
- Positive integers
  Z+ = {1,2, 3.…}
- Rational numbers
  $Q = \{p/q \mid p \in Z, q \in Z, q \neq 0\}$
- Real numbers R

Definition: **Equal sets.**

Two sets are equal if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write A = B if A and B are equal sets.

For example
The sets {1, 3, 5} and {3, 5, 1} are equal, because they have the same elements.

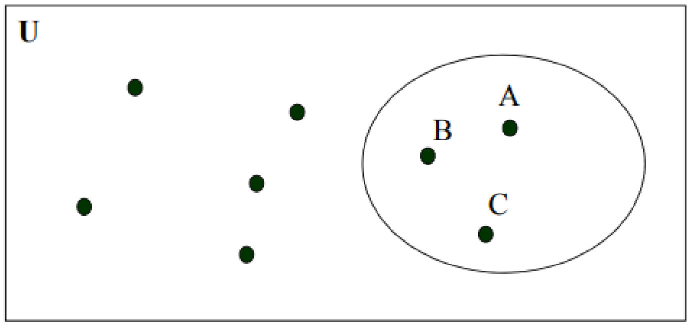
Example 1: Are {1,2,3,4} and {1,2,2,4} equal?
No!

Special sets:

– The **universal set** is denoted by U: the set of all objects under the consideration.
– The **empty set** is denoted as ∅ or { }.

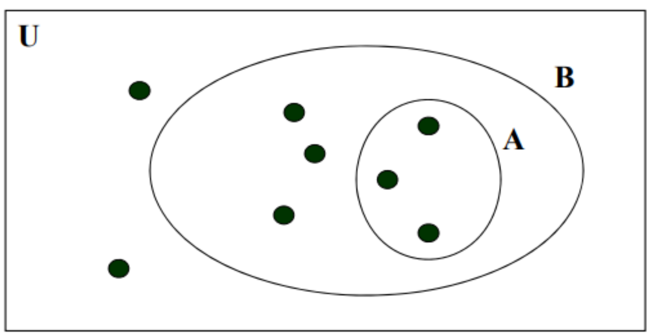
Venn diagrams

- A set can be visualized using Venn Diagrams:
  V={ A, B, C }



Definition: **A Subset**

A set A is said to be a **subset** of B if and only if every element of A is also an element of B. We use **A ⊆ B** to indicate **A is a subset of B**.



• Alternate way to define A is a subset of B:

$\forall x\ (x \in A) \rightarrow (x \in B)$

Empty set properties

Theorem ∅ ⊆ S

• Empty set is a subset of any set.

Proof:

- Recall the definition of a subset: all elements of a set A must be also elements of B: $\forall x\ (x \in A \rightarrow x \in B)$.

- We must show the following implication holds for any S

  $\forall x\ (x \in \emptyset \rightarrow x \in S)$

- Since the empty set does not contain any element, $x \in \emptyset$ is always False
- Then the implication is always True.
- **End of proof**

Subset properties

Theorem: $S \subseteq S$

• Any set S is a subset of itself

Proof:

- the definition of a subset says: all elements of a set A must be also elements of B: $\forall x\ (x \in A \rightarrow x \in B)$.
- Applying this to S we get:
- $\forall x\ (x \in S \rightarrow x \in S)$ which is trivially True

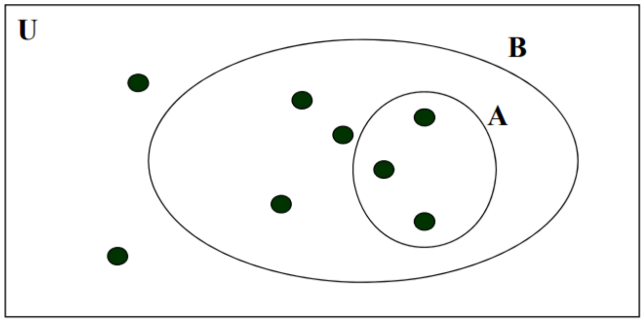
• End of proof

Note on equivalence:

• Two sets are equal if each is a subset of the other set.

Definition: **A proper subset**

A set A is said to be a proper subset of B if and only if $A \subseteq B$ and $A \neq B$. We denote that A is a proper subset of B with the notation $A \subset B$.



Example 2: A={1,2,3} B ={1,2,3,4,5}

Is: A ⊂ B ? Yes.

Cardinality

Definition: Cardinality

Let S be a set. If there are exactly n distinct elements in S, where n is a nonnegative integer, we say S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by | S |.

Example 3 Let A be the set of odd positive integers less than 10.

Then |A| = 5.

Example 4 Let S be the set of letters in the English alphabet.

Then |S| = 26.

Example 5 Because the null set has no elements, it follows that |∅| = 0

Example 6 : Let V={1, 2, 3 ,4, 5}

Then | V | = 5

Example 7 A={1,2,3,4, …, 20}

Then |A| =20

Definition: Infinite set

A set is infinite if it is not finite.

For examples:

• The set of natural numbers is an infinite set.

N = {1, 2, 3, ... }

• The set of reals is an infinite set.

Power sets

Definition: Power set

Given a set S, the power set of S is the set of all subsets of S. The power set is denoted by P(S).

For Examples:

• Assume an empty set ∅

• What is the power set of ∅ ? P(∅) = { ∅ }

• What is the cardinality of P(∅) ? | P(∅) | = 1.

• Assume set {1}

• P({1}) = { ∅, {1} }

• |P({1})| = 2

Assume {1,2}

• P({1,2}) = { ∅, {1}, {2}, {1,2} }

• |P({1,2})| =4

• Assume {1,2,3}

• P({1,2,3}) = {∅, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3} }

• |P({1,2,3} | = 8

• **If S is a set with |S| = n then | P(S) | = 2^n**

N-tuple

• Sets are used to represent unordered collections.

• Ordered-n tuples are used to represent an ordered collection.

Definition: **An ordered n-tuple**

An **ordered n-tuple** (x_1, x_2, ..., x_N) is the ordered collection that has x_1 as its first element, x_2 as its second element, ..., and x_N as its N-th element, $N \geq 2$.

Example:

Coordinates of a point in the 2-D plane (12, 16)

<u>Cartesian product</u>

Definition: **Cartesian product**

Let S and T be sets. The Cartesian product of S and T, denoted by S x T, is the set of all ordered pairs (s,t), where $s \in S$ and $t \in T$.

Hence, $S \times T = \{ (s,t) \mid s \in S \land t \in T\}$

Example 8 : What is the Cartesian product of A = {1, 2} and B = {a, b, c}?

Solution:

The Cartesian product

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$.

- Note: $B \times A = \{ (a,1), (a, 2), (b,1), (b,2), (c,1), (c,2) \}$

$B \times A \neq A \times B$!!!!

Cardinality of the Cartesian product

- $|A \times B| = |A| * |B|$.

Example 9 : Assume the following sets:

A= {John, Peter, Mike}

B ={Jane, Ann, Laura}

What is the Cartesian product of A x B

Solution:

A x B= {(John, Jane),(John, Ann) , (John, Laura), (Peter, Jane), (Peter, Ann) , (Peter, Laura) , (Mike, Jane) , (Mike, Ann) , (Mike, Laura)}

$|A \times B| = 9$

$|A|=3, |B|=3 \rightarrow |A|\ |B|= 9$

Definition:

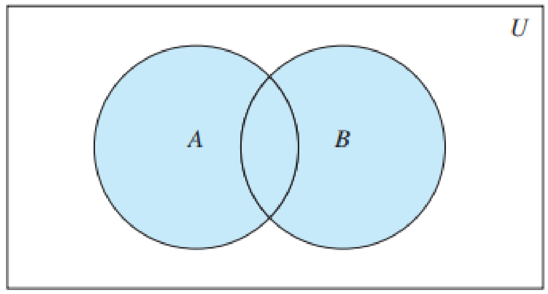
A subset of the Cartesian product A x B is called a relation from the set A to the set B.

2.2 Set operations

Definition: **The union**

Let A and B be sets. The union of A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

• Alternate: A∪ B = { x | x ∈ A ∨ x ∈ B }.



A ∪ B is shaded.

Example 1:

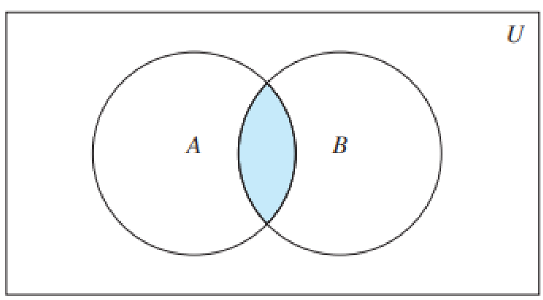
• A = {1,2,3,6} B = { 2,4,6,9}

• A∪ B = { 1,2,3,4,6,9 }

Definition: The intersection

Let A and B be sets. The intersection of A and B, denoted by A ∩ B, is the set that contains those elements that are in both A and B.

• Alternate: A ∩ B = { x | x ∈ A ∧ x ∈ B }.



A ∩ B is shaded.

Example 2:

• A = {1,2,3,6} B = { 2,4,6,9}

• A∩ B = { 2,6 }

Definition: Disjoint set

Two sets are called **disjoint** if their intersection is empty.

• Alternate: A and B are disjoint **if and only if** $A \cap B = \varnothing$.

Example 3 :

• A={1,2,3,6} B={4,7,8} Are these disjoint?

• Yes. Because $A \cap B = \varnothing$

Cardinality of the set union

• $|A \cup B| = |A| + |B| - |A \cap B|$

– Correct for an over-count.

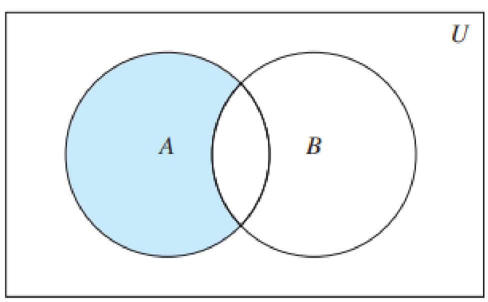
• More general rule:

– The principle of inclusion and exclusion.

Definition: Set difference

Let A and B be sets. The difference of A and B, denoted by **A - B**, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

• Alternate: $A - B = \{ x \mid x \in A \land x \notin B \}$.



$A - B$ is shaded.

Example 4 :

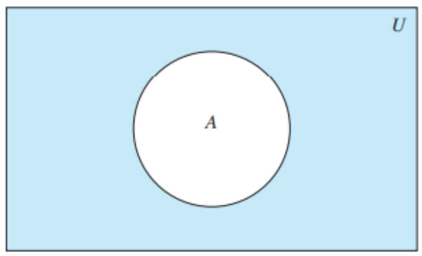
A= {1,2,3,5,7}   B = {1,5,6,8}

• A - B ={2,3,7}

Definition: Complement of a set

The complement of the set A, denoted by $\bar{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is U − A.

• Alternate: $\bar{A} = \{ x \mid x \notin A \}$



$\bar{A}$ is shaded.

Example 5 : Let U={1,2,3,4,5,6,7,8} , A ={1,3,5,7}

Then $\bar{A}$ = {2,4,6,8}

Set identities

Set Identities (analogous to logical equivalences)

• Identity

$A \cup \varnothing = A$

$A \cap U = A$

• Domination

$A \cup U = U$

$A \cap \varnothing = \varnothing$

• Idempotent

$A \cup A = A$

$A \cap A = A$

- Double complement

$\overline{\overline{A}} = A$

- Commutative

$A \cup B = B \cup A$

$A \cap B = B \cap A$

- Associative

$A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

- Distributive

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

– $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

- DeMorgan

$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$

- Absorbtion Laws

$A \cup (A \cap B) = A$

$A \cap (A \cup B) = A$

- Complement Laws

$A \cup \overline{A} = U$

$A \cap \overline{A} = \varnothing$

• **Set identities can be proved using membership tables.**

• List each combination of sets that an element can belong to. Then show that for each such a combination the element either belongs or does not belong to both sets in the identity.

• Prove $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$

A	B	$\bar{A}$	$\bar{B}$	$\overline{A \cap B}$	$\bar{A} \cup \bar{B}$
1	1	0	0	0	0
1	0	0	1	0	0
0	1	1	0	0	0
0	0	1	1	1	1

Example 6 : Let A, B, and C be sets. Show that $\overline{A \cup (B \cap C)} = (\bar{C} \cup \bar{B}) \cap \bar{A}$.

Solution: We have

$\overline{A \cup (B \cap C)} = \bar{A} \cap (\overline{B \cap C})$ by the first De Morgan law

$= \bar{A} \cap (\bar{B} \cup \bar{C})$ by the second De Morgan law

$= (\bar{B} \cup \bar{C}) \cap \bar{A}$ by the commutative law for intersection

$= (\bar{C} \cup \bar{B}) \cap \bar{A}$ by the commutative law for unions

<u>Generalized Unions and Intersections</u>

Definition:

The **union of a collection of sets** is the set that contains those elements that are members of at least one set in the collection.

- We use the notation $A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$

Example 7 :

- Let $A_i = \{1,2,...,i\}$ $i = 1,2,...,n$

- $\bigcup_{i=1}^{n} A_i = \{1, 2, ..., n\}$

Definition:

The **intersection of a collection of sets** is the set that contains those elements that are members of all sets in the collection.

- We use the notation $A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$

Example 8 :

- Let $A_i = \{1,2,...,i\}$ $i = 1,2,...,n$

- $\bigcap_{i=1}^{n} A_i = \{1\}$

Computer representation of sets

- **How to represent sets in the computer?**
- **One solution: Data structures like a list**
- **A better solution:**
- Assign a bit in a bit string to each element in the universal set and set the bit to 1 if the element is present otherwise use 0

Example 9 :

All possible elements: U={1 2 3 4 5}

- Assume A={2,5}
  - Computer representation: A = 01001
- Assume B={1,5}

– Computer representation: B = 10001

Solution:

• A = 01001

• B = 10001

• The **union** is modeled with a bitwise **or**

• $A \vee B$ = 11001

• The **intersection** is modeled with a bitwise **and**

• $A \wedge B$ = 00001

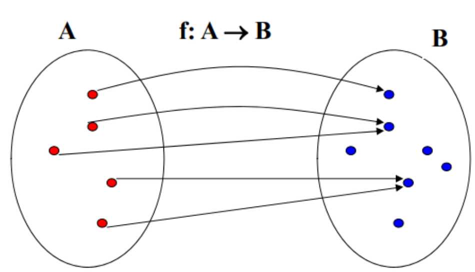
• The **complement** is modeled with a bitwise **negation**

• $\overline{A}$ =10110

2.3 Functions

Definition:

Let A and B be two sets. A function from A to B, denoted $f : A \rightarrow B$, is an assignment of exactly one element of B to each element of A. We write f(a) = b to denote the assignment of b to an element a of A by the function f.



Representing functions

1. Explicitly state the assignments in between elements of the two sets

2. Compactly by a formula. (using ‘standard’ functions)

Example 1 : Let A = {1,2,3} and B = {a,b,c}

Assume f is defined as:

- 1 → c
- 2 → a
- 3 → c
- Is f a function ?
- **Yes**. since f(1)=c, f(2)=a, f(3)=c. each element of A is assigned an element from B.

Example 2: Let A = {1,2,3} and B = {a,b,c}

Assume g is defined as

- 1 → c
- 1 → b
- 2 → a
- 3 → c
- Is g a function?
- **No.** g(1) = is assigned both c and b.

Example 3: Let A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}

Define h: A → B as:

h(x) = x mod 3. (the result is the remainder after the division by 3)

Assignments:

• 0 → 0	• 3 → 0	• 6 → 0
• 1 → 1	• 4 → 1	• 7 → 0

• $2 \to 2$ • $5 \to 2$ …..

Important sets

Definition:

Let f be a function from A to B. We say that A is the domain of f and B is the codomain of f.

• If f(a) = b, b is the image of a and a is a pre-image of b.

• The range of f is the set of all images of elements of A. Also, if f is a function from A to B, we say f maps A to B.

Example 4: Let A = {1,2,3} and B = {a,b,c}

Assume f is defined as: $1 \to c$, $2 \to a$, $3 \to c$

• What is the image of 1?

• $1 \to c$ c is the image of 1

• What is the pre-image of a?

• $2 \to a$ 2 is a pre-image of a.

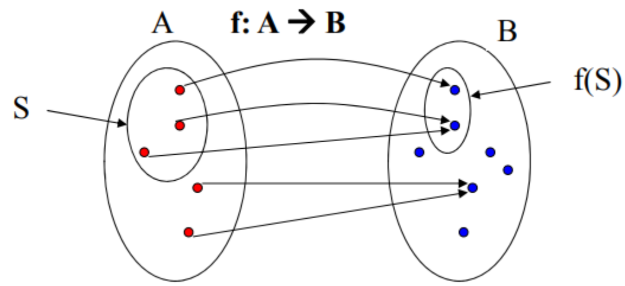
• Domain of f ? {1,2,3}

• Codomain of f ? {a,b,c}

• Range of f ? {a,c}

Definition: **Image of a subset**

Let f be a function from set A to set B and let S be a subset of A. The image of S is a subset of B that consists of the images of the elements of S. We denote the image of S by f(S), so that $f(S) = \{ f(s) \mid s \in S \}$.

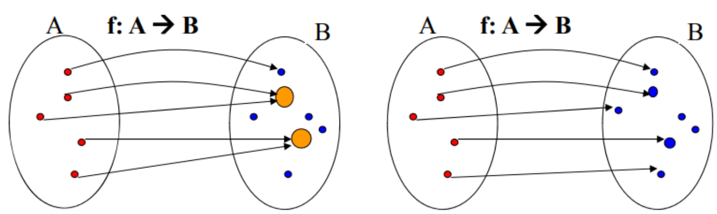


Example 5: Let A = {1,2,3} and B = {a,b,c} and f: 1 → c, 2 → a, 3 → c

• Let S = {1,3} then image f(S) = {c}.

Definition: Injective function

function f is said to be one-to-one, or injective, if and only if f(x) = f(y) implies x = y for all x, y in the domain of f. A function is said to be an injection if it is one-to-one. Alternate: A function is one-to-one if and only if f(x) ≠ f(y), whenever x ≠ y. This is the contrapositive of the definition.



Not injective **Injective function**

Example 6: Let A = {1,2,3} and B = {a,b,c}

Define f as

- 1 → c
- 2 → a
- 3 → c

- Is f one to one?
- **No,** it is not one-to-one since f(1) = f(3) = c, and 1 ≠ 3.

Example 7: Let g : Z → Z, where g(x) = 2x - 1.

• Is g is one-to-one (why?)

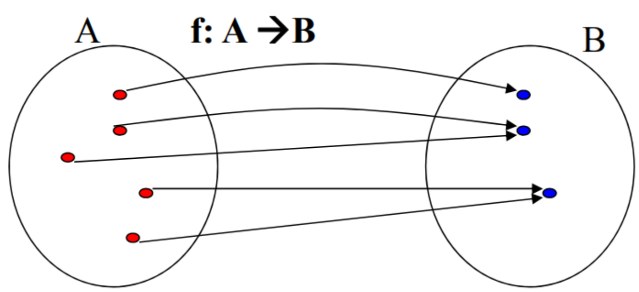
• **Yes.**

• Suppose g(a) = g(b), i.e., 2a - 1 = 2b - 1 => 2a = 2b => a = b.

Surjective function

Definition:

A function f from A to B is called onto, or surjective, if and only if for every b ∈ B there is an element a ∈ A such that f(a) = b. Alternative: all co-domain elements are covered



Example 8: Let A = {1,2,3} and B = {a,b,c}

Define f as • 1 → c • 2 → a • 3 → c

• Is f an onto?

• **No.** f is not onto, since b ∈ B has no pre-image.

Example 9: A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}

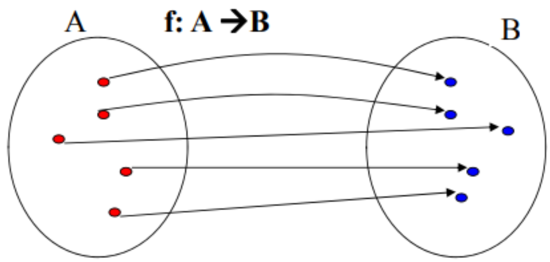
Define h: A → B as h(x) = x mod 3.

• Is h an onto function?

• **Yes.** h is onto since a pre-image of 0 is 6, a pre-image of 1 is 4, a pre-image of 2 is 8.

Definition: Bijective functions

A function f is called a bijection if it is both one-to-one and onto.



Example 10: Let A = {1,2,3} and B = {a,b,c}

Define f as • 1 → c • 2 → a • 3 → b

• Is f is a bijection?

- **Yes**. It is both one-to-one and onto.
- **Note:** Let f be a function from a set A to itself, where A is finite. f is one-to-one if and only if f is onto.
- This is not true for A an infinite set. Define f : Z → Z, where f(z) = 2 * z.
- f is one-to-one but not onto (3 has no pre-image).

Example 11: Define g : W → W (whole numbers), where g(n) = [n/2] (floor function).

• 0 → [0/2] = [0] = 0

• 1 → [1/2] = [1/2] = 0

• 2 → [2/2] = [1] = 1

• 3 → [3/2] = [3/2] = 1

• ...

• Is g a bijection?

– **No.** g is onto but not 1-1 (g(0) = g(1) = 0 however 0 ≠ 1.

Theorem: Let f be a function f: A →A from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Proof:

- ⇒**A is finite** and **f is one-to-one (injective)**
- Is f an onto function (surjection)?
- **Yes.** Every element points to exactly one element. Injection assures they are different. So we have |A| different elements A points to.
- Since f: A → A the co-domain is covered thus the function is also a surjection (and a bijection)
- **⇐ A is finite and f is an onto function**
- Is the function one-to-one?

• Yes. Every element maps to exactly one element and all elements in A are covered. Thus the mapping must be one-to-one

Please note the above is not true when A is an infinite set.

Example 12 : f : Z → Z, where f(z) = 2 * z.

– f is one-to-one but not onto.

• 1 → 2

• 2 → 4

• 3 → 6

– 3 has no pre-image.

Functions on real numbers

Definition: Let f_1 and f_2 be functions from A to R (reals). Then $f_1 + f_2$ and

$f_1 * f_2$ are also functions from A to R defined by

- $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
- $(f_1 * f_2)(x) = f_1(x) * f_2(x)$.

Examples:

- Assume • $f_1(x) = x - 1$
- $f_2(x) = x^3 + 1$ then
- $(f_1 + f_2)(x) = x^3 + x$
- $(f_1 * f_2)(x) = x^4 - x^3 + x - 1$.

Increasing and decreasing functions

Definition: A function f whose domain and codomain are subsets of real numbers is **strictly increasing** if f(x) > f(y) whenever x > y and x and y are in the domain of f. Similarly, f is called **strictly decreasing** if f(x) < f(y) whenever x > y and x and y are in the domain of f.

Example 13: Let $g : R \to R$, where $g(x) = 2x - 1$. Is it increasing ?

Proof . For x>y holds 2x > 2y and subsequently 2x-1 > 2y-1

Thus g is strictly increasing.

Note: Strictly increasing and strictly decreasing functions are oneto-one. **Why?**

One-to-one function: A function is one-to-one if and only if $f(x) \neq f(y)$, whenever $x \neq y$

Identity function

Definition: Let A be a set. The **identity function** on A is the function

$i_A: A \to A$ where $i_A(x) = x$.

Example 14: Let A = {1,2,3} **Then**:

- $i_A(1) = 1$

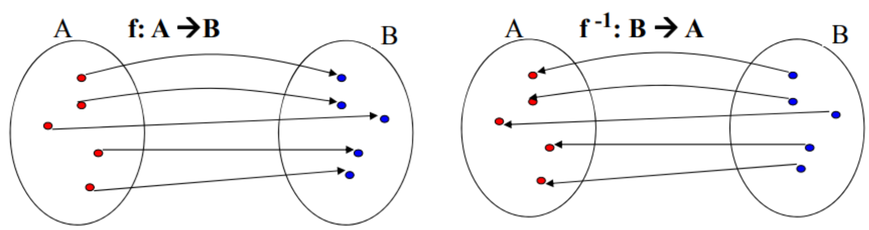
• $i_A(2) = 2$

• $i_A(3) = 3$.

Inverse functions

Definition:

Let f be a bijection from set A to set B. The **inverse** function of f is the function that assigns to an element b from B the unique element a in A such that f(a) = b. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$, when f(a) = b. If the inverse function of f exists, f is called **invertible**.

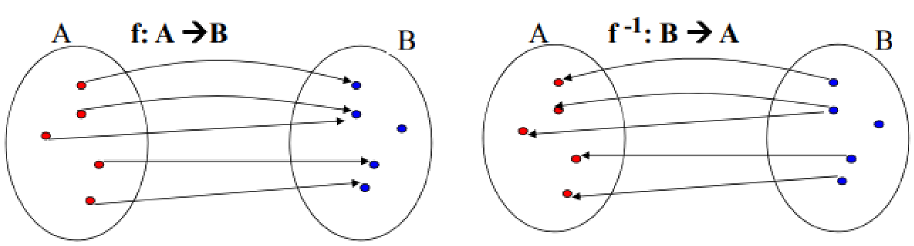


f is bijective **Inverse of f**

Note: if f is not a bijection then it is not possible to define the inverse function of f. **Why?**

Assume f is not one-to-one:

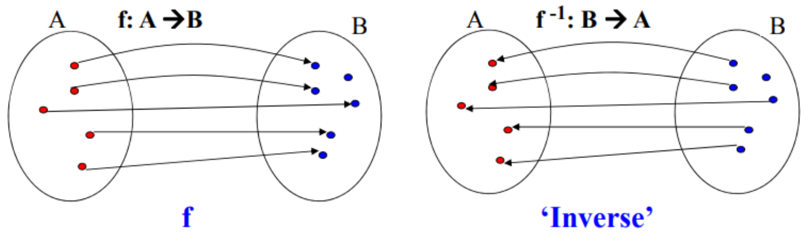
Inverse is not a function. One element of B is mapped to two different elements.



f **'Inverse'**

Assume f is not onto:

Inverse is not a function. One element of B is not assigned any value in B.



Example 15: Let A = {1,2,3} and i_A be the identity function

- $i_A(1) = 1$ $\quad$ $i_A^{-1}(1) = 1$
- $i_A(2) = 2$ $\quad$ $i_A^{-1}(2) = 2$
- $i_A(3) = 3$ $\quad$ $i_A^{-1}(3) = 3$
- Therefore, the inverse function of i_A is i_A.

Example 16: Let $g : R \to R$, where $g(x) = 2x - 1$.

- **What is the inverse function g^{-1} ?**

Approach to determine the inverse:

$y = 2x - 1 \Rightarrow y + 1 = 2x \Rightarrow (y+1)/2 = x$

- Define $g^{-1}(y) = x = (y+1)/2$

Test the correctness of inverse:

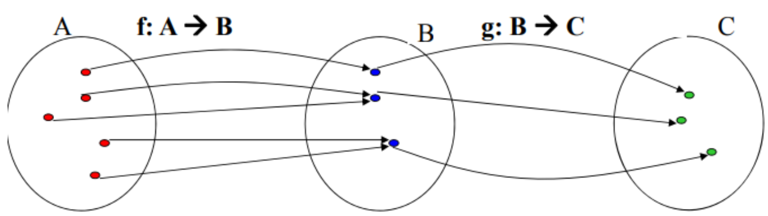
- $g(3) = 2*3 - 1 = 5$
- $g^{-1}(5) = (5+1)/2 = 3$
- $g(10) = 2*10 - 1 = 19$
- $g^{-1}(19) = (19+1)/2 = 10.$

Composition of functions

Definition:

Let f be a function from set A to set B and let g be a function from set B to set C. **The composition of the functions g and f**, denoted by g ϒ f is defined by

- **(g o f)(a) = g(f(a)).**



Example 17: : Let A = {1,2,3} and B = {a,b,c,d}

g : A → A,	f: A → B
1 → 3	1 → b
2 → 1	2 → a
3 → 2	3 → d

f o g : A → B:

- 1 → d
- 2 → b
- 3 → a

Example 18: Let f and g be two functions from Z to Z, where

- f(x) = 2x and g(x) = x^2.
- f o g : Z → Z

• $(f \circ g)(x) = f(g(x))$

$= f(x^2)$

$= 2(x^2)$

• $g \circ f : Z \rightarrow Z$

• $(g \circ f)(x) = g(f(x))$

$= g(2x) = (2x)^2 = 4x^2$

Example 19: $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$, for all x.

• Let $f : \mathbf{R} \rightarrow \mathbf{R}$, where $f(x) = 2x - 1$ and $f^{-1}(x) = (x+1)/2$.

• $(f \circ f^{-1})(x) = f(f^{-1}(x)) = f((x+1)/2) = 2((x+1)/2) - 1 = (x+1) - 1 = x$

• $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(2x - 1) = (2x)/2 = x$

Some functions

Definition:

The **floor function** is denoted by $\lfloor x \rfloor$.

- The **ceiling function** assigns to the real number x the smallest integer that is greater than or equal to x. The ceiling function is denoted by $\lceil x \rceil$.

Other important functions:

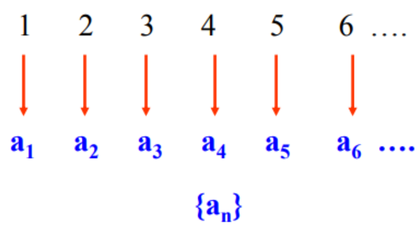
- **Factorials:** $n! = n(n-1)$ such that $1! = 1$

2.4 Sequences and summations

Definition:

A sequence is a function from a subset of the set of integers (typically the set {0,1,2,...} or the set {1,2,3,...} to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.

Notation: $\{a_n\}$ is used to represent the sequence (note {} is the same notation used for sets, so be careful). $\{a_n\}$ represents the ordered list $a_1, a_2, a_3, \ldots$.



Examples:

(1) $a_n = n^2$, where n = 1,2,3...

What are the elements of the sequence? 1, 4, 9, 16, 25, ...

(2) $a^n = (-1)^n$, where n=0,1,2,3,...

What are the elements of the sequence? 1, -1, 1, -1, 1, ...

(3) $a^n = 2^n$, where n=0,1,2,3,...

(4) $a^n = 1/n$. What are the elements of the sequence? 1, 1/2 , 1/3 , 1/4 , ...

What are the elements of the sequence? 1, 2, 4, 8, 16, 32, ...

Arithmetic progression

Definition:

An arithmetic progression is a sequence of the form a, a+d,a+2d, …, a+nd where a is the initial term and d is common difference, such that both belong to R.

Remark: An arithmetic progression is a discrete analogue of the linear function f(x) = dx + a.

Example 1: $s_n = -1+4n$ for n=0,1,2,3, …

• members: -1, 3, 7, 11, …

Definition:

A geometric progression is a sequence of the form: $a, ar, ar^2, ..., ar^k$, where a is the initial term, and r is the common ratio. Both a and r belong to R.

Remark: A geometric progression is a discrete analogue of the exponential function $f(x) = ar^x$.

Example 2: $a_n = (½)^n$ for n = 0,1,2,3, …

- members: 1,½, ¼, 1/8, …..

Given a sequence finding a rule for generating the sequence is not always straightforward.

Example 3: Assume the sequence: 1,3,5,7,9, ….

• What is the formula for the sequence?

- Each term is obtained by adding 2 to the previous term.
  1, 1+2=3, 3+2=5, 5+2=7

• What type of progression this suggest?

- It suggests an arithmetic progression:
  a+nd with a=1 and d=2 $a_n=1+2n$

Example 4: Assume the sequence: 1, 1/3, 1/9, 1/27, …

• What is the sequence?

• The denominators are powers of 3

$(1/3)^0, (1/3)^1, (1/3)^2, (1/3)^3, ….(1/3)^k$

• This suggests a geometric progression: ar^k with a=1 and r=1/3

Recursively defined sequences

• The n-th element of the sequence $\{a_n\}$ is defined recursively in terms of the previous elements of the sequence and the initial elements of the sequence.

Example 5: $a_n = a_{n-1} + 2$ assuming $a_0 = 1$;

$a_1 = 3$, $a_2 = 5$, $a_3 = 7$

• Can you write an non-recursively using n?

$a_n = 1 + 2n$

Fibonacci sequence

• Recursively defined sequence, where $f_0 = 0$; $f_1 = 1$; $f_n = f_{n-1} + f_{n-2}$ for n = 2,3, …

$f_2 = 1$ $f_3 = 2$ $f_4 = 3$ $f_5 = 5$

Summations

Summation of the terms of a sequence:

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \ldots + a_n$$

The variable j is referred to as the index of summation.

• m is the lower limit and

• n is the upper limit of the summation

Example:

- **Sum the first 7 terms of $\{n^2\}$ where n=1,2,3, … .**

$$\sum_{j=1}^{7} a_j = \sum_{j=1}^{7} j^2 = 1 + 4 + 16 + 25 + 36 + 49 = 140$$

- 2) **What is the value of**

$$\sum_{k=4}^{8} a_j = \sum_{k=4}^{8} (-1)^j = 1 + (-1) + 1 + (-1) + 1 = 1$$

Arithmetic series

Definition:

The sum of the terms of the arithmetic progression a, a+d,a+2d, …, a+nd is called an arithmetic series.

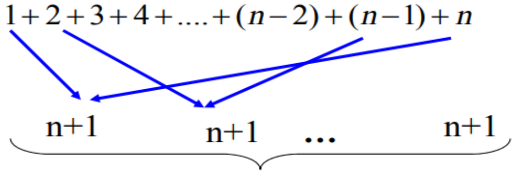
Theorem: The sum of the terms of the arithmetic progression a, a+d,a+2d, …, a+nd is

$$S=\sum_{j=1}^{n}(a+jd)=na+d\sum_{j=1}^{n}j=na+d\frac{n(n+1)}{2}$$

Proof

$$S=\sum_{j=1}^{n}(a+jd)=\sum_{j=1}^{n}a+\sum_{j=1}^{n}jd=na+d\sum_{j=1}^{n}j$$

$$\sum_{j=1}^{n}j=1+2+3+4+....+(n-2)+(n-1)+n$$



$$\frac{n}{2}*(n+1)$$

Example 1:

$$S=\sum_{j=1}^{5}(2+j3)=$$

$$=\sum_{j=1}^{5}2+\sum_{j=1}^{5}j3=$$

$$=2\sum_{j=1}^{5}1+3\sum_{j=1}^{5}j=$$

$$=2*5+3\sum_{j=1}^{5}j=$$

$$=10+3\frac{(5+1)}{2}*5=$$

$$=10+45=55$$

Example 2: $S = \sum_{j=3}^{5} (2 + j3) =$

$$= \left[\sum_{j=1}^{5} (2 + j3)\right] - \left[\sum_{j=1}^{2} (2 + j3)\right]$$

$$= \left[2\sum_{j=1}^{5} 1 + 3\sum_{j=1}^{5} j\right] - \left[2\sum_{j=1}^{2} 1 + 3\sum_{j=1}^{2} j\right]$$

$$= 55 - 13 = 42$$

Double summations

Example 3: $S = \sum_{i=1}^{4} \sum_{j=1}^{2} (2i - j) =$

$$= \sum_{i=1}^{4} \left[\sum_{j=1}^{2} 2i - \sum_{j=1}^{2} j\right] =$$

$$= \sum_{i=1}^{4} \left[2i\sum_{j=1}^{2} 1 - \sum_{j=1}^{2} j\right] =$$

$$= \sum_{i=1}^{4} \left[2i * 2 - \sum_{j=1}^{2} j\right] =$$

$$= \sum_{i=1}^{4} [2i * 2 - 3] =$$

$$= \sum_{i=1}^{4} 4i - \sum_{i=1}^{4} 3 =$$

$$= 4\sum_{i=1}^{4} i - 3\sum_{i=1}^{4} 1 = \quad 4 * 10 - 3 * 4 = 28$$

Geometric series

Definition:

The sum of the terms of a geometric progression a, ar, ar^2, ..., ar^k is called a geometric series.

Theorem: The sum of the terms of a geometric progression a, ar, ar^2, ..., ar^n is

$$S=\sum_{j=0}^{n}(ar^{j})=a\sum_{j=0}^{n}r^{j}=a\left[\frac{r^{n+1}-1}{r-1}\right]$$

Proof:

$$S=\sum_{j=0}^{n}ar^{j}=a+ar+ar^{2}+ar^{3}+\ldots+ar^{n}$$

- multiply S by r

$$rS=r\sum_{j=0}^{n}ar^{j}=ar+ar^{2}+ar^{3}+\ldots+ar^{n+1}$$

- Substract $rS-S=\left[ar+ar^{2}+ar^{3}+\ldots+ar^{n+1}\right]-\left[a+ar+ar^{2}..+ar^{n}\right]$

$$=ar^{n+1}-a$$

$$S=\frac{ar^{n+1}-a}{r-1}=a\left[\frac{r^{n+1}-1}{r-1}\right]$$

Example 4:

$$S=\sum_{j=0}^{3}2(5)^{j}=$$

General formula:

$$S=\sum_{j=0}^{n}(ar^{j})=a\sum_{j=0}^{n}r^{j}=a\left[\frac{r^{n+1}-1}{r-1}\right]$$

$$S=\sum_{j=0}^{3}2(5)^{j}=2*\frac{5^{4}-1}{5-1}=$$

$$=2*\frac{625-1}{4}=2*\frac{624}{4}=2*156=312$$

Infinite geometric series

- **Infinite geometric series can be computed in the closed form for x <1**

- **How?**

$$\sum_{n=0}^{\infty} x^n = \lim_{k\to\infty} \sum_{n=0}^{k} x^n = \lim_{k\to\infty} \frac{x^{k+1}-1}{x-1} = -\frac{1}{x-1} = \frac{1}{1-x}$$

- **Thus:**

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$S = \sum_{j=0}^{3} 2(5)^j = 2 * \frac{5^4-1}{5-1} =$$

$$= 2 * \frac{625-1}{4} = 2 * \frac{624}{4} = 2 * 156 = 312$$

Formulae for commonly occurring sums

Table 2 provides a small table of formulae for commonly occurring sums.

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1}-a}{r-1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Example 5: Find $\sum_{k=50}^{100} k^2$.

Solution: First note that because $\sum_{k=1}^{100} k^2 = \sum_{k=1}^{49} k^2 + \sum_{k=50}^{100} k^2$, we have

$$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2.$$

Using the formula $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$ from Table 2 (and proved in Exercise 38), we see that

$$\sum_{k=50}^{100} k^2 = \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6} = 338{,}350 - 40{,}425 = 297{,}925.$$

2.5 Cardinality of Sets

Recall: The cardinality of a finite set is defined by the number of elements in the set.

Definition:

The sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B. When A and B have the same cardinality, we write |A|=|B|. In other words if there is a bijection from A to B.

Example 1: Assume A = {a,b,c} and B = {α,β,γ} and function f defined as:

- a → α
- b →β
- c → γ

f defines a bijection. Therefore A and B have the same cardinality,

i.e. | A | = | B | = 3.

Countable Sets

Definition:

A set that is either finite or has the same cardinality as the set of positive integers Z+ is called countable. A set that is not countable is called uncountable.

Example 2: Assume A = {0, 2, 4, 6, ... } set of even numbers. Is it countable?

- Using the definition: Is there a bijective function f: $Z^+ \to A$

- Define a function f: x → 2x - 2 (an arithmetic progression)
  - 1 → 2(1)-2 = 0
  - 2 → 2(2)-2 = 2
  - 3 → 2(3)-2 = 4 ...
  - one-to-one (why?) 2x-2 = 2y-2 => 2x = 2y =>x = y.
  - onto (why?) ∀a ∈ A, (a+2) / 2 is the pre-image in Z^+.
  - Therefore | A | = | Z^+ |.

Tet yourself: Show that the set of odd positive integers is a countable.

Example 3: Show that the set of all integers is countable.

Solution:

- We can list all integers in a sequence by starting with 0 and alternating between positive and negative integers: 0, 1, −1, 2, −2,... .
- Alternatively, we could find a one-to-one correspondence between the set of positive integers and the set of all integers (a bijection from Z^+ to Z):
  - f (n) = n/2 when n is even
  - f (n) = −(n − 1)/2 when n is odd
- Consequently, the set of all integers is countable.

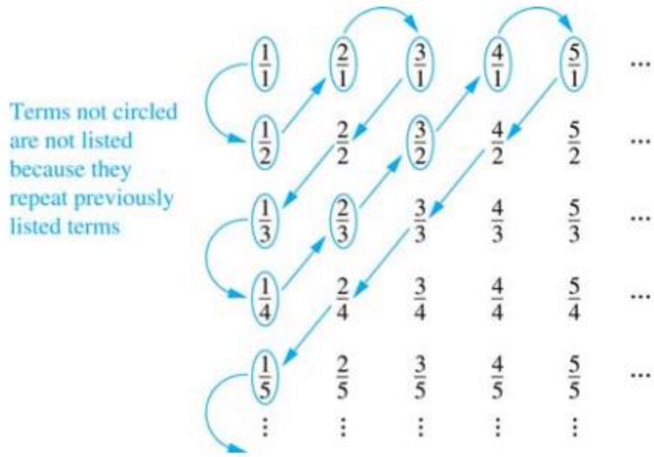
Example 4: Show that the set of positive rational numbers is countable

Solution:

The positive rational numbers are countable since they can be arranged in a sequence:

$r_1 , r_2 , r_3 ,\ldots$

First row $q = 1$.
Second row $q = 2$.
etc.



Constructing the List

First list p/q with $p + q = 2$.
Next list p/q with $p + q = 3$

And so on.