9. 3 Representing Relations using matrices

- **Question:** Can the relation be formed by taking the union or intersection or composition of two relations R_1 and R_2 be represented in terms of matrix operations?
- **Answer: Yes**

Example 1: Suppose that A = {1, 2, 3} and B = {1, 2}. Let R be the relation from A to B containing (a, b) if a ∈ A, b ∈ B, and a > b. What is the matrix representing R if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

Solution: Because R = {(2, 1), (3, 1), (3, 2)}, the matrix for R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Definition:

The **join,** denoted by ∨, of two m-by-n matrices $(a_{ij})$and (b_{ij}) of 0s and 1s is an m-by-n matrix (m_{ij}) where

- $m_{ij} = a_{ij} \vee b_{ij}$ for all i,j

= pairwise or (disjunction)

Example 2

Let A = {1,2,3} and B = {u,v}

R_1 = {(1,u), (2,u), (2,v), (3,u)}

R_2 = {(1,v),(3,u),(3,v)}

- **MR_1 =** $\begin{matrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{matrix}$ **MR_2 =** $\begin{matrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{matrix}$ **$M(R_1 \vee R_2)$=** $\begin{matrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{matrix}$

Definition:

The **meet,** denoted by ∧ , of two m-by-n matrices $(a_{ij})$and (b_{ij}) of 0s and 1s is an m-by-n matrix (m_{ij}) where

- $m_{ij} = a_{ij} \wedge b_{ij}$ for all i,j

= **pairwise and (conjunction)**

Example 3:

- Let A = {1,2,3} and B = {u,v}

  R_1 = {(1,u), (2,u), (2,v), (3,u)}
- R_2 = {(1,v),(3,u),(3,v)}

- **MR_1 =**

1	**0**
1	**1**
1	**0**

MR_2 =

0	**1**
0	**0**
1	**1**

$MR_1 \wedge MR_2$=

0	**0**
0	**0**
1	**0**

Definition: **Composite of relations**

Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite of R and S** is the relation consisting of the ordered pairs (a,c) where a ∈ A and c ∈ C, and for which there is a b ∈ B such that (a,b) ∈ R and (b,c) ∈ S. We denote the composite of R and S by S o R.

Example 4:

- Let A = {1,2,3}, B = {0,1,2} and C = {a,b}.
- R = {(1,0), (1,2), (3,1),(3,2)}
- S = {(0,b),(1,a),(2,b)}
- S o R = {(1,b),(3,a),(3,b)}

Implementation of composite

Definition:

The **Boolean product, denoted by ⊙,** of an m-by-n matrix (a_{ij}) and n-by-p matrix (b_{jk}) of 0s and 1s is an m-by-p matrix (m_{ik}) where

m_{ik} = 1, if a_{ij} = 1 and b_{jk} = 1 for some k=1,2,…,n

0, otherwise

Example 5:

- Let A = {1,2}, B= {1,2,3} C = {a,b}
- R = {(1,2),(1,3),(2,1)} is a relation from A to B
- S = {(1,a),(3,b),(3,a)} is a relation from B to C.
- S o R = {(1,b),(1,a),(2,a)}

M_R =

0	1	1
1	0	0

M_S =

1	0
0	0
1	1

$M_R \odot M_S$ =

1	1
1	0

$M_{S \circ R}$ =

1	1
1	0

Definition:

Let R be a relation on a set A. The **powers R^n**, n =1,2,3,... is defined inductively by

- **$R^1 = R$** and **$R^{n+1} = R^n \circ R$**.

Example 6:

- R = {(1,2),(2,3),(2,4), (3,3)} is a relation on A = {1,2,3,4}.
- R^1 = R = {(1,2),(2,3),(2,4), (3,3)}
- R^2 = {(1,3), (1,4), (2,3), (3,3)}
- R^3 = {(1,3), (2,3), (3,3)}
- R^4 = {(1,3), (2,3), (3,3)}
- $R^k = R^3$, $k > 3$.

Theorem 1: The relation R on a set A is transitive <u>if and only if</u> $R^n \subseteq R$ for n = 1,2,3,... .

Number of reflexive relations

Theorem 2: The number of reflexive relations on a set A, where

$|A| = n$ is: $2^{n(n-1)}$.