Representing binary relations Using graphs

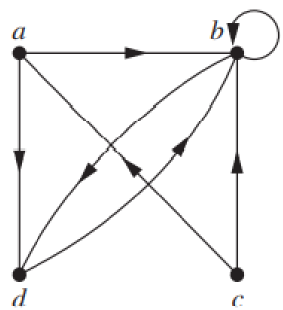
- We have shown that a relation can be represented by listing all of its ordered pairs or by using a zero–one matrix.
- We use such pictorial representations when we think of relations on a finite set as directed graphs, or digraphs.

Definition:

A **directed graph or digraph** consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial vertex of the edge (a,b) and vertex b is the terminal vertexof this edge.

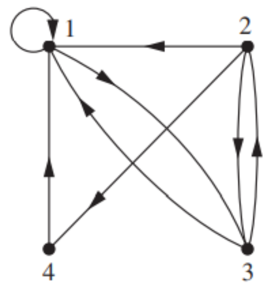
An edge of the form (a, a) is represented using an arc from the vertex a back to itself. Such an edge is called a **loop.**

Example : The directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b), and (d, b) is displayed in Figure down .



Example : Assume the relation R = {(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)} on the set {1, 2, 3, 4}

The directed graph of is shown in down.



9.4 Closures of relations

- **Relations can have different properties:**
  - **reflexive,**
  - **symmetric**
  - **transitive**

- Because of that we define:
  - **symmetric closures.**
  - **reflexive closures.**
  - **transitive closures.**

Definition:

Let R be a relation on a set A. A relation S on A with property P is called **the closure of R with respect to P** if S is asubset of every relation Q ($S \subseteq Q$) with property P that containsR ($R \subseteq Q$).

Example : Let R={(1,1),(1,2),(2,1),(3,2)} on A ={1 2 3}.

- Is this relation reflexive?
- Answer: **No.** Why?
- **(2,2) and (3,3) is not in R.**

- The question is what is **the minimal relation S** $\supseteq$ R that is reflexive?
- How to make R reflexive with minimum number of additions?
- **Answer:** Add (2,2) and (3,3)
  - Then S= {(1,1),(1,2),(2,1),(3,2),(2,2),(3,3)}
  - $R \subseteq S$
  - The minimal set $S \supseteq R$ is called **the reflexive closure of R**

Definition: Reflexive closure

The set S is called **the reflexive closure of R** if it:

– contains R
– has reflexive property
– is contained in every reflexive relation Q that contains R (R $\subseteq$ Q) , that is S $\subseteq$ Q

Definition: Symmetric closure

The set S is called **the reflexive closure of R** if it can be constructed by taking

the union of a relation with its inverse $S = R \cup R^{-1}$

Example (a symmetric closure):

- Assume R={(1,2),(1,3), (2,2)} on A={1,2,3}.
- What is the symmetric closure S of R?
- **S = {(1,2),(1,3), (2,2)} $\cup$ {(2,1), (3,1)}**
  = {(1,2),(1,3), (2,2),(2,1), (3,1)}

Transitive closure

Theorem: The relation R on a set A is transitive <u>if and only if</u> $R^n \subseteq R$ for n = 1,2,3,... .

Example (a transitive closure):

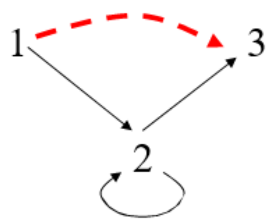
- Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.
- **Is R transitive? No.**
- **How to make it transitive?**
- **S = {(1,2), (2,2), (2,3)} $\cup$ {(1,3)}**
  = {(1,2), (2,2), (2,3),(1,3)}

Thus S is the transitive closure of R

- We can represent the relation on the graph. Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path (or digraph).

Example:
- Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.
- Transitive closure S = {(1,2), (2,2), (2,3),(1,3)}.

R



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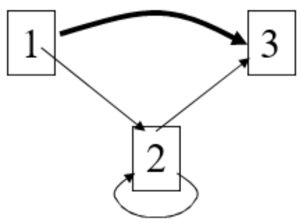


Paths in Directed Graphs

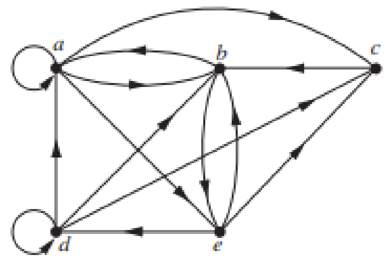
- Constructing the transitive closure of a relation is more complicated than constructing either the reflexive or symmetric closure.
- We now introduce some terminology that we will use for this purpose.

Definition: Paths in Directed Graphs

A path from a to b in the directed graph **G** is a sequence of edges $(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ in **G**, where n is a nonnegative integer, and x_0 = **a** and x_n = **b**, that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path. This path is denoted by $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ and has length n. We view the empty set of edges as a path of length zero from **a to a**. A path of length $n \geq 1$ that begins and ends at the same vertex is called a circuit or **cycle.**

Note: A path in a directed graph can pass through a vertex more than once. Moreover, an edge in a directed graph can occur more than once in a path.

Example: Which of the following are paths in the directed graph shown in the Figure down: a, b, e, d; a, e, c, d, b; b, a, c, b, a, a, b; d,c; c, b, a; e, b, a, b, a, b, e? What are the lengths of those that are paths? Which of the paths in this list are circuits?



Solution:

- Because each of (a, b), (b, e), and (e, d) is an edge, a, b, e, d is a path of length three.
- Because (c, d) is not an edge, a, e, c, d, b is not a path.
- Also, b, a, c, b, a, a, b is a path of length six because (b, a), (a, c), (c, b), (b, a), (a, a), and (a, b) are all edges.
- We see that d,c is a path of length one, because (d, c) is an edge.
- Also c, b, a is a path of length two, because (c, b) and (b, a) are edges.
- All of (e, b), (b, a), (a, b), (b, a), (a, b), and (b, e) are edges, so e, b, a, b, a, b, e is a path of length six.
- The two paths b, a, c, b, a, a, b and e, b, a, b, a, b, e are circuits because they begin and end at the same vertex.
- The paths a, b, e, d; c, b, a; and d,c are not circuits.

Theorem (Path length) : Let R be a relation on a set A. There is a path of length n from a to b if and only if $(a,b) \in R^n$

Example:

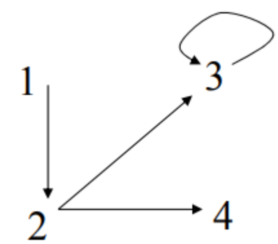
R = {(1,2),(2,3),(2,4), (3,3)} is a relation on A = {1,2,3,4}.

R^1 = R = {(1,2),(2,3),(2,4), (3,3)}

R^2 = {(1,3), (1,4), (2,3), (3,3)}

What does R^2 represent? **Paths of length 2**

R^3 = {(1,3), (2,3), (3,3)} **Paths of length 3**



Definition: **Connectivity relation**

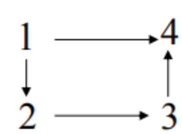
Let R be a relation on a set A. The **connectivity relation** R* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R.

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

Example:

A = {1,2,3,4}

R = {(1,2),(1,4),(2,3),(3,4)}



R^2 = {(1,3),(2,4)}

R^3 = {(1,4)}

R^4 = ∅

...

R* = {(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)}

Theorem: The transitive closure of a relation R **equals** the connectivity relation R*.

Theorem: Let M_R be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure R^* is $M_{R^*} = M_R \vee M^{[2]}_R \vee M^{[3]}_R \vee \cdots \vee M^{[n]}_R$.

Example: Find the zero–one matrix of the transitive closure of the relation R where

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution: By Theorem, it follows that the zero–one matrix of R^* is $M_{R^*} = M_R \vee M^{[2]}_R \vee M^{[3]}_R$. Because

$$\mathbf{M}_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

it follows that

$$\mathbf{M}_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Definition: **Equivalence relation**

A relation R on a set A is called an **equivalencerelation** if it is **reflexive, symmetric and transitive.**

Example: Let A = {0,1,2,3,4,5,6} and

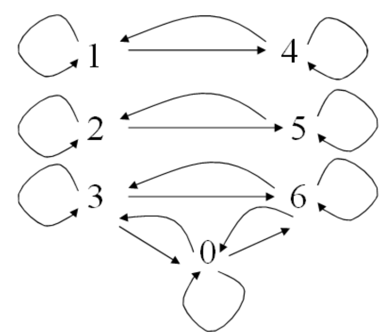
R= {(a,b)| a,b $\in$ A, a $\equiv$ b mod 3}   (a is congruent to b modulo 3)

Congruencies:

- 0 mod 3 = 0   1 mod 3 = 1   2 mod 3 = 2   3 mod 3 = 0
- 4 mod 3 = 1   5 mod 3 = 2   6 mod 3 = 0

Relation R has the following pairs:

- (0,0)   (0,3), (3,0), (0,6), (6,0)
- (3,3), (3,6) (6,3),   (1,1),(1,4), (4,1), (4,4) ,(6,6)
- (2,2), (2,5), (5,2), (5,5)



Is R reflexive? **Yes.**

Is R symmetric? **Yes.**

Is R transitive. **Yes.**

Then

R is an equivalence relation.