Ch10 Graphs

10.1 Graphs and Graph Models

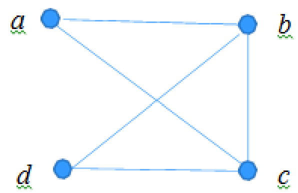
Definition:

A *graph* $G = (V, E)$ consists of a nonempty set V of *vertices* (or *nodes*) and a set E of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge issaid to *connect* its endpoints.

Remark:

The set of vertices V of a graph G may be **infinite**. A graph with an infinite vertex set or an infinite number of edges is called an **infinite graph**, and in comparison, a graph with **a finite** vertex set and a finite edge set is called **a finite graph.**

Example:

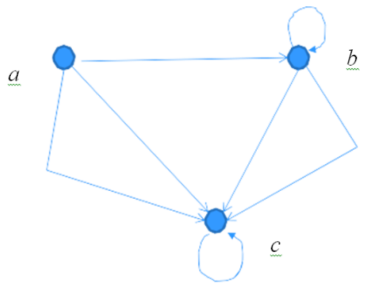


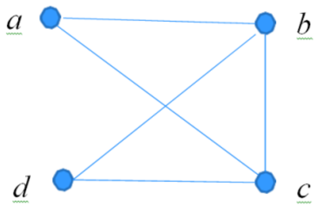
Basic types of graphs:

- **Directed graphs**

  Undirected graphs

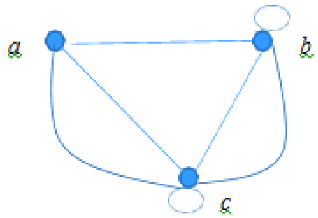
  - Graphs where the end points of an edge are not ordered





Terminology

- In a ***simple graph*** each edge connects two different vertices and no two edges connect the same pair of vertices.
- ***Multigraphs*** may have multiple edges connecting the same two vertices. When *m* different edges connect the vertices *u* and *v*, wesay that {*u*,*v*} is an edge of *multiplicity m*.
- An edge that connects a vertex to itself is called a ***loop***.
- A ***pseudograph*** may include loops, as well as multiple edges connecting the same pair of vertices.

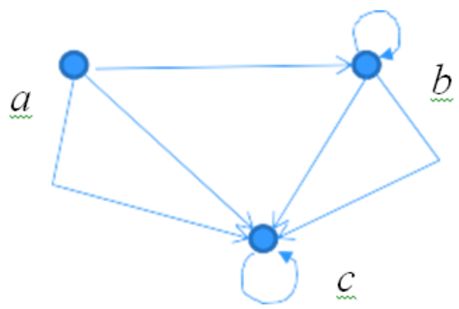


Directed graph

- A *simple directed graph* has no loops and no multiple edges.

Example:

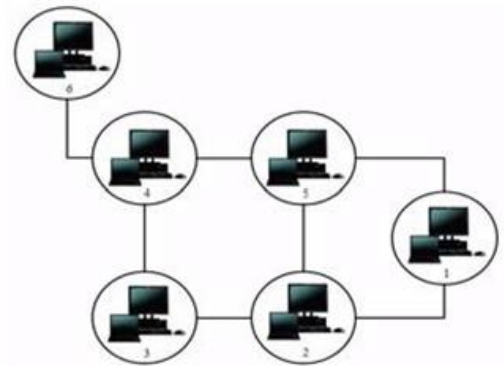
- multiplicity of (*a*,*b*) is ?   1
- and the multiplicity of (b,c) is 2



- **Graphs and graph theory can be used to model:**
  - Computer networks
  - Social networks
  - Communications networks
  - Information networks
  - Software design
  - Transportation networks
  - Biological networks

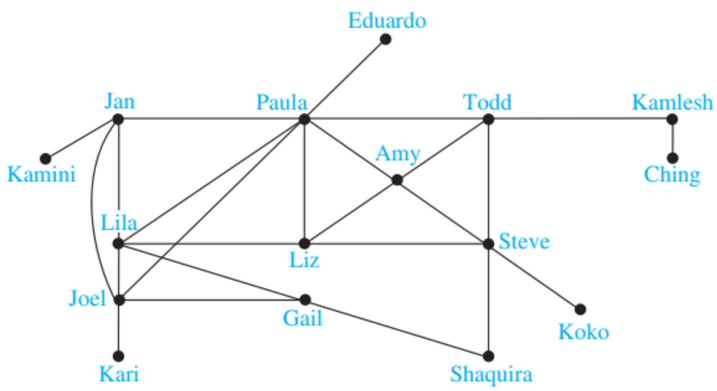
Graph models

- **Computer networks:**
  - **Nodes – computers**
  - **Edges - connections**

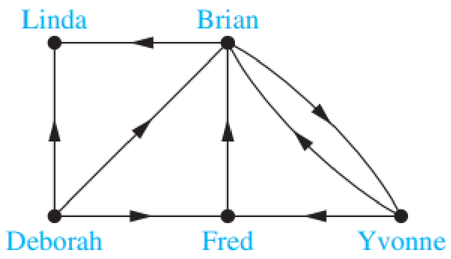


- **Social networks:**

- Graphs can be used to model social structures based on different kinds of relationships between people or groups.
- ***Social network***, vertices represent individuals or organizations and edges represent relationships between them.
- Useful graph models of social networks include:
  - ***friendship graphs*** - undirected graphs where two people are connected if they are friends (in the real world, on Facebook, or in a particular virtual world, and so on.)



- Useful graph models of social networks include:
  - ***influence graphs*** - directed graphs where there is an edge from one person to another if the first person can influence the second person



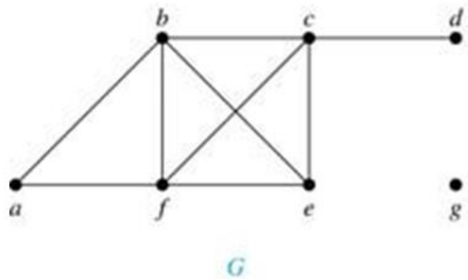
Graph characteristics: Undirected graphs

Definition 1. Two vertices *u*, *v* in an undirected graph *G* are called ***adjacent* (or *neighbors*)** in *G* if there is an edge *e* between *u* and *v*. Such an edge *e* is called *incident with* the vertices *u* and *v* and *e* is said to *connect u* and *v*.

Definition 2. The set of all neighbors of a vertex *v* of *G* = (*V*, *E*), denoted by *N*(*v*), is called **the *neighborhood* of *v***. If *A* is a subset of *V*, we denote by *N*(*A*) the set of all vertices in *G* that are adjacent to at least one vertex in *A*.

Definition 3. The ***degree of a vertex in a undirected graph*** is the number of edges incident with it, except that a loop at a vertex contributes two to the degree of that vertex. The degree of the vertex *v* is denoted by deg(*v*).

Example: What are the degrees and neighborhoods of the vertices in the graphs *G*?



Solution:

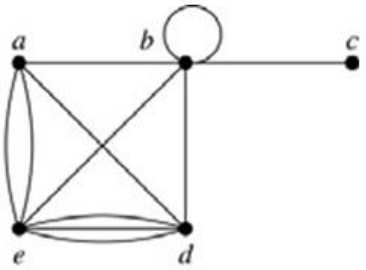
G: deg(*a*) = 2, deg(*b*) = deg(*c*) = deg(*f*) = 4, deg(*d*) = 1,

deg(*e*) = 3, deg(*g*) = 0.

N(*a*) = {*b, f* }, *N*(*b*) = {*a, c, e, f* }, *N*(*c*) = {*b, d, e, f* },

N(*d*) = {*c*}, *N*(*e*) = {*b, c , f* }, *N*(*f*) = {*a, b, c, e*}, *N*(*g*) = ∅ .

Example: What are the degrees and neighborhoods of thevertices in the graphs *H*?



H

Solution:

H: deg(*a*) = 4, deg(*b*) = deg(*e*) = 6, deg(*c*) = 1, deg(*d*) = 5.

N(*a*) = {*b, d, e*}, *N*(*b*) = {*a, b, c, d, e*}, *N*(*c*) = {*b*},

N(*d*) = {*a, b, e*}, *N*(*e*) = {*a, b ,d*}

Theorem 1 (*Handshaking Theorem*): If *G* = (*V*,*E*) is anundirected graph with *m* edges, then

$$2m = \sum_{v \in V} \deg(v)$$

Proof:

Each edge contributes twice to the degree count of all vertices.Hence, both the left-hand and right-hand sides of this equation equal twice the number of edges.

Theorem 2: An undirected graph has an even number of verticesof odd degree.

Proof: Let V_1 be the vertices of even degree and V_2 be the verticesof odd degree in an undirected graph *G* = (*V*, *E*) with *m* edges.

Then

$$2m = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v).$$

Graph characteristics: Directed graphs

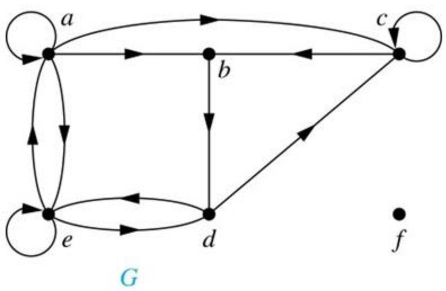
Definition: An ***directed graph*** *G* = (*V*, *E)* consists of *V*, a nonempty set of *vertices* (or *nodes*), and *E*, a set of *directed edges*or *arcs*. Each edge is an ordered pair of vertices. The directed edge (*u*,*v*) is said to start at *u* and end at *v*.

Definition: Let (*u*,*v*) be an edge in *G*. Then *u* is the *initial vertex* of this edge and is *adjacent to v* and *v* is the *terminal* (or *end*) *vertex* of this edge and is *adjacent from u*. The initial and terminalvertices of a loop are the same.

Definition: The *in-degree of a vertex v*, denoted *deg* $^-$(*v*), isthe number of edges which terminate at *v*. The *out-degree of v*, denoted *deg*$^+$(*v*), is the number of edges with *v* as their initial vertex. Note that a loop at a vertex contributes 1 to both the in- degree and the out-degree of the vertex.

Example: Assume graph *G:*

What are in-degrees of vertices: ?



Deg $^-$(*a*) = 2, deg $^-$(*b*) = 2,

deg $^-$(*c*) = 3,

Deg $^-$(*d*) = 2, deg $^-$(*e*) = 3,
deg $^-$(*f*) = 0.

What are out-degrees of vertices: ?

deg$^+$(*a*) = 4, deg$^+$(*b*) = 1,
deg$^+$(*c*) = 2,

deg$^+$(*d*) = 2, deg$^+$ (*e*) = 3,
deg$^+$(*f*) = 0.

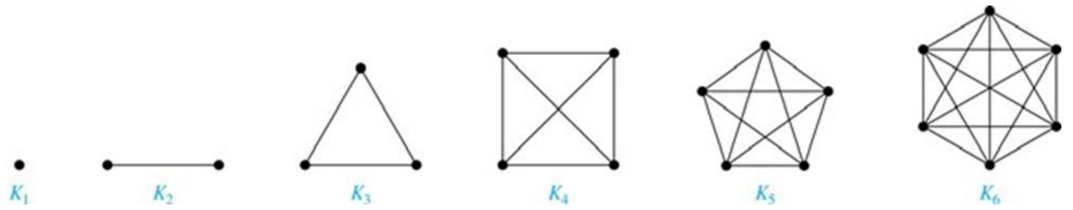
Theorem: Let $G = (V, E)$ be a graph with directed edges. Then:

$$|E| = \sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v).$$

Some Special Simple Graphs

Complete graphs

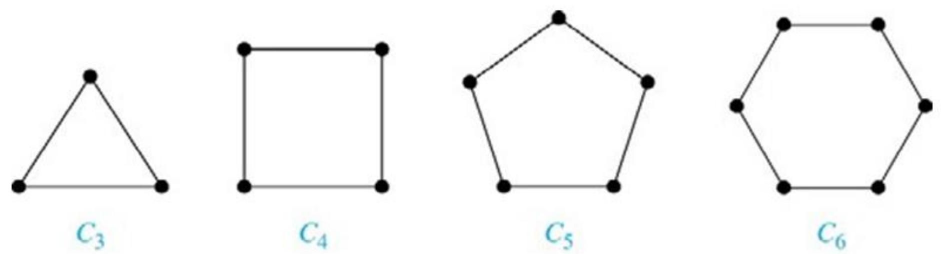
A *complete graph on n vertices*, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinctvertices.



A cycle

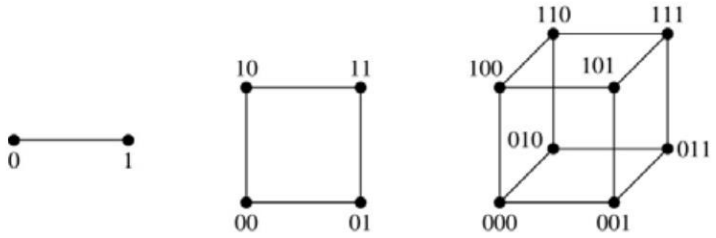
A *cycle* C_n for $n \geq 3$ consists of n vertices $v_1, v_2, \cdots, v_n$, and edges

$\{v_1, v_2\}, \{v_2, v_3\}, \cdots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.



N-dimensional hypercube

An *n-dimensional hypercube*, or *n-cube*, Q_n, is a graph with 2^n vertices representing all bit strings of length *n*, where there is anedge between two vertices that differ in exactly one bit position.

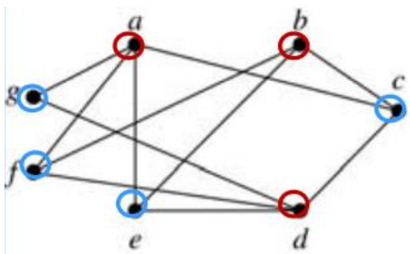


Q_1 Q_2 Q_3

Bipartite graphs

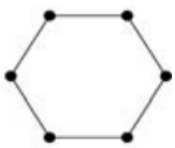
Definition: A simple graph G is **bipartite** if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects avertex in V_1 and a vertex in V_2. In other words, there are no edges which connect two vertices in V_1 or in V_2.

Note: An equivalent definition of a bipartite graph is a graph where it is possible to color the vertices red or blue so that no twoadjacent vertices are the same color.



G

Example: Show that C_6 is bipartite.

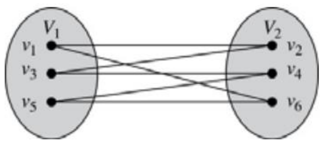


C_6

Solution:

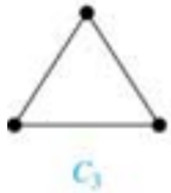
- We can partition the vertex set into
  $V_1 = \{v_1, v_3, v_5\}$ and
  $V_2 = \{v_2, v_4, v_6\}$

so that every edge of C_6 connects a vertex in V_1 and V_2 .



Example: Show that C_3 is not bipartite.

Solution:



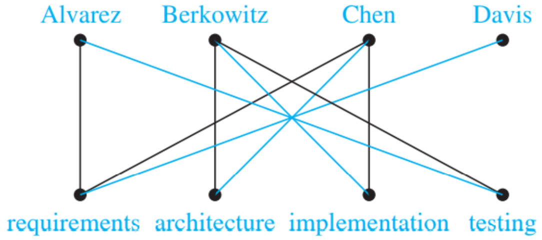
If we divide the vertex set of C_3 into two nonempty sets, one of the two must contain two vertices. But in C_3 every vertex is connected to every other vertex. Therefore, the two vertices in the same partition are connected. Hence, C_3 is not bipartite.

Bipartite graphs and matching

Bipartite graphs are used to model applications that involve **matching**

the elements of one set to elements in another, for example:

Example: *Job assignments* - vertices represent the jobs and the employees, edges link employees with those jobs they have been trained to do. A common goal is to match jobs to employees so thatthe most jobs are done.



(a)