## 1.4 Predicates and Quantifiers

**Propositional logic:** the world is described in terms of elementary propositions and their logical combinations

**Elementary statements:**

**• Typically refer to objects, their properties and relations. But these are not explicitly represented** in the propositional logic

– **For example:**

• "Omar is a Ptuk student."

**Omar** → **a Ptuk student**

object → a property

Objects and properties are hidden in the statement, it is not possible to reason about them.

**Limitations of the propositional logic**

**(1) Statements that must be repeated for many objects**

In propositional logic these must be exhaustively enumerated

▪ **For example:**

– If Omar is an AC Ptuk graduate then Omar has passed Calculus.

**Translation:**

– Omar is an AC Ptuk graduate → Omar has passed Calculus.

Similar statements can be written for other Ptuk graduates:

– Adnan is an AC Ptuk graduate → Adnan has passed Calculus

– Amal is an AC Ptuk graduate → Amal has passed Calculus

– …

**• Solution:** make statements with **variables**

– If x is an AC Ptuk graduate then x has passed Calculus.

– x is an AC Ptuk graduate → x has passed Calculus.

**(2) Statements that define the property of the group of objects**

▪ **For example:**

– "Every computer connected to the university network is functioning properly."

– All new cars must be registered.

– "There is a computer on the university network that is under attack by an intruder."

– Some of the AC graduates graduate with honors.

• **Solution:** make statements with **quantifiers.**

## Predicate logic

To understand predicate logic, we first need to introduce the concept of a predicate.

**Predicates**

**Predicates** represent properties or relations among objects.

**For examples:**

Statements involving variables, such as

"$x > 3$," "$x = y + 3$," "$x + y = z$,"

and

"computer *x* is under attack by an intruder,"

and

"computer *x* is functioning properly,"

- The statement "*x* is greater than 3" has two parts.
    - The first part, the variable *x*, is the subject of the statement(**object**).
    - The second part—the **predicate**, "is greater than 3"—refers to a **property** that the subject of the statement can have.
- We can denote the statement "*x* is greater than 3" by $P(x)$.
- The statement $P(x)$ is also said to be the value of the **propositional function** *P* at *x*.

  **Let the following examples:**
  - Student(x) denotes the statement "x is a student"
  - Person(x) denotes the statement "x is a person"
  - University(x) denotes the statement "x is a university"
- Once a value has been assigned to the variable *x*, the statement $P(x)$ becomes a proposition and has a truth value.
  - **Student**(John) …. T (if John is a student)
  - **Student**(Ann) …. T (if Ann is a student)
  - **Student**(Jane) ….. F (if Jane is not a student)
  - **University**(Ptuk) …. T
  - **Person**(Ahmad) …. T

**Example 1:** Let $P(x)$ denote the statement "$x > 3$." What are the truth values of $P(4)$ and $P(2)$?

***Solution:***

We obtain the statement *P(4)* by setting $x = 4$ in the statement "$x > 3$." Hence, *P(4)*, which is the statement "4 > 3," is true. However, *P(2)*, which is the statement "2 > 3," is false(F).

**Example 2:** Assume a predicate P(x) that represents the statement:
" x is a prime number "

1. What are the truth values of , P(x) for x= 3, 4, 5, 6, and 7 .
2. **Is P(x) a proposition?**

***Solution:***

1.
    - P(3) T
    - P(4) F
    - P(5) T
    - P(6) F
    - P(7) T

    **All statements P(2), P(3), P(4), P(5), P(6), P(7) are propositions**
2. **No. Many possible substitutions are possible.**

Predicates can have **more arguments (variables)** which represent the **relations between objects.**

- A predicate with two arguments is denoted by *Q(x, y)*, where *x* and *y* are variables is a **predicate**.
- Once a values has been assigned to the variable *x* and *y*, the statement *Q(x, y)* becomes a ***proposition*** and has a truth value.

  **For example: Let Older**(x,y) denotes the statement "x is older than y"
    - **Older** (John, Peter) : "John is older than Peter"
      – this is a proposition because it is either true or false
    - **Older** (x, y) : " x is older than y"
      – not a proposition, but after the substitution it becomes one.
- **Similarly** a predicate with three arguments is denoted by *R(x, y, z)*, where *x, y* and *z* are variables.

- Once a values has been assigned to the variable *x* , *y*, and z the statement *Q(x, y,z)* becomes a ***proposition*** and has a truth value.

  **For example:** Let **StudyAt**(x,y, z) denotes the statement "x study at university y major z "
    - **StudyAt**(Amjad,Ptuk, AC): Amjad study at Ptuk major AC "
      – this is a proposition because it is either true or false

- **In general**, a statement involving the $n$ variables $x_1, x_2, \ldots, x_n$ can be denoted by $P(x_1, x_2, \ldots, x_n)$.
  - When values are assigned to the variables $x_1, x_2, \ldots, x_n$ the statement $P(x_1, x_2, \ldots, x_n)$ has a truth value.

**Example 3:** Let $Q(x, y)$ denote the statement "$x = y + 3$." What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

***Solution:***
- To obtain $Q(1, 2)$, set $x = 1$ and $y = 2$ in the statement $Q(x, y)$. Hence, $Q(1, 2)$ is the statement "1 = 2 + 3," which is false.
- The statement $Q(3, 0)$ is the proposition "3 = 0 + 3," which is true.

**Example 4:** Let Q(x,y) denote "x+5 >y"

**1.** Is Q(x,y) a proposition? **No!**
2. Is Q(3,7) a proposition? **Yes.** It is true.
3. What is the truth value of:
   a) Q(3,7) T
   b) Q(1,6) F
   c) Q(2,2) T
4. Is Q(3,y) a proposition? **No!** We cannot say if it is true or false.

***Solution:***
1. **No!**
2. **Yes.** It is true.
3. the truth value of:
   a) Q(3,7) T
   b) Q(1,6) F
   c) Q(2,2) T
4. **No!** We cannot say if it is true or false.

**Example 5:** let $R(x, y, z)$ denote the statement " $x + y = z$."
What are the truth values of the propositions $R(1, 2, 3)$ and $R(0, 0, 1)$?

***Solution:***
- The proposition $R(1, 2, 3)$ is obtained by setting $x = 1$, $y = 2$, and $z = 3$ in the statement $R(x, y, z)$.
- $R(1, 2, 3)$ is the statement "1 + 2 = 3," which is true.

- Note that $R(0, 0, 1)$, which is the statement "0 + 0 = 1," is false.

## Compound statements in predicate logic

**Compound statements are obtained via logical connectives**

**For examples:**

- Student(Ann) ∧ Student(Jane)
  - **Translation:** "Both Ann and Jane are students"
  - **Proposition:** yes.
- Country(Sienna) ⋁ River(Sienna)
  - **Translation:** "Sienna is a country or a river"
  - **Proposition:** yes.
- AC -major(x) → Student(x)
  - **Translation:** "if x is an AC-major then x is a student"
  - **Proposition:** no.

## Quantifiers

**Predicate logic lets us to make statements about groups of objects by using special quantified expressions.**

First we want to define the **domain of quantification.**

**Definition**:

The **domain of quantification**; i.e., what the quantifiers (or variables) range over. The domain must be nonempty. (The domain is sometimes also called the **universe of discourse** or the **domain of discourse**.)

The universe of discourse can be people, students, numbers, etc.

Two types of quantified statements:

- **Universal quantifier** –the property is satisfied by all members of the group.
  - **For example:** " all AC Ptuk graduates have to pass Calculus" – the statement is true for all graduates.
- **Existential quantifier** – at least one member of the group satisfy the property.
  - **For example:** "Some AC Ptuk students graduate with honor." – the statement is true for some people.

## Universal quantifier

**Definition**:

The universal quantification of P(x) is the proposition: "*P(x) is true for all values of x in the domain of discourse*." The notation **∀ x P(x)** denotes the universal quantification of P(x), and is expressed as **for every x, P(x).** An element for which *P(x)* is false is called a **counterexample** of ∀*xP(x)*.

**Example 1:** Let *P(x)* be the statement "$x + 1 > x$." What is the truth value of the quantification ∀*xP(x)*, where the domain consists of all real numbers?
***Solution:***
Because *P(x)* is true for all real numbers *x*, the quantification ∀*xP(x)* **is true**.

***Remarks:***
1. If the domain is empty, then ∀*xP(x)* is true for any propositional function *P(x)* because there are no elements *x* in the domain for which *P(x)* is false.
2. Remember that the truth value of ∀*xP(x)* depends on the domain!
3. Besides "for all" and "for every," universal quantification can be expressed in many other ways, including "**all of**," "**for each**," "**given any**," "**for arbitrary**," "**for each**," and "**for any**."
4. A statement ∀*xP(x)* is false, where*P(x)* is a propositional function, if and only if *P(x)* is not always true when *x* is in the domain.

**Example 2:** Let *Q(x)* be the statement "$x < 2$." What is the truth value of the quantification ∀*xQ(x)*, where the domain consists of all real numbers?
***Solution:***
*Q(x)* is not true for every real number *x*, because, for instance, *Q*(3) is false. That is, $x = 3$ is a counterexample for the statement ∀*xQ(x)*. Thus ∀*xQ(x)* is false.

**Example 3:** Suppose that *P(x)* is "$x^2 > 0$." Show that the statement ∀*xP(x)* is false.
***Solution:***
To show that the statement ∀*xP(x)* is false where the universe of discourse consists of all integers, we give a counterexample. We see that $x = 0$ is a

counterexample because $x^2 = 0$ when $x = 0$, so that $x^2$ is not greater than 0 when $x = 0$.

***Remark :*** When all the elements in the domain can be listed say, $x_1, x_2, \ldots, x_n$ it follows that the universal quantification $\forall x P(x)$ is the same as the conjunction $P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$, because this conjunction is true if and only if $P(x_1), P(x_2), \ldots, P(x_n)$ are all true.

**Example 4:** What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?

***Solution:***

The statement $\forall x P(x)$ is the same as the conjunction

$$P(1) \wedge P(2) \wedge P(3) \wedge P(4),$$

because the domain consists of the integers 1, 2, 3, and 4. Because $P(4)$, which is the statement "$4^2 < 10$," is false, it follows that $\forall x P(x)$ is false.

**Example 5:**What does the statement $\forall x N(x)$ mean if $N(x)$ is "Computer $x$ is connected to the network" and the domain consists of all computers on campus?

***Solution:*** The statement $\forall x N(x)$ means that for every computer $x$ on campus, that computer $x$ is connected to the network. This statement can be expressed in English as "Every computer on campus is connected to the network."

**Example 6:**What is the truth value of $\forall x(x^2 \geq x)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers?

***Solution:***

The universal quantification $\forall x(x^2 \geq x)$, where the domain consists of all real numbers, is false. For example, $\left(\frac{1}{2}\right)^2 < \frac{1}{2}$.

## Existential quantifier

**Definition**:

The **existential quantification** of P(x) is the proposition "*There exists an element in the domain (universe) of discourse such that P(x) is true*." The notation **∃x P(x)** denotes the existential quantification of P(x), and is expressed as **there is an x such that P(x) is true**.

**Example 7:** Let $P(x)$ denote the statement "$x > 3$." What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

***Solution:*** Because "$x > 3$" is sometimes true—for instance, when $x = 4$ , the existential quantification of $P(x)$, which is $\exists x P(x)$, is true.

***Remarks :***

1. the statement $\exists x P(x)$ is false if and only if there is no element $x$ in the domain for which $P(x)$ is true.
2. we can also express existential quantification in many other ways, such as by using the words "**for some**," "**for at least one**," or "**there is**."
3. The existential quantification $\exists x P(x)$ is read as
   "**There is an *x* such that *P(x)*,**"
   "**There is at least one *x* such that *P(x)*,**"

   or

   "**For some *xP(x)*.**"

**Example 8:** Let $Q(x)$ denote the statement "$x = x + 1$."What is the truth value of the quantification $\exists x Q(x)$, where the domain consists of all real numbers?

***Solution:***
Because $Q(x)$ is false for every real number $x$, the existential quantification of $Q(x)$, which is $\exists x Q(x)$, is false.

**Example 9:** Let T(x) denote x > 5 and x is from Real numbers. What is the truth value of $\exists x$ T(x)?
***Solution:***
Since 10 > 5 is true. Therefore, it is **true that ∃x T(x).**

***Remark :*** When all elements in the domain can be listed—say, $x_1, x_2, \ldots, x_n$ the existential quantification $\exists x P(x)$ is the same as the disjunction

$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n),$$

because this disjunction is true if and only if at least one of $P(x_1), P(x_2), \ldots, P(x_n)$ is true.

**Example 10:** What is the truth value of ∃*xP(x)*, where *P(x)* is the statement "*x*2 > 10" and the universe of discourse consists of the positive integers not exceeding 4?

***Solution:***

Because the domain is {1, 2, 3, 4}, the proposition ∃*xP(x)* is the same as the disjunction

$$P(1) \vee P(2) \vee P(3) \vee P(4).$$

Because *P*(4), which is the statement "$4^2 > 10$," is true, it follows that ∃*xP(x)* is true.

- **Recall** that **quantification** is another important way to create a proposition from a propositional function.

**Example 11: Determine whether the following** statements proposition or not

1. AC-major(x) → Student(x)
2. ∀x AC -major(x) → Student(x)

***Solution***

1. **Translation:** "if x is an AC-major then x is a student"
   It is not a proposition.
2. **Translation:** "(For all people it holds that) if a person is an AC-major then she is a student."
   It is a proposition.

**Example 12: Determine whether the following** statements proposition or not

1. AC-Ptuk- graduate (x) ∧ Honor-student(x)
2. ∃*x* AC-Ptuk- graduate (x) ∧ Honor-student(x)

***Solution***

1. **Translation:** "x is a AC-Ptuk- graduate and x is an honor student"
   It is not a proposition.
2. **Translation:** "There is a person who is a AC-Ptuk- graduate and who is also an honor student."
   It is a proposition.

## Summary of quantified statements

- When ∀**x P(x)** and ∃**x P(x)** are true and false?

| Statement | When True? | When False? |
|---|---|---|
| $\forall x P(x)$ | $P(x)$ is true for every $x$. | There is an $x$ for which $P(x)$ is false. |
| $\exists x P(x)$ | There is an $x$ for which $P(x)$ is true. | $P(x)$ is false for every $x$. |

## Negating Quantified Expressions

### ❖ Negation of a quantified expression.

- For instance, consider the negation of the statement :
    - *"Every student in your class has taken a course in calculus."*
- This statement is a universal quantification, namely,
    - $\forall x P(x)$,

where $P(x)$ is the statement "$x$ has taken a course in calculus"
and the domain consists of the students in your class.

- **The negation of this statement is**
    - *"It is not the case that every student in your class has taken a course in calculus."*
- This is equivalent to
    - *"There is a student in your class who has not taken a course in calculus."*
- And this is simply the existential quantification of the negation of the original propositional function, namely,
    - $\exists x\ \neg P(x)$.
- This example illustrates the following logical equivalence:
    - $\neg\forall x P(x) \equiv \exists x\ \neg P(x)$.

- To show that $\neg\forall x P(x)$ and $\exists x P(x)$ are logically equivalent
    - first note that $\neg\forall x P(x)$ is true iff $\forall x P(x)$ is false.
    - Next, note that $\forall x P(x)$ is false iff there is an element $x$ in the domain for which $P(x)$ is false.

- This holds iff there is an element $x$ in the domain for which $\neg P(x)$ is true.
- Finally, note that there is an element $x$ in the domain for which $\neg P(x)$ is true iff $\exists x\ \neg P(x)$ is true.
- we can conclude that $\neg\forall x P(x)$ is true iff $\exists x\ \neg P(x)$ is true.
- It follows that $\neg\forall x P(x) \equiv \exists x\ \neg P(x)$.

### ❖ Negation of an existential expression.

- For instance, consider the negation of the statement :
  - "*There is a student in this class who has taken a course in calculus.*"
- This statement is an existential quantification, namely,
  - $\exists x Q(x)$,

where $Q(x)$ is the statement "$x$ has taken a course in calculus"
and the domain consists of the students in your class.

- **The negation of this statement is**
  - "*It is not the case that there is student in this who has taken a course in calculus.*"
- This is equivalent to
  - "*Every student in this class has not taken calculus.*"
- And this is simply the universal quantification of the negation of the original propositional function, namely,
  - $\forall x\ \neg Q(x)$.
- This example illustrates the following logical equivalence:
  - $\neg\exists x\ Q(x) \equiv \forall x\ \neg Q(x)$.

- To show that $\neg\exists x Q(x)$ and $\forall x Q(x)$ are logically equivalent
  - first note that $\neg\exists x Q(x)$ is true iff $\exists x Q(x)$ is false.
  - This holds iff no $x$ exists in the domain for which $Q(x)$ is true.
  - Next, note that no $x$ exists in the domain for which $Q(x)$ is true if and only if $Q(x)$ is false for every $x$ in the domain.

- Finally, note that $Q(x)$ is false for every $x$ in the domain if and only if $\neg Q(x)$ is true for all $x$ in the domain,
- we can conclude that $\neg\exists x Q(x)$ is true iff $\forall x\ \neg Q(x)$ is true.
- It follows that $\neg\exists x Q(x) \equiv \forall x\ \neg Q(x)$.

The rules for negations for quantifiers are called **De Morgan's laws for quantifiers**. These rules are summarized in following Table.

| *Negation* | *Equivalent Statement* | *When Is Negation True?* | *When False?* |
|---|---|---|---|
| $\neg\exists x P(x)$ | $\forall x \neg P(x)$ | For every $x$, $P(x)$ is false. | There is an $x$ for which $P(x)$ is true. |
| $\neg\forall x P(x)$ | $\exists x \neg P(x)$ | There is an $x$ for which $P(x)$ is false. | $P(x)$ is true for every $x$. |

**Example 13:** What are the negations of the statements "

1. "There is an honest politician"
2. "All Americans eat cheeseburgers"?

***Solution:***

1. Let $H(x)$ denote "$x$ is honest."
   Then the statement

   *"There is an honest politician"*

   is represented by

   $\exists x H(x)$,

   where the domain consists of all politicians. The negation of this statement is

   $\neg\exists x H(x)$,

   which is equivalent to

   $\forall x\ \neg H(x)$.

   This negation can be expressed as

   "*Every politician is dishonest*."

2. Let $C(x)$ denote "$x$ eats cheeseburgers."
   Then the statement

   "*All Americans eat cheeseburgers*"

   is represented by

   $\forall x C(x)$,

   where the domain consists of all Americans.

The negation of this statement is

$$\neg\forall x C(x),$$

which is equivalent to

$$\exists x\, \neg C(x).$$

This negation can be expressed in several different ways, including

*"Some American does not eat cheeseburgers"*

and

*"There is an American who does not eat cheeseburgers."*

**Example 14:** What are the negations of the statements

1. $\forall x(x^2 > x)$
2. $\exists x(x^2 = 2)$

***Solution:***

1. The negation of $\forall x(x^2 > x)$ is the statement

$$\neg\forall x(x^2 > x),$$

which is equivalent to

$$\exists x\, \neg(x^2 > x).$$

This can be rewritten as

$$\exists x(x^2 \leq x).$$

2. The negation of $\exists x(x^2 = 2)$ is the statement

$$\neg\exists x(x^2 = 2),$$

which is equivalent to

$$\forall x\, \neg(x^2 = 2).$$

This can be rewritten as

$$\forall x(x^2 \neq 2).$$

The truth values of these statements depend on the domain.

**Example 15:** Show that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent.

***Solution:***

By De Morgan's law for universal quantifiers, we know that

$$\neg\forall x(P(x) \rightarrow Q(x)) \equiv \exists x(\neg(P(x) \rightarrow Q(x))).$$

We know that

$$\neg(P(x) \rightarrow Q(x)) \equiv P(x) \wedge \neg Q(x) \text{ for every } x.$$