It follows that

$$\neg\forall x(P(x) \rightarrow Q(x)) \equiv \exists x(P(x) \wedge \neg Q(x))$$

Translating from English into Logical Expressions

Translating from English to logical expressions becomes even more complex when quantifiers are needed.

Example 16: Express the statement

"Every student in this class has studied calculus"

using predicates and quantifiers.

Solution:

- **Assume:** the domain of discourse are all students in this class. we rewrite the statement to obtain:

  "For every student in this class, that student has studied calculus."

  Next, we introduce a variable x so that our statement becomes

  "For every student x in this class, x has studied calculus."

  Then $C(x)$ is the statement "*x has studied calculus*."

- **Translation:**

  $\forall x C(x).$

- **Assume:** the domain of discourse consists of all people. we will need to express our statement as

  "For every person x, if person x is a student in this class, then x has studied calculus."

  If $S(x)$ represents the statement that person x is in this class.

- **Translation:**

  $\forall x(S(x) \rightarrow C(x)).$

[*Caution!* Our statement *cannot* be expressed as $\forall x(S(x) \wedge C(x))$!]

Assume: we are interested in the background of people in subjects besides calculus.

Then we would replace *C(x)* by *Q(x,* calculus) in both approaches to obtain

$\forall x Q(x, calculus)$
or
$\forall x(S(x) \rightarrow Q(x, calculus))$.

where *Q(x, y) is* two-variable quantifier for the statement "student *x* has studied subject *y*."

Example 17: Express the statements:

1. "Some student in this class has visited Mexico"
2. "Every student in this class has visited either Canada or Mexico"

using predicates and quantifiers.

Solution:

1. The statement "Some student in this class has visited Mexico"
means that

"There is a student in this class with the property that the student has visited Mexico."

When introducing a variable *x* the statement becomes

"There is a student x in this class having the property that x has visited Mexico."

Let *M(x)*: "*x* has visited Mexico."

- **Assume:** the domain for *x* consists of the students in this class.
- **Translation:** we rewrite the statement to obtain:

$\exists x M(x)$.

- **Assume:** the domain of discourse consists of all people. The statement can be expressed as:

"There is a person x having the properties that x is a student in this class and x has visited Mexico."

Let *S(x)* be "*x* is a student in this class."
Our solution becomes $\exists x(S(x) \wedge M(x))$

2. the second statement can be expressed as

"For every x in this class, x has the property that x has visited Mexico or x has visited Canada."

Let *C(x)* be "*x* has visited Canada."

- **Assume:** the domain for *x* consists of the students in this class.

Translation: this second statement can be expressed as:

$$\forall x(C(x) \lor M(x)).$$

- **Assume:** the domain of discourse consists of all people. The statement can be expressed as:

  "For every person x, if x is a student in this class, then x has visited Mexico or x has visited Canada."

In this case, the statement can be expressed as

$$\forall x(S(x) \rightarrow (C(x) \lor M(x))).$$

- We can use, V *(x,* Mexico) and *V (x,* Canada) as the same meaning as *M(x)* and *C(x)*,

where *V (x, y) is* a two-place predicate represent *"x has visited country y."*

Additional examples

Write the following informal statements in a formal language:

1. "All Ptuk students are smart."
2. "Someone at Ptuk is smart."
3. "All triangles have three sides"
4. "No dogs have wings"
5. "Some programs are structured"
6. "If a real number is an integer, then it is a rational number"
7. "All bytes have eight bits"
8. "No fire trucks are green"
9. "People who like Homos are smart"
10. "If a number is an integer, then it is a rational number"
11. "All Palestinian like Jerusalem "

Solution:

1. **Sentence***:* *"All Ptuk students are smart".*

- **Assume:** the domain of discourse of x are Ptuk students
- **Translation:**
  - $\forall x\ Smart(x)$
- **Assume:** the universe of discourse are students (all students):
- **Translation:**
  - $\forall x\ at(x,Ptuk) \rightarrow Smart(x)$

- **Assume:** the universe of discourse are people:
- **Translation:**

  - $\forall x\ student(x) \wedge at(x,Ptuk) \rightarrow Smart(x)$

2. **Sentence:** " Someone at Ptuk is smart."

- **Assume:** the domain of discourse are all Ptuk students
- **Translation:**

$\exists x\ \ Smart(x)$

- **Assume:** the universe of discourse are people:

$\exists x\ \ at(x,Ptuk) \rightarrow Smart(x)$

3. **Sentence:** **"**All triangles have three sides"
   - **Assume:** the domain of discourse are all triangles
   - **Translation:**

     $\forall x$ ThreeSided(x) *or* $\forall x \in Triangle \cdot$ ThreeSided(x)

4. **Sentence:** "No dogs have wings"
   - **Translation:**

     $\forall d \in Dog \cdot \neg\ HasWings(d)$
5. **Sentence:** **"**Some programs are structured"
   - **Translation:**

     $\exists p \in Program \cdot structured(p)$
6. **Sentence:** "If a real number is an integer, then it is a rational number"
   - **Translation:**

     $\forall n \in RealNumber \cdot Integer(n) \rightarrow Rational(n)$
7. "All bytes have eight bits"

   $\forall b \in$ Byte $\cdot$ EightBits(b)
8. "No fire trucks are green"

   $\forall t \in$ FireTruck $\cdot \neg$ Green(t)

9. "People who like Homos are smart"

$\forall x \in \text{Person} \cdot \text{Like}(x, \textit{Homos}) \rightarrow \text{Smart}(x)$

$\forall x \in \text{Person} \cdot \text{LikeHomos}(x) \rightarrow \text{Smart}(x)$

10. "If a number is an integer, then it is a rational number"

$\forall n \;\; \text{Integer}(n) \rightarrow \text{Rational}(n)$

11. "All Palestinian like Jerusalem "

- **Assume:** the domain of discourse are all Palestinian
- **Translation:**

  $\forall p \in \textit{Palestinian} \;.\; \textit{Likes}(p, \textit{Jerusalem})$

- **Assume:** the domain of discourse are all person
- **Translation:**

  $\forall p \;.\; \textit{Palestinian}(p) \wedge \textit{Likes}(p, \textit{Jerusalem})$

1.5 Nested quantifiers

More than one quantifier may be necessary to capture the meaning of a statement in the predicate logic.

For Examples: Assume that the domain for the variables x and y consists of all real numbers.

- The statement

$$\forall x \forall y (x + y = y + x)$$

says that $x + y = y + x$ for all real numbers x and y. This is the **commutative law** for addition of real numbers.

- Likewise, the statement

$$\forall x \exists y (x + y = 0)$$

says that for every real number x there is a real number y such that $x + y = 0$. This states that every real number has an **additive inverse**.

- Similarly, the statement

$$\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$$

is **the associative law** for addition of real numbers.

Example 1: Translate this statement into a logical expression

"Every real number has its corresponding negative."

Solution

• **Translation:**

– **Assume:**

• a real number is denoted as x and its negative as y

• A predicate P(x,y) denotes: "x + y =0"

• Then we can write:

$$\forall x \exists y P(x,y)$$

Example 2: Translate this statement into a logical expression

" There is a person who loves everybody. ""

• **Translation:**

– **Assume:**

• Variables x and y denote people

• A predicate L(x,y) denotes: "x loves y"

• Then we can write in the predicate logic:

$$\exists x\ \forall y\ L(x,y)$$

Example 3: Translate into English the statement

$$\forall x \forall y((x > 0) \land (y < 0) \rightarrow (xy < 0)),$$

where the domain for both variables consists of all real numbers.

Solution:

- This statement says that for every real number x and for every real number y, if $x > 0$ and $y < 0$, then $xy < 0$.
- That is, this statement says that for real numbers x and y, if x is positive and y is negative, then xy is negative.
- In the summary this statement says that

  "The product of a positive real number and a negative real number is always a negative real number."

Order of quantifiers

- **The order of nested quantifiers matters if quantifiers are of different type**
  - **$\forall x \exists y$ L(x,y)** is not the same as **$\exists y\ \forall x$ L(x,y)**

For Example:

• **Assume:**

  - L(x,y) denotes "x loves y"

- Then: $\forall x \exists y\, L(x,y)$
- Translates to: Everybody loves somebody.
- And: $\exists y\, \forall x\, L(x,y)$
- Translates to: There is someone who is loved by everyone.

⇒The meaning of the two is different.

- **The order of nested quantifiers does not matter if quantifiers are of the same type**

For Example:

" For all x and y, if x is a parent of y then y is a child of x"

- **Assume:**
  - Parent(x,y) denotes "x is a parent of y"
  - Child(x,y) denotes "x is a child of y"
- Two equivalent ways to represent the statement:
  - $\forall x\, \forall y\, \text{Parent}(x,y) \rightarrow \text{Child}(y,x)$
  - $\forall y\, \forall x\, \text{Parent}(x,y) \rightarrow \text{Child}(y,x)$

Example 4 : Let $P(x, y)$ be the statement "$x + y = y + x$." What are the truth values of the quantifications $\forall x \forall y P(x, y)$ and $\forall y \forall x P(x, y)$ where the domain for all variables consists of all real numbers?

Solution:

The quantification

$$\forall x \forall y P(x, y)$$

denotes the proposition

"For all real numbers *x*, for all real numbers *y*, $x + y = y + x$."

Because $P(x, y)$ is true for all real numbers x and y (it is the commutative law for addition, which is an axiom for the real numbers , the proposition

$\forall x \forall y P(x, y)$ is **true**.

Note that the statement

$\forall y \forall x P(x, y)$ says

"For all real numbers y, for all real numbers x, $x + y = y + x$."

This has the same meaning as the statement

"For all real numbers x, for all real numbers y, $x + y = y + x$."

That is, $\forall x \forall y P(x, y)$ and $\forall y \forall x P(x, y)$ have the same meaning, and both are true.

Example 5 : Let $Q(x, y)$ denote "$x + y = 0$." What are the truth values of the quantifications $\exists y\forall xQ(x, y)$ and $\forall x\exists yQ(x, y)$, where the domain for all variables consists of all real numbers?

Solution:

The quantification $\exists y\forall xQ(x, y)$ denotes the proposition

"There is a real number y such that for every real number x, $Q(x, y)$."

I.e., no matter what value of y is chosen, there is only one value of x for which $x + y = 0$.

Because there is no real number y such that $x + y = 0$ for all real numbers x, the statement $\exists y\forall xQ(x, y)$ is **false.**

The quantification $\forall x\exists yQ(x, y)$

denotes the proposition

"For every real number x there is a real number y such that $Q(x, y)$."

Given a real number x, there is a real number y such that $x + y = 0$; namely, $y = -x$.

Hence, the statement $\forall x\exists yQ(x, y)$ is **true**.

Translating Mathematical Statements into Statements Involving Nested Quantifiers

Example 6 : Translate the statement "The sum of two positive integers is always positive" into a logical expression.

Solution:

- **Assume:**
  - the domain for both variables consists of all integers.
  - we first rewrite it so that the implied quantifiers as

    "For every two integers, if these integers are both positive, then the sum of these integers is positive."
  - Next, we introduce the variables x and y to obtain

    "For all positive integers x and y, $x + y$ is positive."

- **Translation:**
  - we can express this statement as

$$\forall x\forall y((x > 0) \wedge (y > 0) \rightarrow (x + y > 0)),$$

- **Assume:**

– the domain for both variables consists of all positive integers.
– Then the statement becomes
"For every two positive integers, the sum of these integers is positive."
– We can express this as

$$\forall x \forall y(x + y > 0),$$

However, we avoided sentences whose translation into logical expressions required the use of nested quantifiers.

Example 7: Express the statement
"If a person is female and is a parent, then this person is someone's mother"
as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.

Solution:

- **Assume:**

– the domain consisting of all people.
– variables x and y denote people
– We introduce the propositional functions
$F(x)$ to represent "x is female,"
$P(x)$ to represent "x is a parent,"
and
$M(x, y)$ to represent "x is the mother of y."
– The statement can be expressed as
"For every person x, if person x is female and person x is a parent, then there exists a person y such that person x is the mother of person y."
– Then the original statement can represented as

$$\forall x((F(x) \wedge P(x)) \rightarrow \exists y M(x, y)).$$

– We can move $\exists y$ to the left so that it appears just after $\forall x$, because y does not appear in $F(x) \wedge P(x)$.
– We obtain the logically equivalent expression

$$\forall x \exists y((F(x) \wedge P(x)) \rightarrow M(x, y)).$$

Translation exercise

Suppose:
– Variables x,y denote people
– L(x,y) denotes "x loves y".

Translate:

• Everybody loves Raymond. $\forall x\ L(x, Raymond)$

- Everybody loves somebody. $\forall x \exists y\, L(x,y)$
- There is somebody whom everybody loves. $\exists y \forall x\, L(x,y)$
- There is somebody who Raymond doesn't love. $\exists y\, \neg L(Raymond,y)$
- There is somebody whom no one loves. $\exists y\, \forall x\, \neg L(x,y)$

Translating from Nested Quantifiers into English

Example 8: Translate the statement $\forall x(C(x) \vee \exists y(C(y) \wedge F(x, y)))$ into English, where
$C(x)$ is "x has a computer,"
$F(x, y)$ is "x and y are friends,"
and the domain for both x and y consists of all students in your school.

Solution:
The statement says that for every student x in your school, x has a computer or there is a student y such that y has a computer and x and y are friends.
In other words,

"every student in your school has a computer or has a friend who has a computer."

Negating Nested Quantifiers

Example 9: Express the negation of the statement $\forall x \exists y(xy = 1)$ so that no negation precedes a quantifier.
Solution:
We find that

$$\forall x \exists y(xy = 1)$$

is equivalent to

$$\exists x\, \neg \exists y(xy = 1),$$

which is equivalent to

$$\exists x \forall y\, \neg(xy = 1).$$

Because $\neg(xy = 1)$ can be expressed more simply as $xy \neq 1$, we conclude that our negated statement can be expressed as

$$\exists x \forall y(xy \neq 1).$$

1.6 Rules of Inference

Valid Arguments in Propositional Logic

Consider the following argument involving propositions:
"If you have a current password, then you can log onto the network."
"You have a current password."
Therefore,
"You can log onto the network."

We would like to determine whether the conclusion *"You can log onto the network"* must be true when the premises "If you have a current password, then you can log onto the network" and "You have a current password" are both true.

Use **p** to represent "You have a current password"
and
q to represent "You can log onto the network."
Then, the argument has the form

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

where ∴ is the symbol that denotes "therefore."

We want to show that the statement $((p \to q) \land p) \to q$ is a tautology.

Definition:

An argument in propositional logic is a sequence of propositions. All but the final proposition in the argument are called premises and the final proposition is called the conclusion.

An argument is valid if the truth of all its premises implies that the conclusion is true