– Computer representation: B = 10001

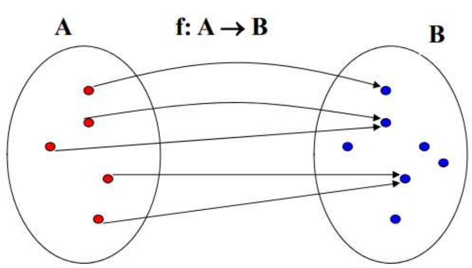
Solution:

- A = 01001
- B = 10001
- The **union** is modeled with a bitwise **or**
- $A \vee B = 11001$
- The **intersection** is modeled with a bitwise **and**
- $A \wedge B = 00001$
- The **complement** is modeled with a bitwise **negation**
- $\overline{A} = 10110$

2.3 Functions

Definition:

Let A and B be two sets. A function from A to B, denoted $f : A \to B$, is an assignment of exactly one element of B to each element of A. We write f(a) = b to denote the assignment of b to an element a of A by the function f.



Representing functions

1. Explicitly state the assignments in between elements of the two sets

2. Compactly by a formula. (using 'standard' functions)

Example 1 : Let A = {1,2,3} and B = {a,b,c}

Assume f is defined as:

- 1 → c
- 2 → a
- 3 → c
- Is f a function ?
- **Yes**. since f(1)=c, f(2)=a, f(3)=c. each element of A is assigned an element from B.

Example 2: Let A = {1,2,3} and B = {a,b,c}

Assume g is defined as

- 1 → c
- 1 → b
- 2 → a
- 3 → c
- Is g a function?
- **No.** g(1) = is assigned both c and b.

Example 3: Let A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}

Define h: A → B as:

h(x) = x mod 3. (the result is the remainder after the division by 3)

Assignments:

- 0 → 0
- 3 → 0
- 6 → 0
- 1 → 1
- 4 → 1
- 7 → 0

• $2 \rightarrow 2$ • $5 \rightarrow 2$ …..

Important sets

Definition:

Let f be a function from A to B. We say that A is the domain of f and B is the codomain of f.

• If f(a) = b, b is the image of a and a is a pre-image of b.

• The range of f is the set of all images of elements of A. Also, if f is a function from A to B, we say f maps A to B.

Example 4: Let A = {1,2,3} and B = {a,b,c}

Assume f is defined as: $1 \rightarrow c$, $2 \rightarrow a$, $3 \rightarrow c$

• What is the image of 1?

• $1 \rightarrow c$ c is the image of 1

• What is the pre-image of a?

• $2 \rightarrow a$ 2 is a pre-image of a.

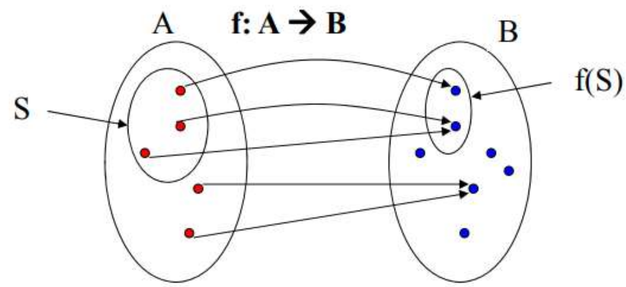
• Domain of f ? {1,2,3}

• Codomain of f ? {a,b,c}

• Range of f ? {a,c}

Definition: **Image of a subset**

Let f be a function from set A to set B and let S be a subset of A. The image of S is a subset of B that consists of the images of the elements of S. We denote the image of S by f(S), so that $f(S) = \{ f(s) \mid s \in S \}$.

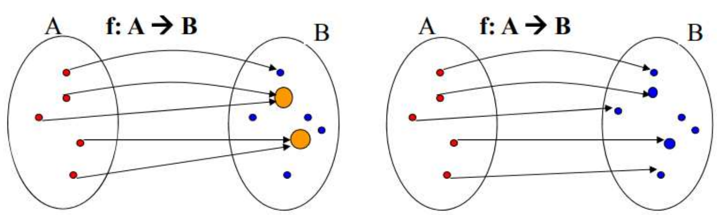


Example 5: Let A = {1,2,3} and B = {a,b,c} and f: 1 → c, 2 → a, 3 → c

• Let S = {1,3} then image f(S) = {c}.

Definition: Injective function

function f is said to be one-to-one, or injective, if and only if f(x) = f(y) implies x = y for all x, y in the domain of f. A function is said to be an injection if it is one-to-one. Alternate: A function is one-to-one if and only if f(x) ≠ f(y), whenever x ≠ y. This is the contrapositive of the definition.



Not injective **Injective function**

Example 6: Let A = {1,2,3} and B = {a,b,c}

Define f as

- 1 → c
- 2 → a
- 3 → c

- Is f one to one?
- **No,** it is not one-to-one since f(1) = f(3) = c, and 1 ≠ 3.

Example 7: Let g : Z → Z, where g(x) = 2x - 1.

• Is g is one-to-one (why?)

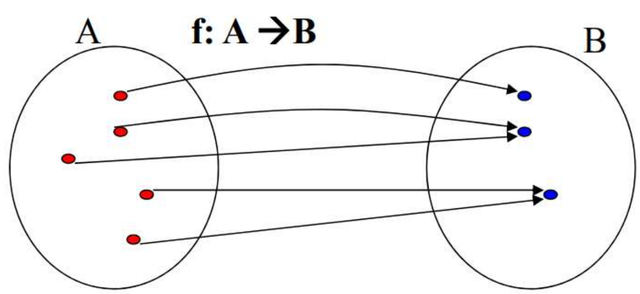
• **Yes.**

• Suppose g(a) = g(b), i.e., 2a - 1 = 2b - 1 => 2a = 2b => a = b.

Surjective function

Definition:

A function f from A to B is called onto, or surjective, if and only if for every b ∈ B there is an element a ∈ A such that f(a) = b. Alternative: all co-domain elements are covered



Example 8: Let A = {1,2,3} and B = {a,b,c}

Define f as • 1 → c • 2 → a • 3 → c

• Is f an onto?

• **No.** f is not onto, since b ∈ B has no pre-image.

Example 9: A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}

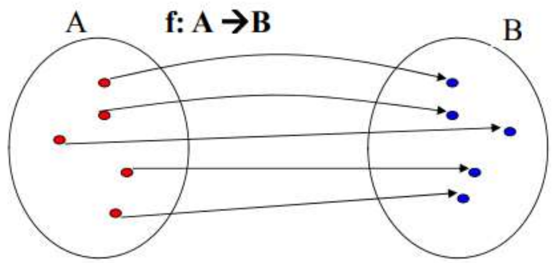
Define h: A → B as h(x) = x mod 3.

• Is h an onto function?

• **Yes.** h is onto since a pre-image of 0 is 6, a pre-image of 1 is 4, a pre-image of 2 is 8.

Definition: Bijective functions

A function f is called a bijection if it is both one-to-one and onto.



Example 10: Let A = {1,2,3} and B = {a,b,c}

Define f as • 1 → c • 2 → a • 3 → b

• Is f is a bijection?

- **Yes**. It is both one-to-one and onto.
- **Note:** Let f be a function from a set A to itself, where A is finite. f is one-to-one if and only if f is onto.
- This is not true for A an infinite set. Define f : Z → Z, where f(z) = 2 * z.
- f is one-to-one but not onto (3 has no pre-image).

Example 11: Define g : W → W (whole numbers), where g(n) = [n/2] (floor function).

• 0 → [0/2] = [0] = 0

• 1 → [1/2] = [1/2] = 0

• 2 → [2/2] = [1] = 1

• 3 → [3/2] = [3/2] = 1

• ...

• Is g a bijection?

– **No.** g is onto but not 1-1 (g(0) = g(1) = 0 however 0 ≠ 1.

Theorem: Let f be a function f: A →A from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Proof:

- ⇒**A is finite** and **f is one-to-one (injective)**
- Is f an onto function (surjection)?
- **Yes.** Every element points to exactly one element. Injection assures they are different. So we have |A| different elements A points to.
- Since f: A → A the co-domain is covered thus the function is also a surjection (and a bijection)
- **⇐ A is finite and f is an onto function**
- Is the function one-to-one?

• Yes. Every element maps to exactly one element and all elements in A are covered. Thus the mapping must be one-to-one

Please note the above is not true when A is an infinite set.

Example 12 : f : Z → Z, where f(z) = 2 * z.

– f is one-to-one but not onto.

• 1 → 2

• 2 → 4

• 3 → 6

– 3 has no pre-image.

Functions on real numbers

Definition: Let f_1 and f_2 be functions from A to R (reals). Then $f_1 + f_2$ and

$f_1 * f_2$ are also functions from A to R defined by

- $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
- $(f_1 * f_2)(x) = f_1(x) * f_2(x)$.

Examples:

- Assume • $f_1(x) = x - 1$
- $f_2(x) = x^3 + 1$ then
- $(f_1 + f_2)(x) = x^3 + x$
- $(f_1 * f_2)(x) = x^4 - x^3 + x - 1$.

Increasing and decreasing functions

Definition: A function f whose domain and codomain are subsets of real numbers is **strictly increasing** if $f(x) > f(y)$ whenever $x > y$ and x and y are in the domain of f. Similarly, f is called **strictly decreasing** if $f(x) < f(y)$ whenever $x > y$ and x and y are in the domain of f.

Example 13: Let $g : R \rightarrow R$, where $g(x) = 2x - 1$. Is it increasing ?

Proof . For $x>y$ holds $2x > 2y$ and subsequently $2x-1 > 2y-1$

Thus g is strictly increasing.

Note: Strictly increasing and strictly decreasing functions are oneto-one. **Why?**

One-to-one function: A function is one-to-one if and only if $f(x) \neq f(y)$, whenever $x \neq y$

Identity function

Definition: Let A be a set. The **identity function** on A is the function

$i_A: A \rightarrow A$ where $i_A(x) = x$.

Example 14: Let $A = \{1,2,3\}$ **Then**:

- $i_A(1) = 1$

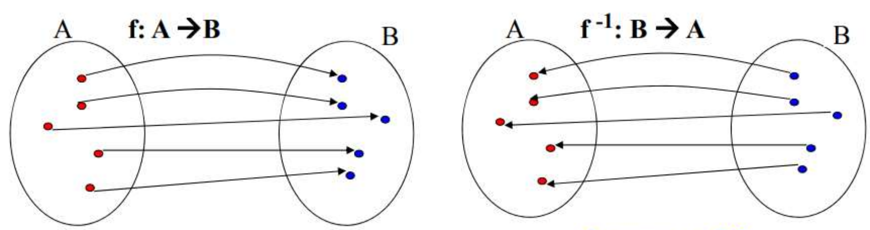
• $i_A(2) = 2$

• $i_A(3) = 3$.

Inverse functions

Definition:

Let f be a bijection from set A to set B. The **inverse** function of f is the function that assigns to an element b from B the unique element a in A such that f(a) = b. The inverse function of f is denoted by $\mathbf{f^{-1}}$. Hence, $f^{-1}(b) = a$, when f(a) = b. If the inverse function of f exists, f is called **invertible**.

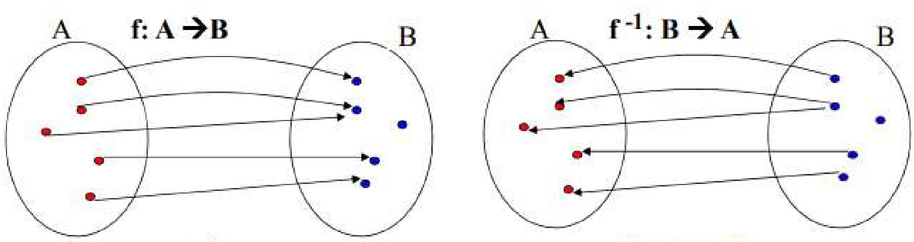


f is bijective **Inverse of f**

Note: if f is not a bijection then it is not possible to define the inverse function of f. **Why?**

Assume f is not one-to-one:

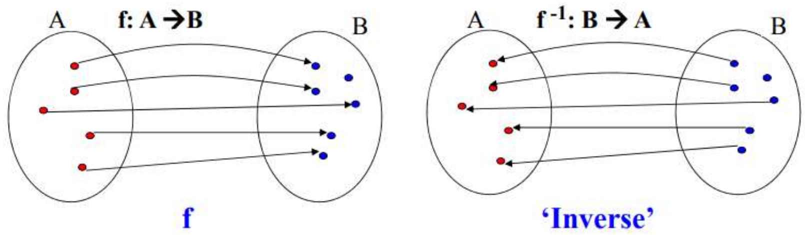
Inverse is not a function. One element of B is mapped to two different elements.



f **'Inverse'**

Assume f is not onto:

Inverse is not a function. One element of B is not assigned any value in B.



Example 15: Let A = {1,2,3} and i_A be the identity function

- $i_A(1) = 1$ $\quad$ $i_A^{-1}(1) = 1$
- $i_A(2) = 2$ $\quad$ $i_A^{-1}(2) = 2$
- $i_A(3) = 3$ $\quad$ $i_A^{-1}(3) = 3$
- Therefore, the inverse function of i_A is i_A.

Example 16: Let $g : R \to R$, where $g(x) = 2x - 1$.

- **What is the inverse function g^{-1} ?**

Approach to determine the inverse:

$y = 2x - 1 \Rightarrow y + 1 = 2x \Rightarrow (y+1)/2 = x$

- Define $g^{-1}(y) = x = (y+1)/2$

Test the correctness of inverse:

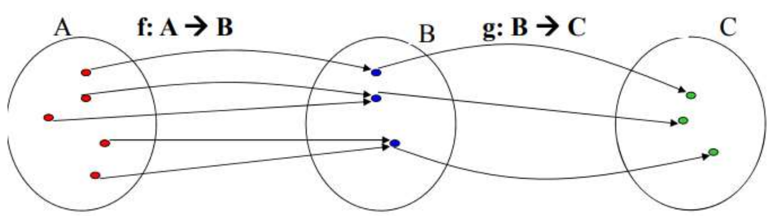
- $g(3) = 2*3 - 1 = 5$
- $g^{-1}(5) = (5+1)/2 = 3$
- $g(10) = 2*10 - 1 = 19$
- $g^{-1}(19) = (19+1)/2 = 10$.

Composition of functions

Definition:

Let f be a function from set A to set B and let g be a function from set B to set C. **The composition of the functions g and f**, denoted by g ϒ f is defined by

- **(g o f)(a) = g(f(a)).**



Example 17: : Let A = {1,2,3} and B = {a,b,c,d}

g : A → A,	f: A → B
1 → 3	1 → b
2 → 1	2 → a
3 → 2	3 → d

f o g : A → B:

- 1 → d
- 2 → b
- 3 → a

Example 18: Let f and g be two functions from Z to Z, where

- f(x) = 2x and g(x) = x^2.
- f o g : Z → Z

- $(f \circ g)(x) = f(g(x))$

  $= f(x^2)$

  $= 2(x^2)$

- $g \circ f : Z \to Z$
- $(g \circ f)(x) = g(f(x))$

  $= g(2x) = (2x)^2 = 4x^2$

Example 19: $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$, for all x.

- Let $f : \mathbf{R} \to \mathbf{R}$, where $f(x) = 2x - 1$ and $f^{-1}(x) = (x+1)/2$.
- $(f \circ f^{-1})(x) = f(f^{-1}(x)) = f((x+1)/2) = 2((x+1)/2) - 1 = (x+1) - 1 = x$
- $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(2x - 1) = (2x)/2 = x$

Some functions

Definition:

The **floor function** is denoted by $\lfloor x \rfloor$.

- The **ceiling function** assigns to the real number x the smallest integer that is greater than or equal to x. The ceiling function is denoted by $\lceil x \rceil$.

Other important functions:

- **Factorials:** $n! = n(n-1)$ such that $1! = 1$

2.4 Sequences and summations

Definition:

A sequence is a function from a subset of the set of integers (typically the set {0,1,2,...} or the set {1,2,3,...} to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.