Chapter 9 Relations

9.1 Relations and Their Properties

Definition: Binary relation

Let A and B be two sets. A **binary relation from A toB** is a subset of a Cartesian product A x B.

- Let R $\subseteq$ A x B means R is a set of ordered pairs of the form (a,b)where a $\in$ A and b $\in$ B.
- We use the notation **a R b to denote (a,b)** $\in$ **R** and **a R b todenote (a,b)** $\notin$ **R**. If **a R b**, we say a is related to b by R

Example 1: Let A={a,b,c} and B={1,2,3}.

- Is R={(a,1),(b,2),(c,2)} a relation from A to B? **Yes.**
- Is Q={(1,a),(2,b)} a relation from A to B? **No.**
- Is P={(a,a),(b,c),(b,a)} a relation from A to A? **Yes**

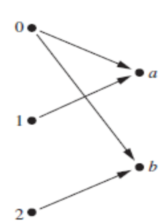
Representing binary relations

- We can graphically represent a binary relation R as follows:
  - if **a R b** then draw an arrow from a to b.

$$\mathbf{a} \rightarrow \mathbf{b}$$

Example 2:

- Let A = {0, 1, 2}, B = {a,b}
  and R = { (0,a), (0,b), (1,a), (2,b) } is a relation from A to B.
- **Graph:**



- We can represent a binary relation R by a **table** showing(marking) the ordered pairs of R.

Example 3:

- Let A = {0, 1, 2}, B = {u,v} and R = { (0,u), (0,v), (1,v), (2,u) }
- **Table:**

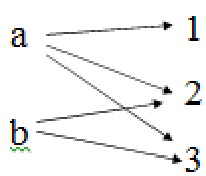
R	u	v
0	x	x
1		x
2	x	

or

R	u	v
0	1	1
1	0	1
2	1	0

Functions as Relations

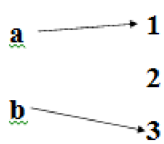
- Relations represent **one to many relationships** betweenelements in A and B.
- **For example:**



- What is the difference between a **relation and a function from A to B**?
- A function defined on sets A, B

A → B assigns to each element in the domain set A exactly one element from B.

- So it is **a special relation.**



Relation on the set

Definition:

A relation on the set A is a relation from A to itself.

Example 4:

- Let A = {1,2,3,4} and R_{div} = {(a,b)| a divides b}
- What does R_{div} consist of?

R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

R	1	2	3	4
1	x	x	x	x
2		x		x
3			x	
4				x

Example 5:

- Let A = {1,2,3,4}.
- Define a $R_{\neq}$ b if and only if a $\neq$ b.

$R_{\neq}$ ={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

R	1	2	3	4
1		x	x	x
2	x		x	x
3	x	x		x
4	x	x	x	

Theorem: The number of binary relations on a set A, where | A | = n is : 2^{n^2}

Proof

If | A | = n then the cardinality of the Cartesian product

| A x A | = n^2.

- R is a binary relation on A if R $\subseteq$ A x A (that is, R is a subsetof A x A).
- The number of subsets of a set with k elements : 2^k
- The number of subsets of A x A is : $2^{|AxA|} = 2^{n^2}$

Example 6: Let A = {1,2}

- What is A x A ?

A x A = **{(1,1),(1,2),(2,1),(2,2)}**

- **List of possible relations (subsets of A x A):**
  - ∅
  - {(1,1)} {(1,2)} {(2,1)} {(2,2)}
  - {(1,1), (1,2)} {(1,1),(2,1)}
    {(1,1),(2,2)} {(1,2),(2,1)}
    {(1,2),(2,2)} {(2,1),(2,2)}
  - {(1,1),(1,2),(2,1)}
    {(1,1),(1,2),(2,2)}
    {(1,1),(2,1),(2,2)}
    {(1,2),(2,1),(2,2)}
  - {(1,1),(1,2),(2,1),(2,2)}

  - Use formula: $\mathbf{2^4 = 16}$

Properties of relations

Definition:(reflexive relation) :

A relation R on a set A is called **reflexive** if (a,a) ∈ R for every element a ∈ A.

Example 7:

- Assume relation R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- **Is R_{div} reflexive?**
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}
- **Answer: Yes.** (1,1), (2,2), (3,3), and (4,4) ∈ R_{div} .

1 1 1 1

$$MR_{div} = \begin{matrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

- **A relation R is reflexive** if and only if MR has 1 in everyposition on its main diagonal.

Example 8:

- Relation R_{fun} on A = {1,2,3,4} defined as:
  - R_{fun} = {(1,2),(2,2),(3,3)}.
- **Is R_{fun} reflexive?**

  No. It is not reflexive since $(1,1) \notin R_{fun}$.

Definition:(irreflexive relation) :

A relation R on a set A is called **irreflexive** if **$(a,a) \notin$ R for every $a \in$ A.**

Example 9:

- Assume relation $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and onlyif $a \neq b$.
- **Is $R_{\neq}$ irreflexive?**
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**
- **Answer:** Yes. Because (1,1),(2,2),(3,3) and $(4,4) \notin R_{\neq}$
- **A relation R is irreflexive** if and only if MR has 0 in everyposition on its main diagonal.

$$MR = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

Example 10:

- R_{fun} on A = {1,2,3,4} defined as:
    - R_{fun} = {(1,2),(2,2),(3,3)}.
- **Is R_{fun} irreflexive?**
- **Answer: No**. Because (2,2) and (3,3) $\in R_{fun}$

Definition:(symmetric relation):

A relation R on a set A is called **symmetric** if $\forall$ a, b $\in$ A (a,b)$\in$ R $\rightarrow$ (b,a) $\in$ R.

Example 11:

- R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- **Is R_{div} symmetric?**
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}
- **Answer: No.** It is not symmetric since (1,2) $\in R_{div}$ but (2,1) $\notin R_{div}$.

Example 12:

- $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if a ≠ b.
- **Is $R_{\neq}$ symmetric ?**
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**
- **Answer: Yes. If** (a,b) $\in R_{\neq} \rightarrow$ (b,a) $\in R_{\neq}$

- **A relation R is symmetric** if and only if $m_{ij} = m_{ji}$ for all i,j.

$$MR = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

Example 13:

- Relation R_{fun} on A = {1,2,3,4} defined as:
    - R_{fun} = {(1,2),(2,2),(3,3)}.

- **Is R_{fun} symmetric?**
  Answer: No. For $(1,2) \in R_{fun}$ there is no $(2,1) \notin R_{fun}$

Definition:(anti-symmetric relation):

A relation on a set A iscalled **anti-symmetric** if

- $[(a,b) \in R \text{ and } (b,a) \in R] \rightarrow a = b$ where $a, b \in A$.

- **Example 14:**
- Relation R_{fun} on A = {1,2,3,4} defined as:
  - R_{fun} = {(1,2),(2,2),(3,3)}.
- **Is R_{fun} anti-symmetric?**
- **Answer: Yes.** It is anti-symmetric

$$MR_{fun} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

- A relation is **antisymmetric** if and only if $m_{ij} = 1 \rightarrow m_{ji} = 0$ for $i \neq j$.

Definition:(transitive relation):

A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

Example 15:

- R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}
- **Is R_{div} transitive?**
- **Answer:** Yes

Example 16:

- $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if a ≠ b.
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**
- **Is $R_{\neq}$ transitive?**
- **Answer: No.** It is not transitive since (1,2) ∈ R and (2,1) ∈ R but (1,1) is not an element of R.

- ## Example 17:

- Relation R_{fun} on A = {1,2,3,4} defined as:
  - R_{fun} = {(1,2),(2,2),(3,3)}.
- **Is R_{fun} transitive?**
- **Answer: Yes.** It is transitive.

Properties of relations on A:

- Reflexive
- Irreflexive
- Symmetric
- **Anti-symmetric**
- **Transitive**

Combining relations

Definition: Let A and B be sets. A **binary relation from A to B** isa subset of a Cartesian product A x B.

- Let R $\subseteq$ A x B means R is a set of ordered pairs of the form (a,b)where a $\in$ A and b $\in$ B.

Combining Relations

- **Relations are sets → combinations via set operations**
- Set operations of: **union, intersection, difference and symmetric difference.**

Example:

- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

What is:

- R1 $\cup$ R2 = {(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)}
- R1 $\cap$ R2 = {(3,u)}
- R1 - R2 = {(1,u),(2,u),(2,v)}
- R2 - R1 = {(1,v),(3,v)}