Engineering Economy

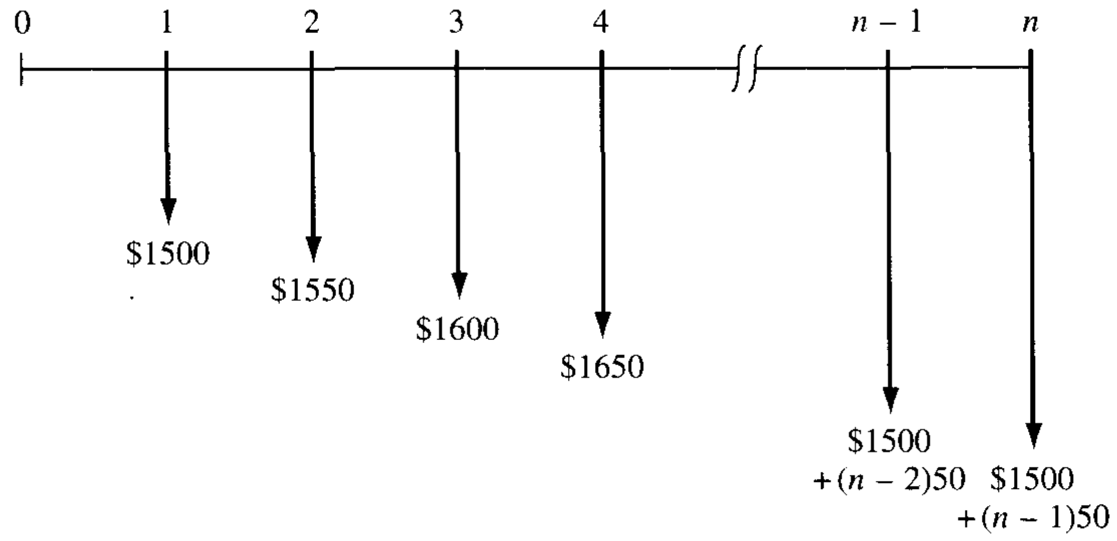
[2-3]
Time Value of Money

Arithmetic Gradient Series

Arithmetic (Linear) Gradient Series

- An *arithmetic* gradient is a cash flow series that either increases or decreases by a constant amount
- The cash flow, whether income or disbursement, changes by the same arithmetic amount each period
- The amount of the increase or decrease is the gradient (G)
- For example, if an engineer predicts that the cost of maintaining a machine will increase by $500 per year until the machine is retired, a gradient series is involved and the amount of the gradient is $500

Arithmetic (Linear) Gradient Series



- The diagram is of an arithmetic gradient series with a ***base amount*** of \$1,500 and a gradient of \$50

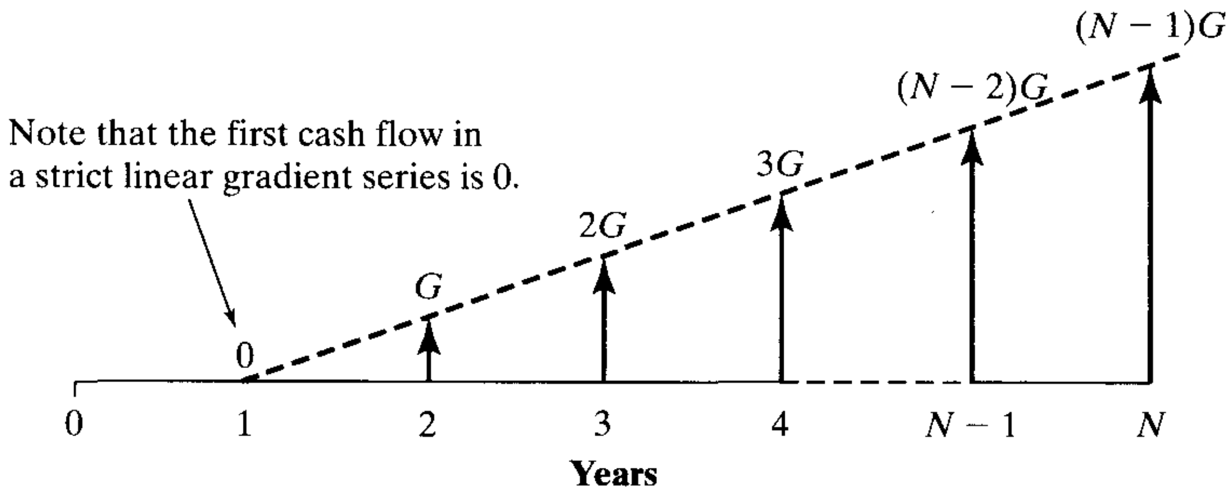
- The *origin* of the series is at the end of the first period

- G is the constant arithmetic change in the magnitude of *receipts* or *disbursements* from one time period to the next

Strict Linear Gradient Series

- The *strict* linear gradient series has the ***origin*** *at the end of the first period* with a zero value

- The *gradient* G can be either positive or negative. If $G > 0$, the series is referred to as an increasing gradient series. If $G < 0$, it is a decreasing gradient series



Arithmetic (Linear) Gradient Series
Example

- A company expects a revenue of \$80,000 in fees next year. Fees are expected to increase uniformly to a level of \$200,000 in nine years

- Determine the arithmetic gradient and construct the cash flow diagram

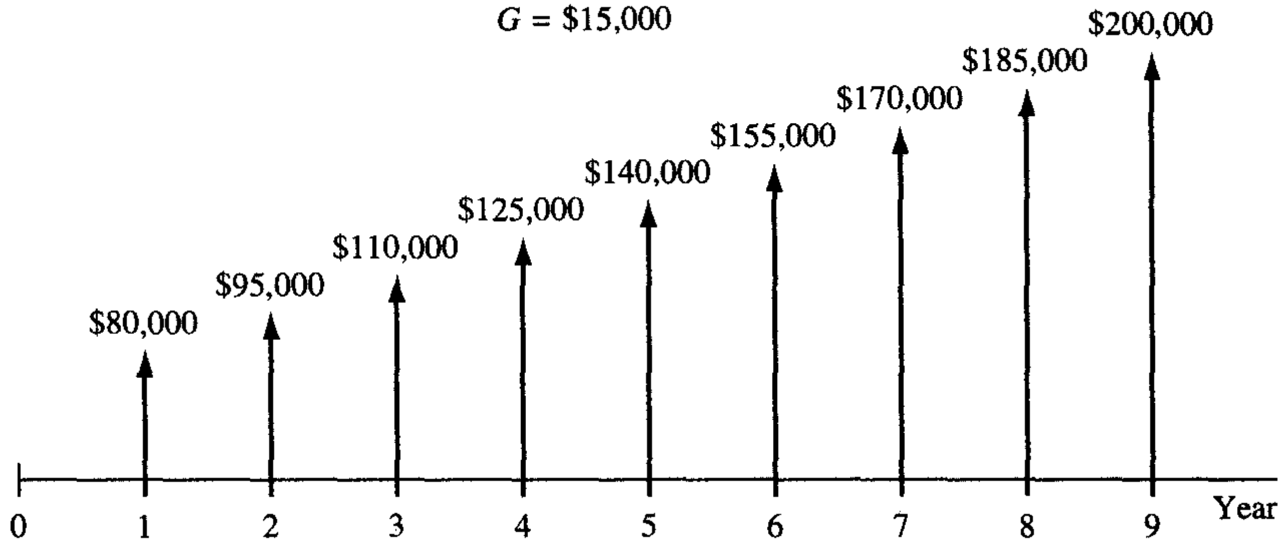
Arithmetic (Linear) Gradient Series
Example

- The cash flow in year n (CFn) may be calculated as:

  CFn = base amount + (n-1)G

- The base amount (generally A1) is $80,000 and the total revenue increase in 9 years = 200,000 – 80,000 = 120,000

- G = increase/(n-1) = 120,000/(9-1) = $15,000



Arithmetic (Linear) Gradient Series
Analysis

Three factors will be considered for arithmetic gradient ***strict*** series:

- P/G factor for present worth: G(P/G,i,n)

  Convert an arithmetic gradient G (*without the base amount*) for n years into a present worth at year 0

- A/G factor for annual series: G(A/G,i,n)

  Convert an arithmetic gradient G (*without the base amount*) for n years into an equivalent uniform series of A value

- F/G factor for future worth: G(F/G,i,n)

  Convert an arithmetic gradient G (*without the base amount*) for n years into an equivalent future value at year n

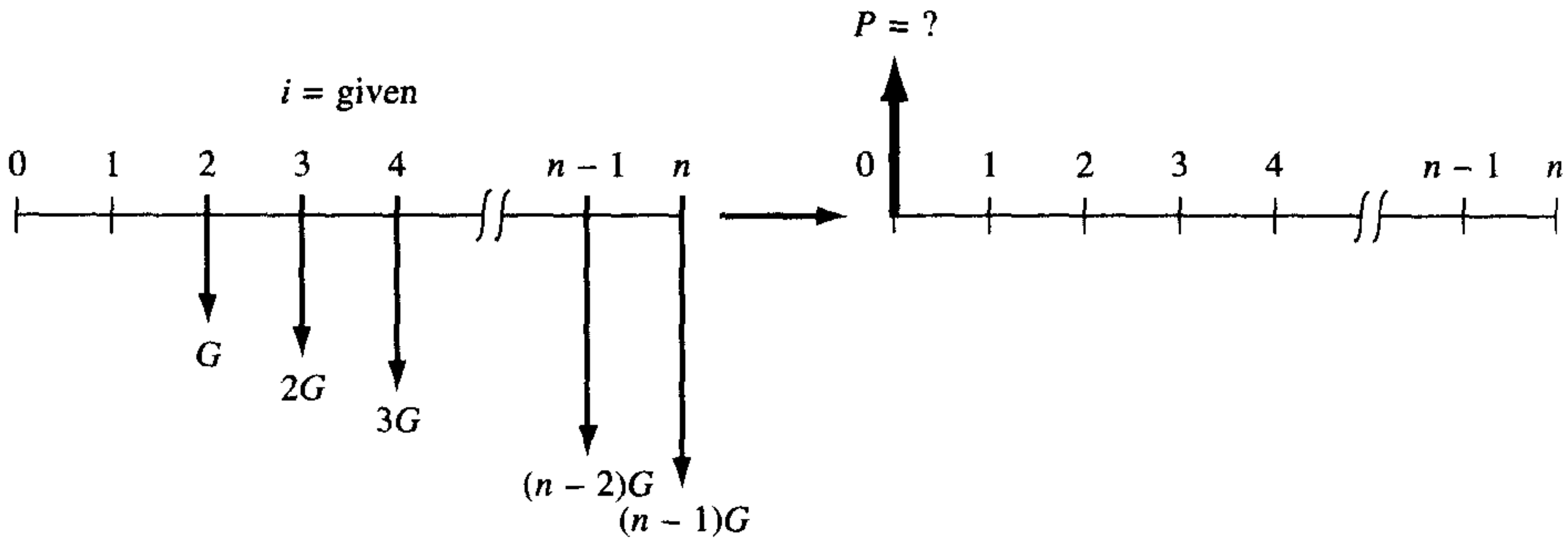
Arithmetic (Linear) Gradient Series
Present Worth Factor – P/G Factor

The Present worth factor (P/G) can be expressed in the following form:

P = G(P/G,i,n)→

gradient series present-worth factor

$$P = G\left[\frac{(1+i)^n - in - 1}{i^2(1+i)^n}\right]$$



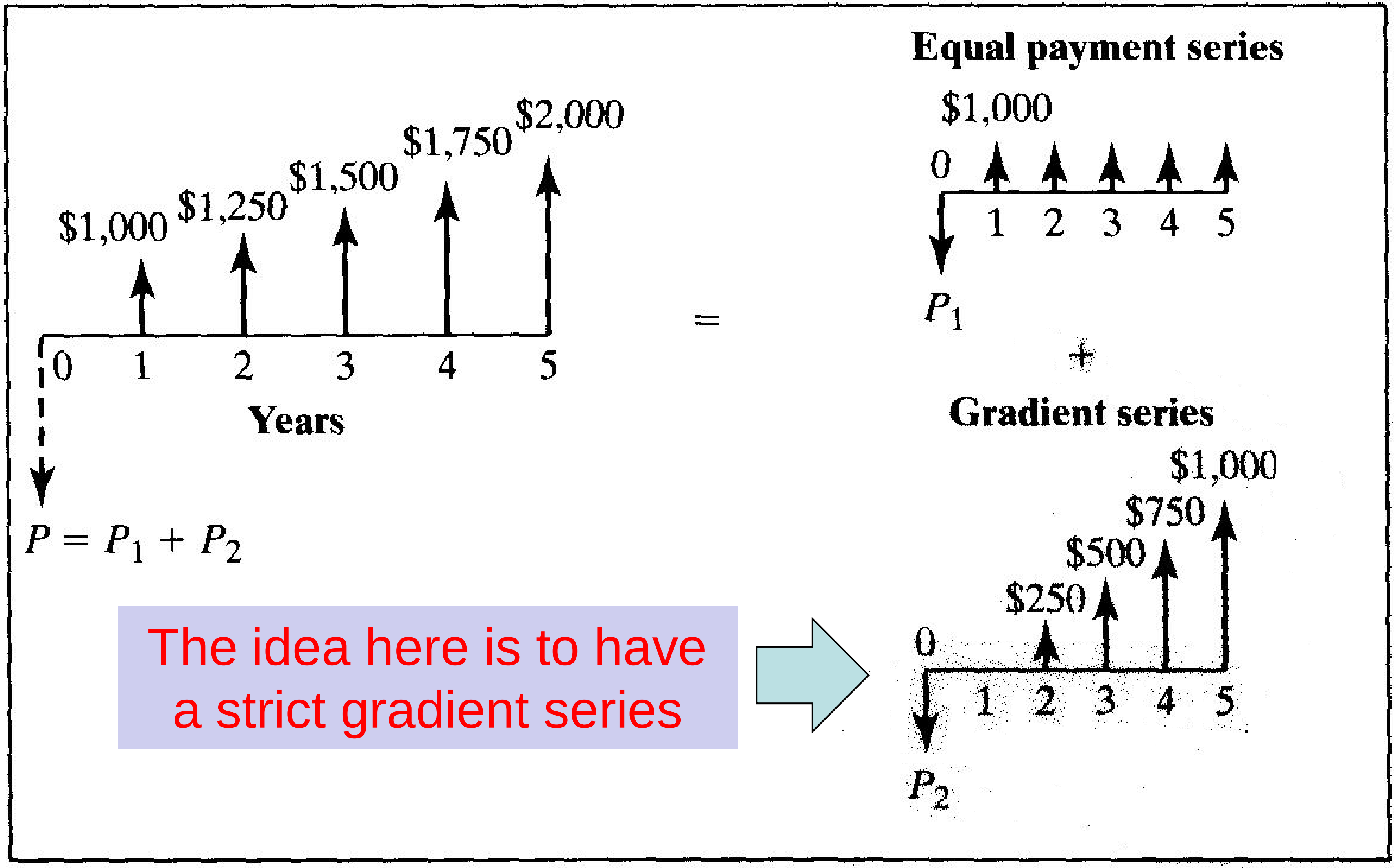
Arithmetic (Linear) Gradient Series
Present Worth Factor – Example

- A textile mill has just purchased a lift truck that has a useful life of five years. The engineer estimates that maintenance costs for the truck during the first year will be $1,000

- As the truck ages, maintenance costs are expected to increase at a rate of $250 per year over the remaining life

- Assume that the maintenance costs occur at the end of each year. The firm wants to set up a maintenance account that earns 12% annual interest. All future maintenance expenses will be paid out of this account. How much does the firm have to deposit in the account now?

Arithmetic (Linear) Gradient Series
Present Worth Factor – Example



The idea here is to have a strict gradient series

Arithmetic (Linear) Gradient Series
Present Worth Factor – Example

- We have: A1=$1,000; G=$250; i=12%; and n=5 years. Find P

- The cash flow can be broken into two components where the first is an equal uniform payment series (A1) and the second is a strict linear gradient series (G)

- P = P1 + P2

  P = A1(P/A,12%,5) + G(P/G,12%,5) =

  $1,000(3.6048) + $250(6.397) = $5,204

Arithmetic (Linear) Gradient Series

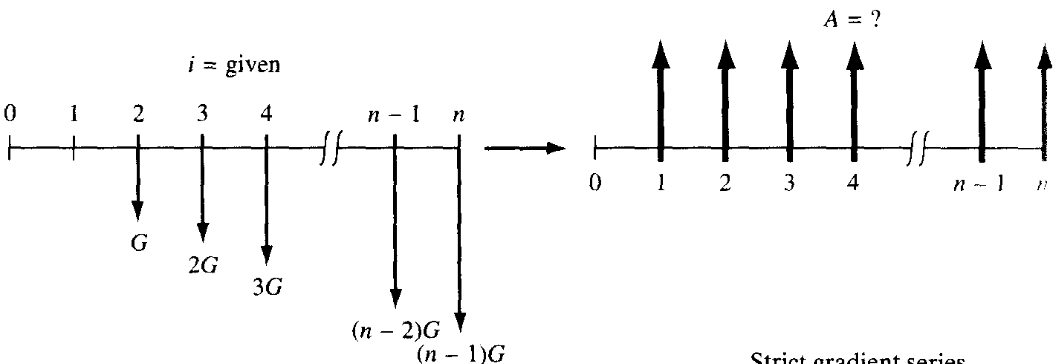
Annul Series Factor – A/G Factor

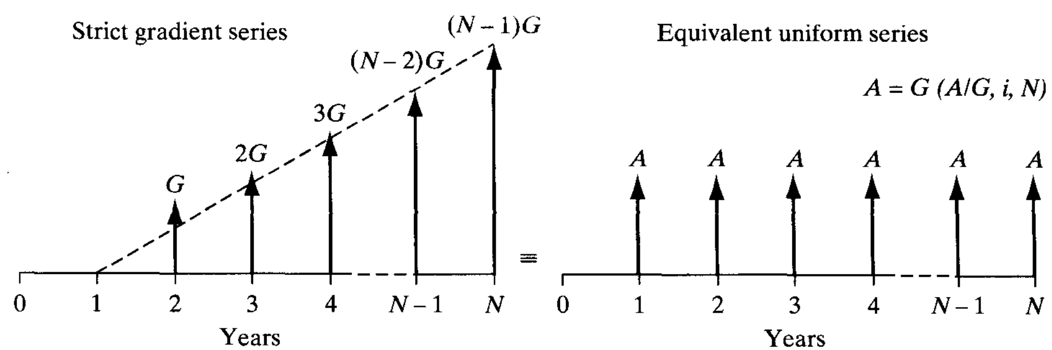
The equivalent uniform annual series (A value) for an arithmetic gradient G is found by the following formula:

A = G(A/G,i,n) →

$$A = G\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right]$$

Arithmetic-gradient uniform-series factor



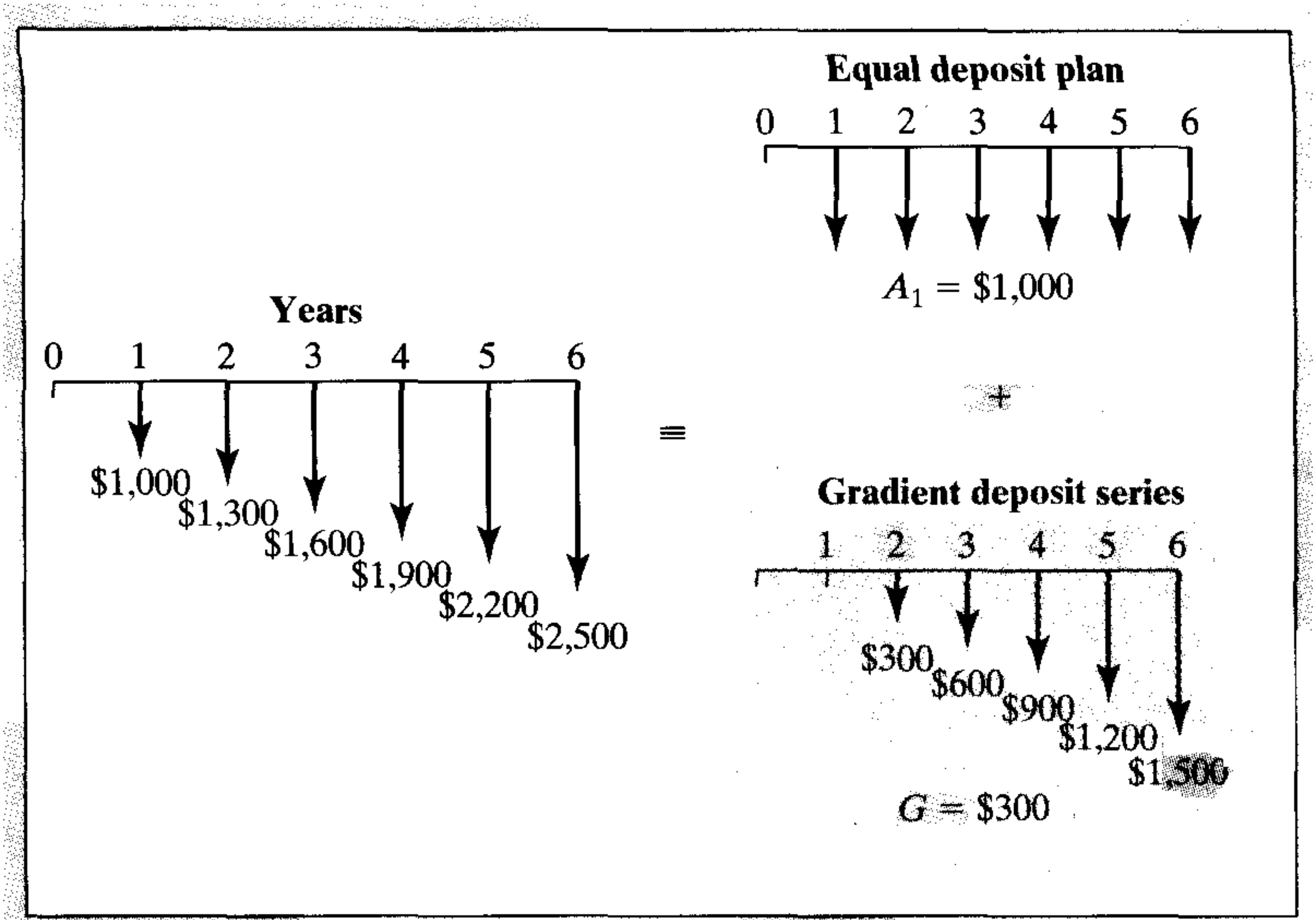


Arithmetic (Linear) Gradient Series
Annul Series Factor – Example

- You want to deposit $1,000 in your saving account at the end of the first year and increase this amount by $300 for each of the next five years

- Then what should be the size of an annual uniform deposit that yields an equal balance with the above by the end of six years if the interest rate is 10%?

Arithmetic (Linear) Gradient Series
Annul Series Factor – Example



Arithmetic (Linear) Gradient Series
Annul Series Factor – Example

- We have: A1=\$1,000; G=\$300; i=10%, and n=6. <u>Find A</u>

- We have to separate the constant portion of \$1,000 from the series leaving the gradient series of 0; 0; 300; 600; ….; 1,500

- To find the equal payment series beginning at the end of year 1 and ending at year 6 we consider:

  $A = \$1,000 + \$300(A/G,10\%,6) =$

  $\$1,000 + \$300(2.2236) = \$1,667.08$

Arithmetic (Linear) Gradient Series

Annul Series Factor – Example

Year	F	P
0	0	$0.00
1	$1,000.00	$909.09
2	$1,300.00	$1,074.38
3	$1,600.00	$1,202.10
4	$1,900.00	$1,297.73
5	$2,200.00	$1,366.03
6	$2,500.00	$1,411.18
Total	-	$7,260.51

An alternative way to solve this question is by finding the present worth of all the payments and then to convert P to a uniform series of A

A/P	0.2296074
A	$1,667.07

Arithmetic (Linear) Gradient Series
Future Worth Factor – F/G Factor

The future worth factor (F/G) can be expressed in the following form:

$$F = G(F/G,i,n) \rightarrow$$

Arithmetic-gradient future worth factor

$$F = G\left[\left(\frac{1}{i}\right)\left(\frac{(1+i)^n - 1}{i} - n\right)\right]$$

Arithmetic (Linear) Gradient Series
Future Worth Factor – Example

- Suppose that you make a series of annual deposits into a bank account that pays 10% interest. The initial deposit at the end of the first year is $1,200

- The deposit amounts decline by $200 in each of the next four years

- How much would you have immediately after the fifth deposit?

Arithmetic (Linear) Gradient Series

Future Worth Factor – Example

F = F1 – F2

F = A1(F/A,10%,5) – $200(F/G,10%,5) =

$1,200(6.105) – $200(11.051) = $5,115