Engineering Economy

[4]

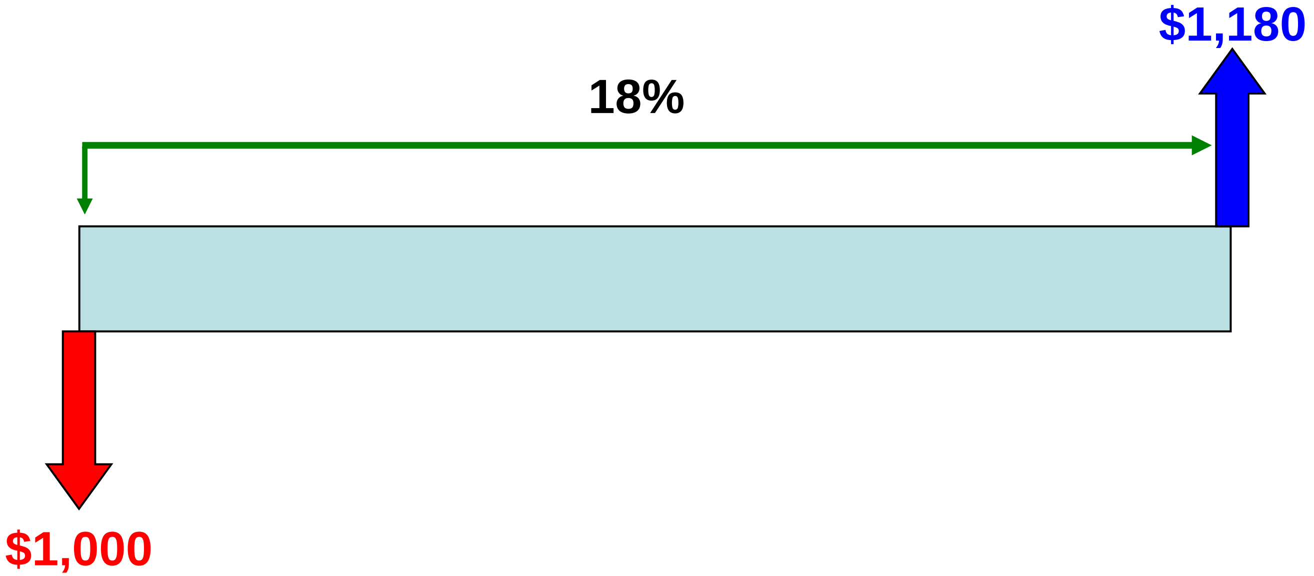
Effective Interest Rates

General Questions

- Suppose that you deposited $1,000 in a savings account at the beginning of a year where the annual interest rate is 18% (not realistic!!)
- [1] What would be the future worth?
- [2] What would be the future worth if the interest is compounded monthly?

General Questions

[1] When compounding yearly and with an interest rate of 18% per year

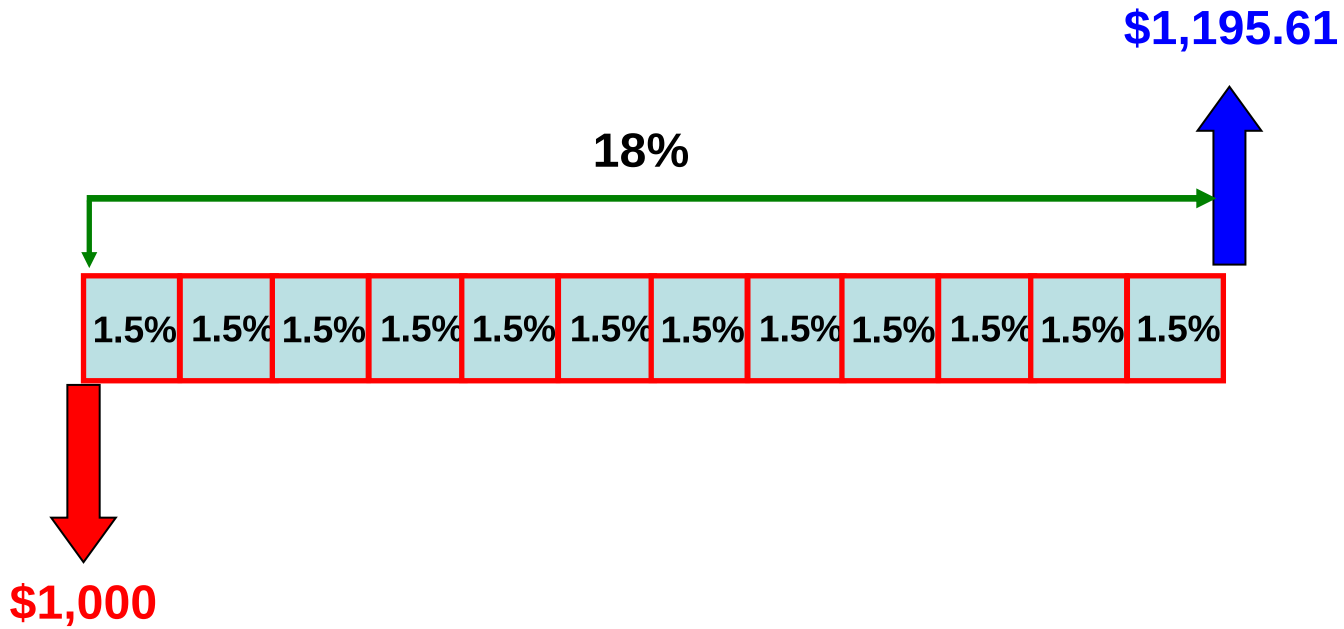


Apparently, the end-of-year amount = 1,000(1+18%) = $1,180

General Questions

[2] When compounding monthly

- For each month, we have an interest rate of 0.18/12 = 1.5%
- The future worth F = $P(1+i)^{12}$ = $1{,}000(1+0.015)^{12}$ = \$1,195.61
- Thus, the annual effective interest rate = 19.561%



General Questions

Thus,

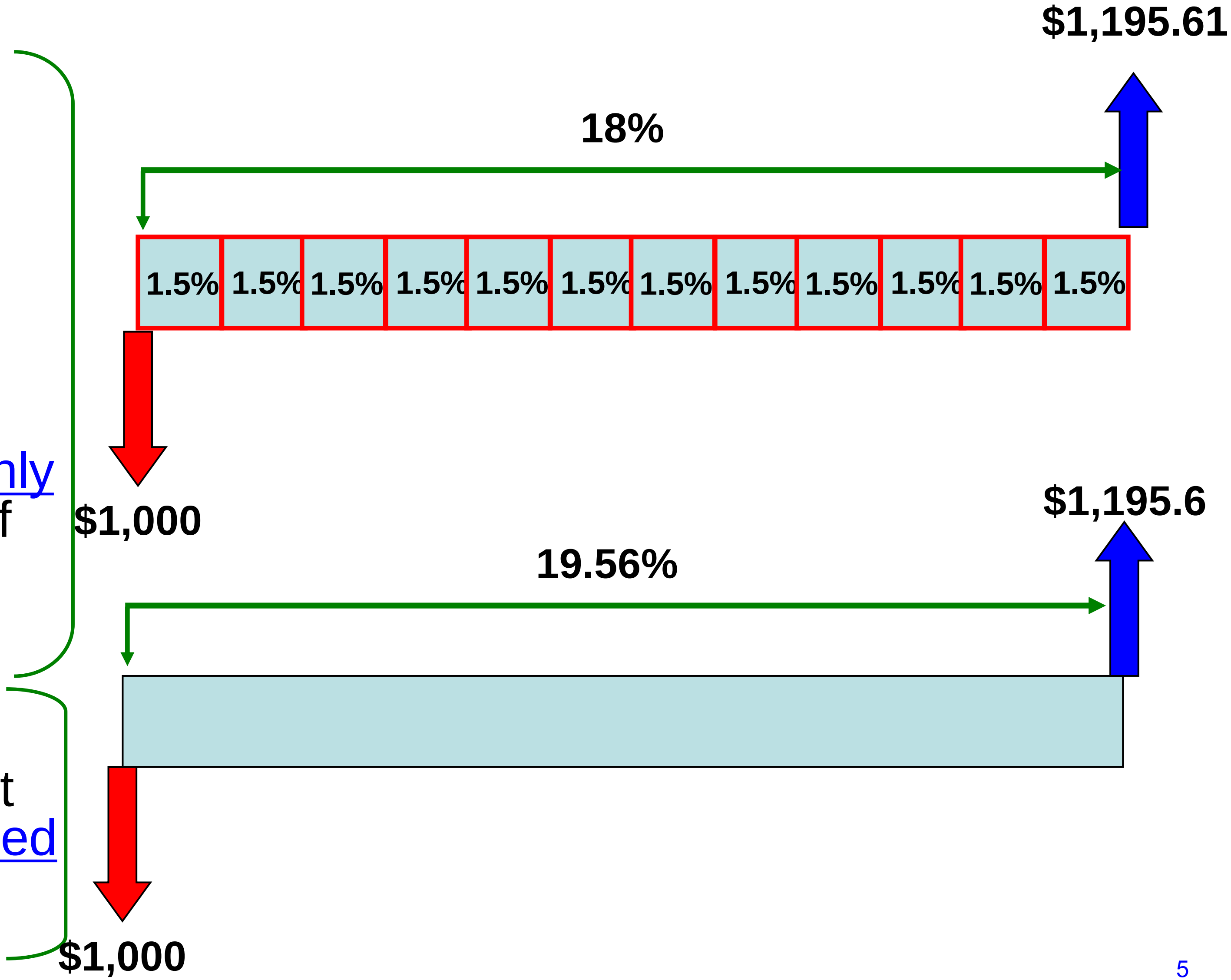
- We have an annual interest rate of 18% compounded monthly

Or

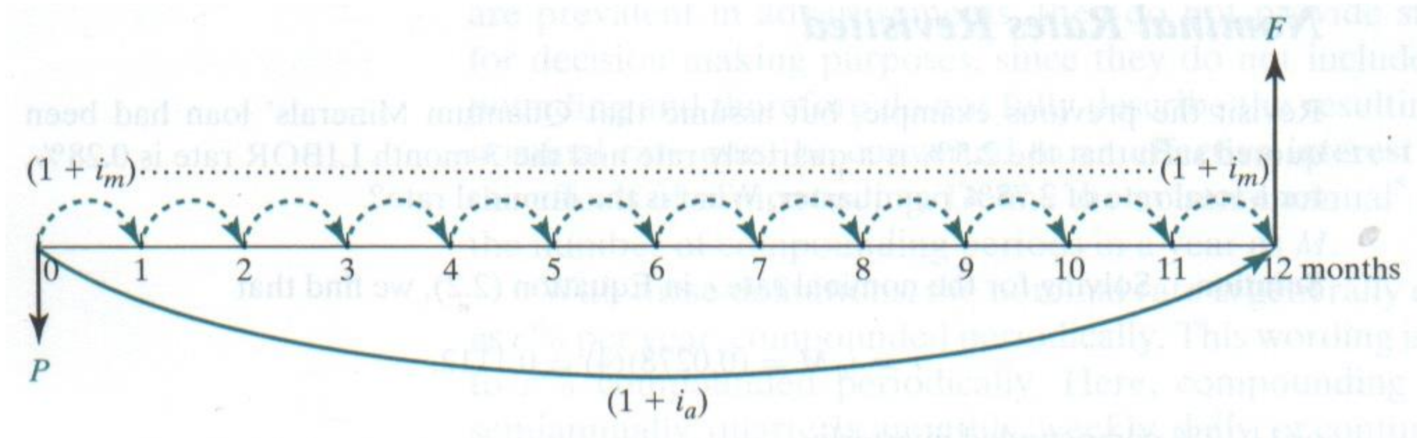
- We have monthly interest rates of 1.5% for 12 months

Or

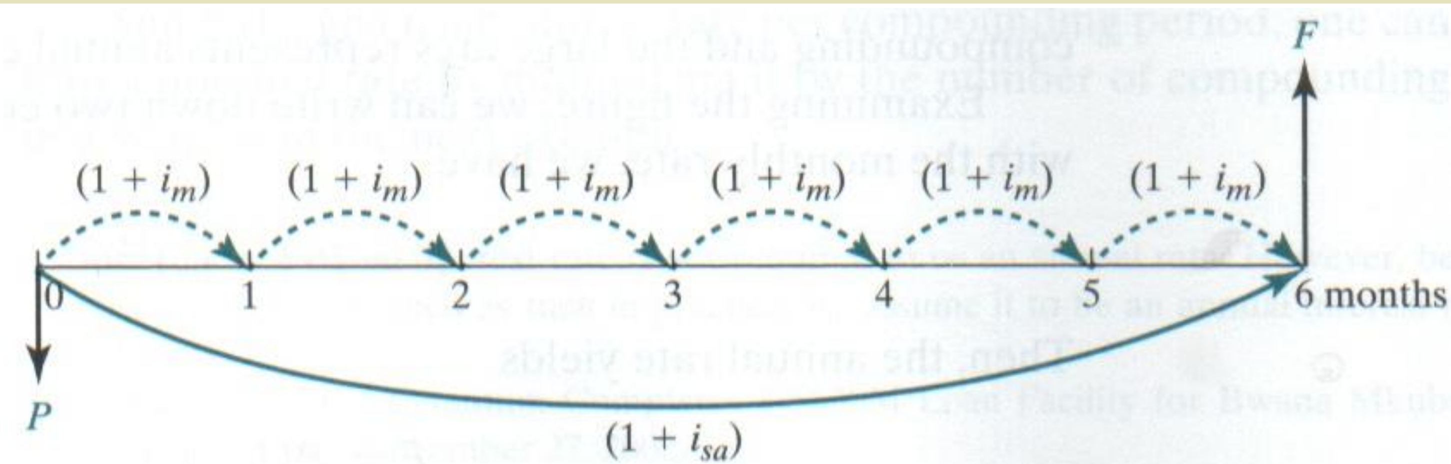
- We have a 19.56% interest rate compounded annually



Compounding and Effective Interest Rates

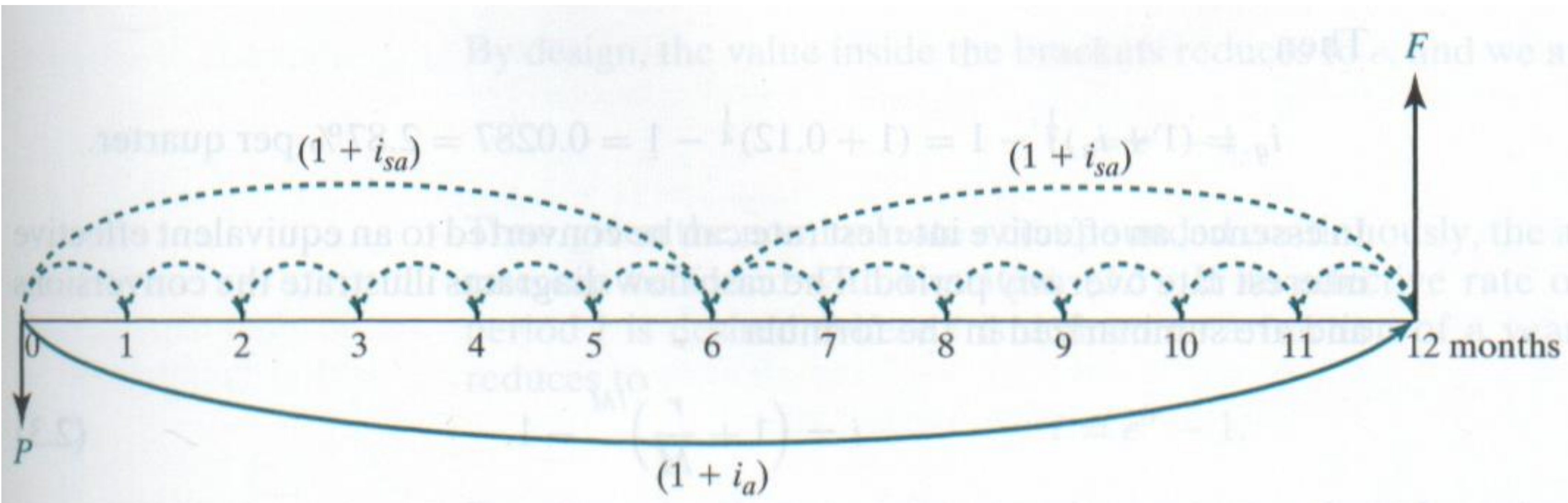


Compounding with monthly and annual interest rates



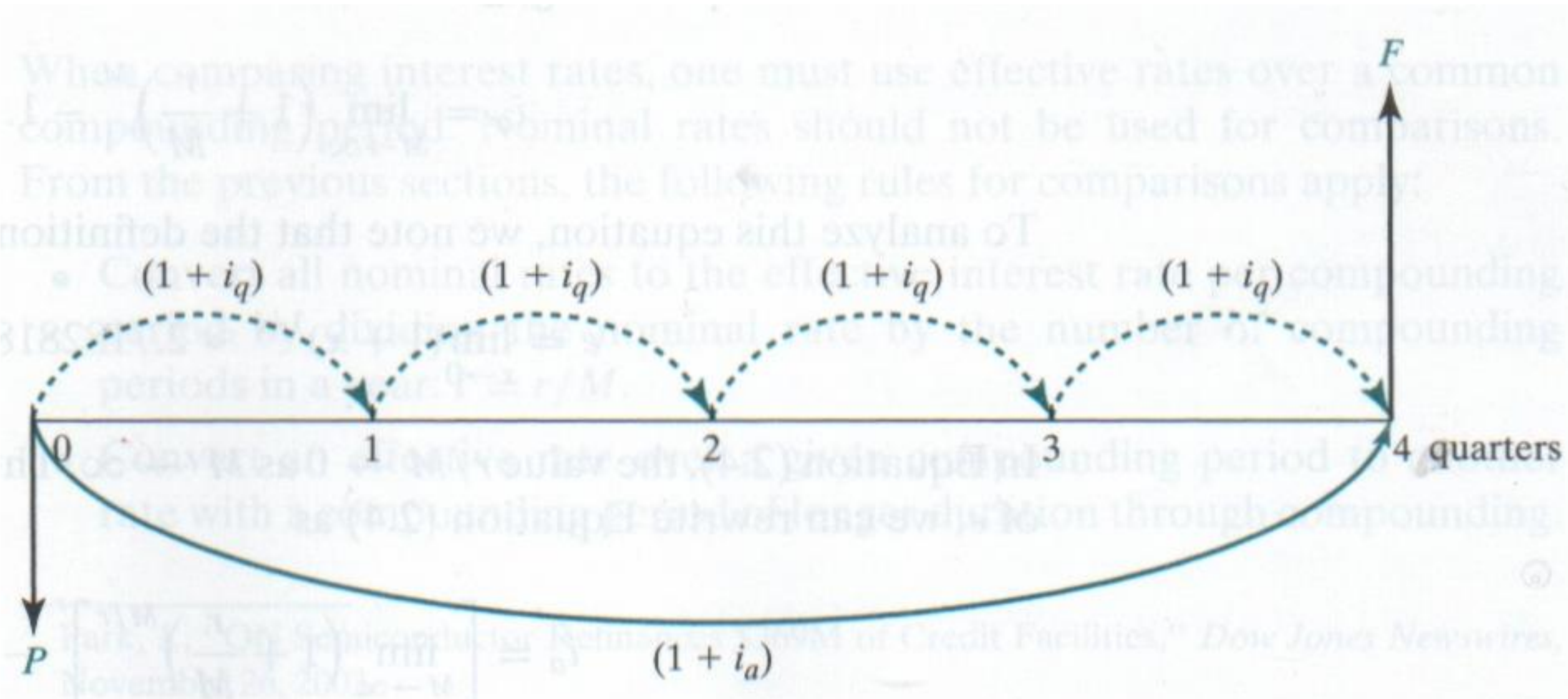
Compounding with monthly and semi-annual interest rates

Compounding and Effective Interest Rates



compounding with monthly, semiannual, and annual interest rates

Compounding and Effective Interest Rates

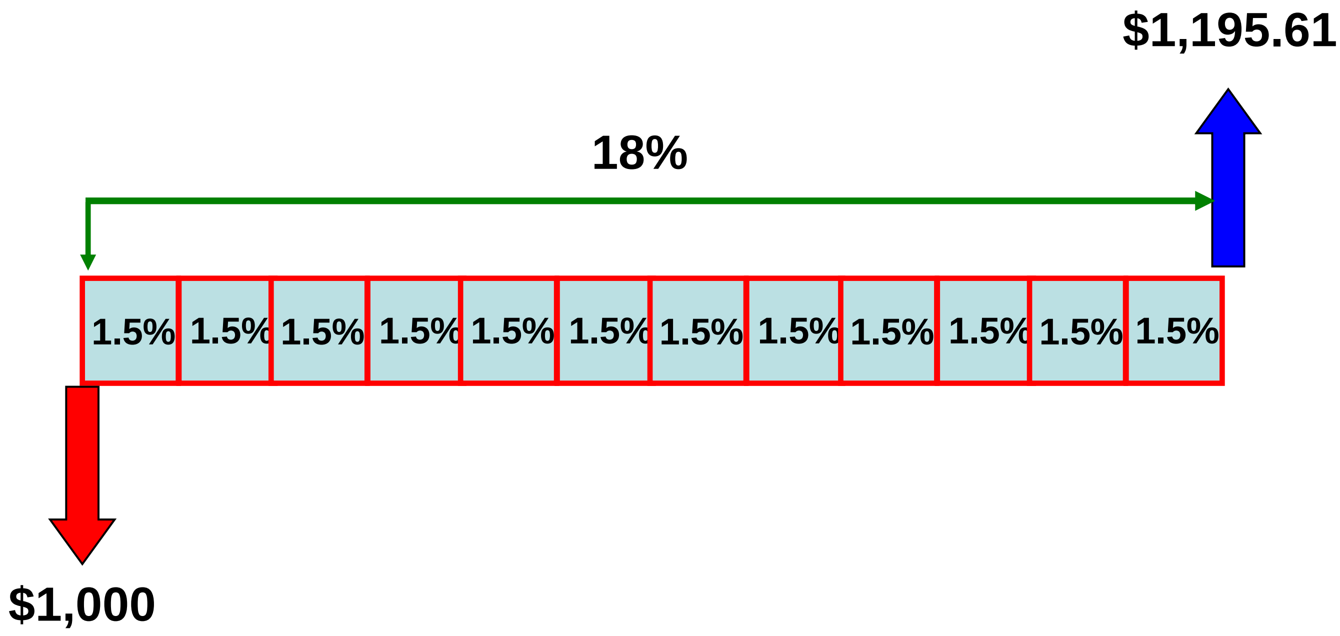


compounding with quarterly and annual interest rates

Necessary Definitions

- Time period (t): the period over which the interest is expressed (1% per month or 18% per year)
- Compounding period (CP): the ***shortest*** time unit over which interest is charged or earned or COMPUTED (18% per year compounded monthly)
- Compounding frequency: the number of times compounding occurs within the time period (or payment period)
- Payment period (PP): frequency of the payments or receipts which is the cash flow transaction period

Necessary Definitions



- Time period = ? months
- Payment period = ? months
- Compounding period = ? month
- Compounding frequency = ?
- Interest rate per compounding period = ? %

Nominal and Effective Interest Rates

- Nominal interest rate: this is the ***annual interest rate***. It does not consider any compounding

- Effective interest rate: the actual rate that should be considered for the payment period. It implies the compounding of the interest within the time period of interest

Calculation of the Effective Interest Rate

- r = nominal interest rate per year (or per payment period)
- m = number of *compounding periods* per year (or per payment period)
- i = effective interest rate per *compounding period* which can be viewed as the nominal interest rate per the compounding period
- i_a = effective interest rate per year (or per payment period)

$$i = \frac{r}{m}$$

$$i_a = (1+i)^m - 1 = (1+\frac{r}{m})^m - 1$$

Calculation of the Effective Interest Rate

Determine the payment period [PP]

Determine the nominal interest rate per payment period [r/PP]

Determine the compounding period [CP]

Determine the compounding frequency within PP [m]

Calculation of the Effective Interest Rate

Nominal interest rate per year

r = 18% per year, compounded *m*-ly

Compounding Period	Times Compounded per Year, *m*	Rate per Compound Period, *i*	Distribution of *i* over the Year of Compounding Periods	Effective Annual Rate, i_a
Year	1	18%	18% (1)	$(1.18)^1 - 1 = 18\%$
6 months	2	9%	9% (1), 9% (2)	= 18.81%
Quarter	4	4.5%	4.5% (1), 4.5% (2), 4.5% (3), 4.5% (4)	= 19.252%
Month	12	1.5%	1.5% in each: 1 2 3 4 5 6 7 8 9 10 11 12	= 19.562%
Week	52	0.34615%	0.34615% in each: 1 2 3 … 24 26 28 … 50 52	= 19.684%

Effective i_a for Selected i

TABLE 4–3 Effective Annual Interest Rates for Selected Nominal Rates

Nominal Rate r%	Semiannually ($m = 2$)	Quarterly ($m = 4$)	Monthly ($m = 12$)	Weekly ($m = 52$)	Daily ($m = 365$)	Continuously ($m = \infty$; $e^r - 1$)
0.25	0.250	0.250	0.250	0.250	0.250	0.250
0.50	0.501	0.501	0.501	0.501	0.501	0.501
1.00	1.003	1.004	1.005	1.005	1.005	1.005
1.50	1.506	1.508	1.510	1.511	1.511	1.511
2	2.010	2.015	2.018	2.020	2.020	2.020
3	3.023	3.034	3.042	3.044	3.045	3.046
4	4.040	4.060	4.074	4.079	4.081	4.081
5	5.063	5.095	5.116	5.124	5.126	5.127
6	6.090	6.136	6.168	6.180	6.180	6.184
7	7.123	7.186	7.229	7.246	7.247	7.251
8	8.160	8.243	8.300	8.322	8.328	8.329
9	9.203	9.308	9.381	9.409	9.417	9.417
10	10.250	10.381	10.471	10.506	10.516	10.517
12	12.360	12.551	12.683	12.734	12.745	12.750
15	15.563	15.865	16.076	16.158	16.177	16.183
18	18.810	19.252	19.562	19.684	19.714	19.722
20	21.000	21.551	21.939	22.093	22.132	22.140
25	26.563	27.443	28.073	28.325	28.390	28.403
30	32.250	33.547	34.489	34.869	34.968	34.986
40	44.000	46.410	48.213	48.954	49.150	49.182
50	56.250	60.181	63.209	64.479	64.816	64.872

TABLE 3.1 Various Interest Statements and Their Interpretations

(1) Interest Rate Statement	(2) Interpretation	(3) Comment
i = 12% per year i = 1% per month i = 3½% per quarter	i = *effective* 12% per year compounded yearly	When no compounding period is given, interest rate is an effective rate, with compounding period assumed to be equal to stated time period.
i = 8% per year, compounded monthly i = 4% per quarter compounded monthly i = 14% per year compounded semiannually	i = *nominal* 8% per year compounded monthly	When compounding period is given without stating whether the interest rate is nominal or effective, it is assumed to be nominal. Compounding period is as stated.
i = effective 10% per year compounded monthly i = effective 6% per quarter i = effective 1% per month compounded daily	i = *effective* 10% per year compounded monthly	If interest rate is stated as an effective rate, then it is an effective rate. If compounding period is not given, compounding period is assumed to coincide with stated time period.

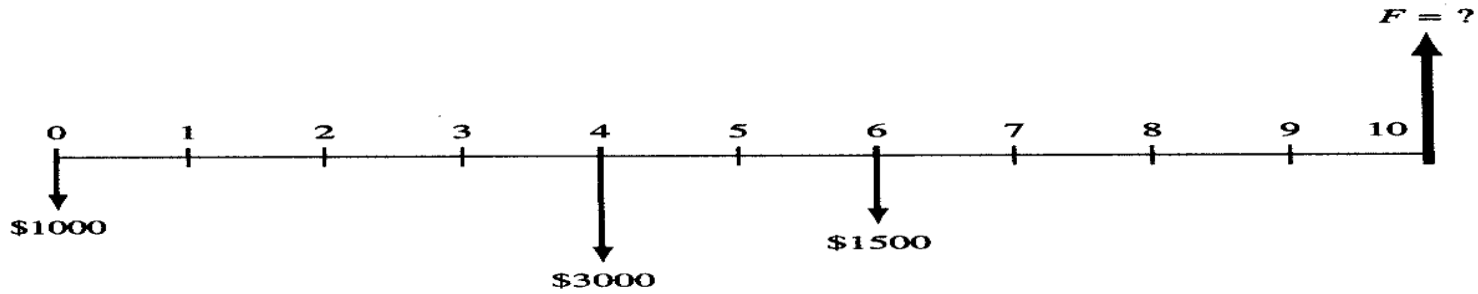
TABLE 3.2 Specific Examples of Interest Statements and Interpretations

(1) Interest Rate Statement	(2) Nominal or Effective Interest	(3) Compounding Period
15% per year compounded monthly	Nominal	Monthly
15% per year	Effective	Yearly
Effective 15% per year compounded monthly		
20% per year compounded quarterly		
Nominal 2% per month compounded weekly		
2% per month		
2% per month compounded monthly		
Effective 6% per quarter		
Effective 2% per month compounded daily		
1% per week compounded continuously		

Example [1]

- Find the amount after 10 years for the given cash flow diagram if the interest rate is 12% per year compounded semiannually

- We need to find the effective interest rate per year (for the payment period). Thus, r = 12% for a period of 1 year that includes 2 compounding periods m = 2 → $i_a = (1+0.12/2)^2 - 1 = 12.36\%$
- F = 1,000(F/P,12.36%,10)+3,000(F/P,12.36,6) +1,500(F/P,12.36%,4) = $11,634



Example [1]

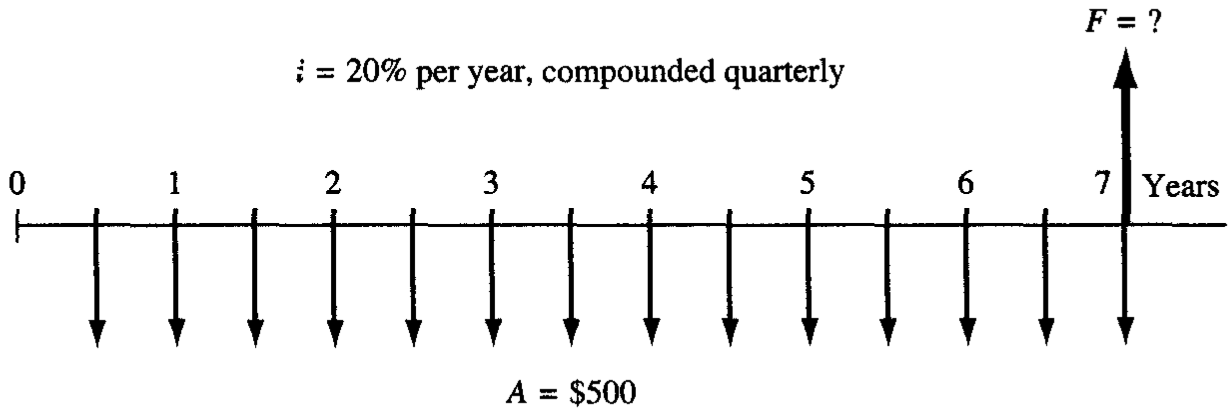
- A second way to solve this question is as follows:

  F = 1,000(F/P,6%,20)+3,000(F/P,6%,12)+1,500(F/P,6%,8) = $11,634

Example [2]

- For the past 7 years, an engineer was paying every 6 months for the software maintenance. What is the equivalent amount after the last payment. Assume an interest rate of 20% per year and that the interest is compounded quarterly
- We have: a payment period of ½ year through which the nominal interest rate is r = 10% and we have within this ½ year 2 compounding periods (m=2)
- The effective interest rate per 6 months (½ year)

  $i_a = (1+0.1/2)^2 - 1 = 10.25\%$
- F =A(F/A,i, n) = 500(F/A,10.25%,14) = $14,244

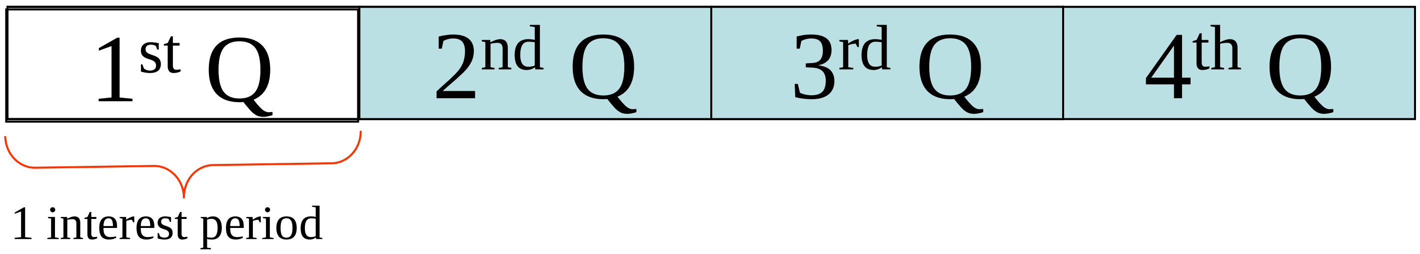


Example [3]

- Find the effective interest rate per quarter at a nominal rate of 8% compounded (a) quarterly, (b) monthly, (c) weekly, and (d) daily

- Quarterly compounded

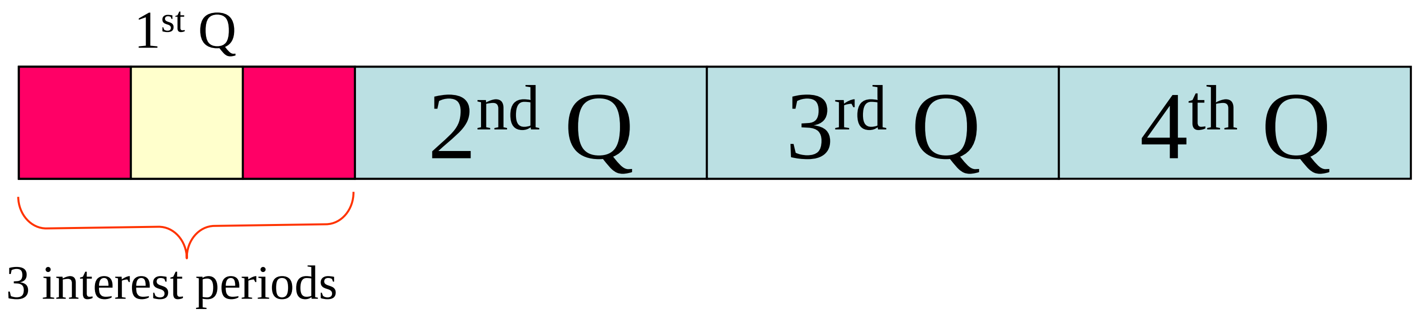
$r = 8\%/4 = 2\%,\ m = 1,\ i_q = (1+0.02/1)^1 - 1 = 2\%$



Example [3]

- Monthly compounded

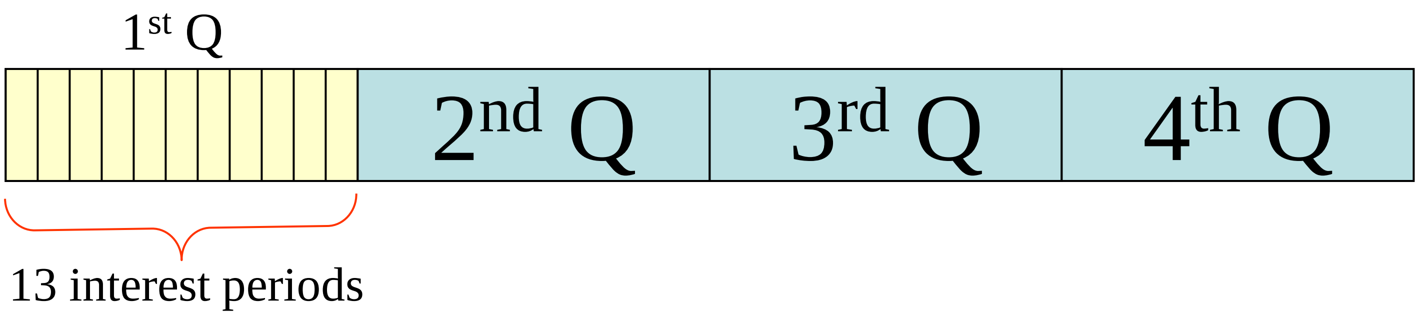
$r = 2\%$, $m = 3$, $i_q = (1+0.02/3)^3 - 1 = 2.013\%$



Example [3]

- Weekly compounded

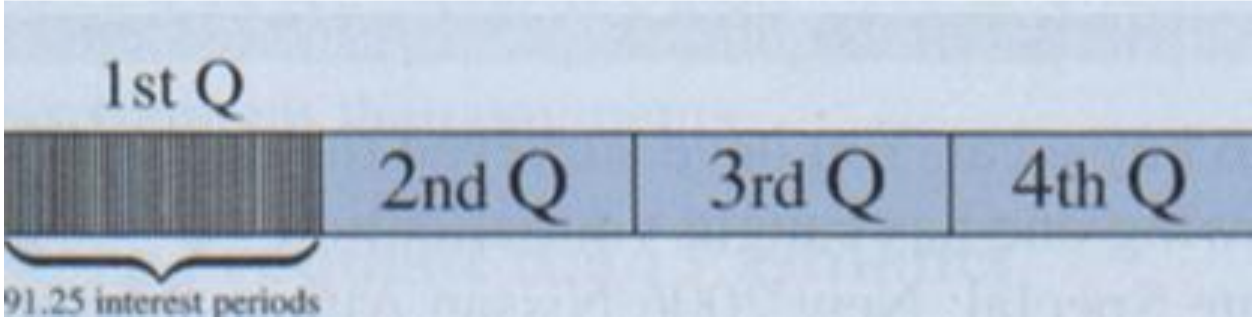
$r = 2\%$, $m = 13$, $i_q = (1+0.02/13)^{13} - 1 = 2.0186\%$



Example [3]

- Daily compounded

$r = 2\%$, $m = 91.25$, $i_q = (1+0.02/91.25)^{91.25} - 1$
=2.0199%



Continuous Compounding

- When we have a large number of compounding periods (m), the interest rate per compounding period (r/m) becomes very small

- If m approaches infinity, r/m approaches zero

- In this case, we have a continuous compounding that can be expressed using the following formula:

$$i_a = e^r - 1$$

- As an example, the effective ***annual*** interest rate for a nominal interest rate of 12% compounded *continuously* is $i_a = e^{0.12} - 1 = 12.7497\%$

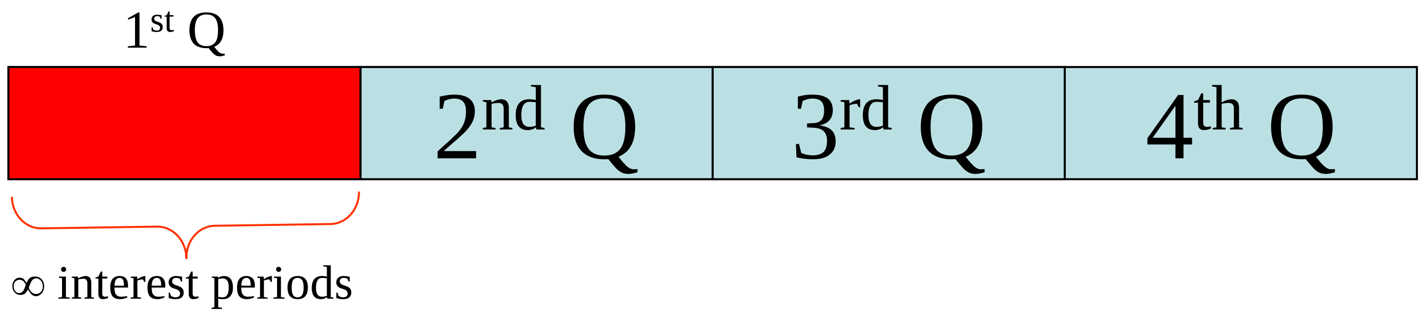
Continuous Compounding - Example

- Find the effective interest rate per quarter at a nominal rate of 8% compounded continuously

- Continuous compounding

$r = 8\%/4 = 2\%$ (because r is for a quarter)

$i_a = e^{0.02} - 1 = 2.0201\%$



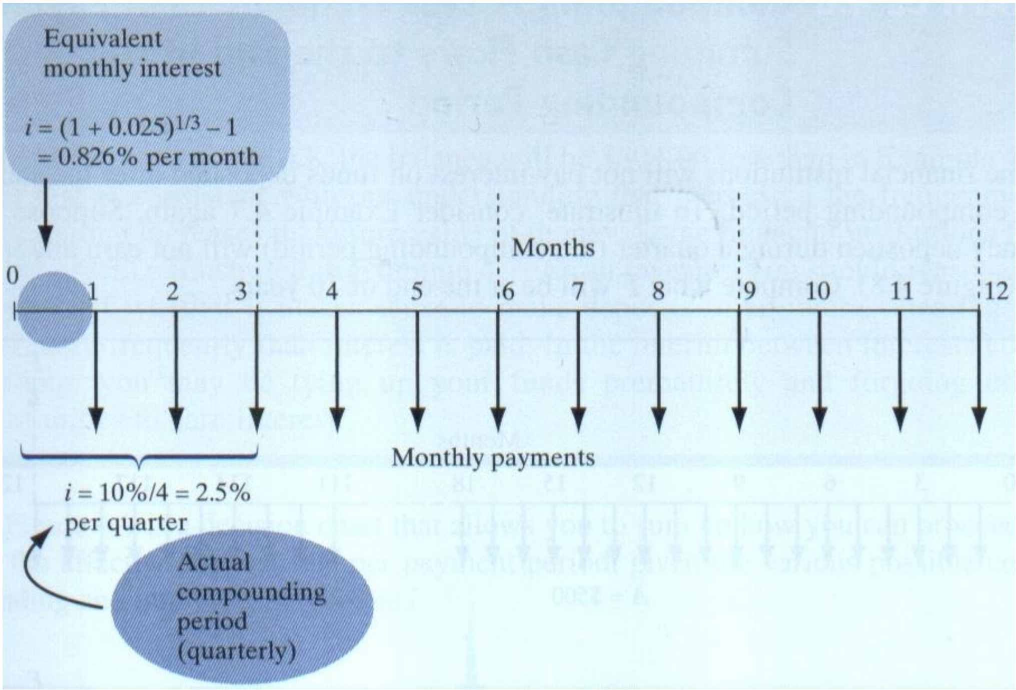
Summary of the Past Example

8% compounded quarterly	8% compounded monthly	8% compounded weekly	8% compounded daily	8% compounded continuously
Payments occur quarterly	Payments occur quarterly	Payments occur quarterly	Payments occur quarterly	Payments occur quarterly
2.000% per quarter	2.013% per quarter	2.0186% per quarter	2.0199% per quarter	2.0201% per quarter

Example [4]

- Suppose you make $500 monthly deposits to a savings account that pays interest at a rate of 10% compounded quarterly

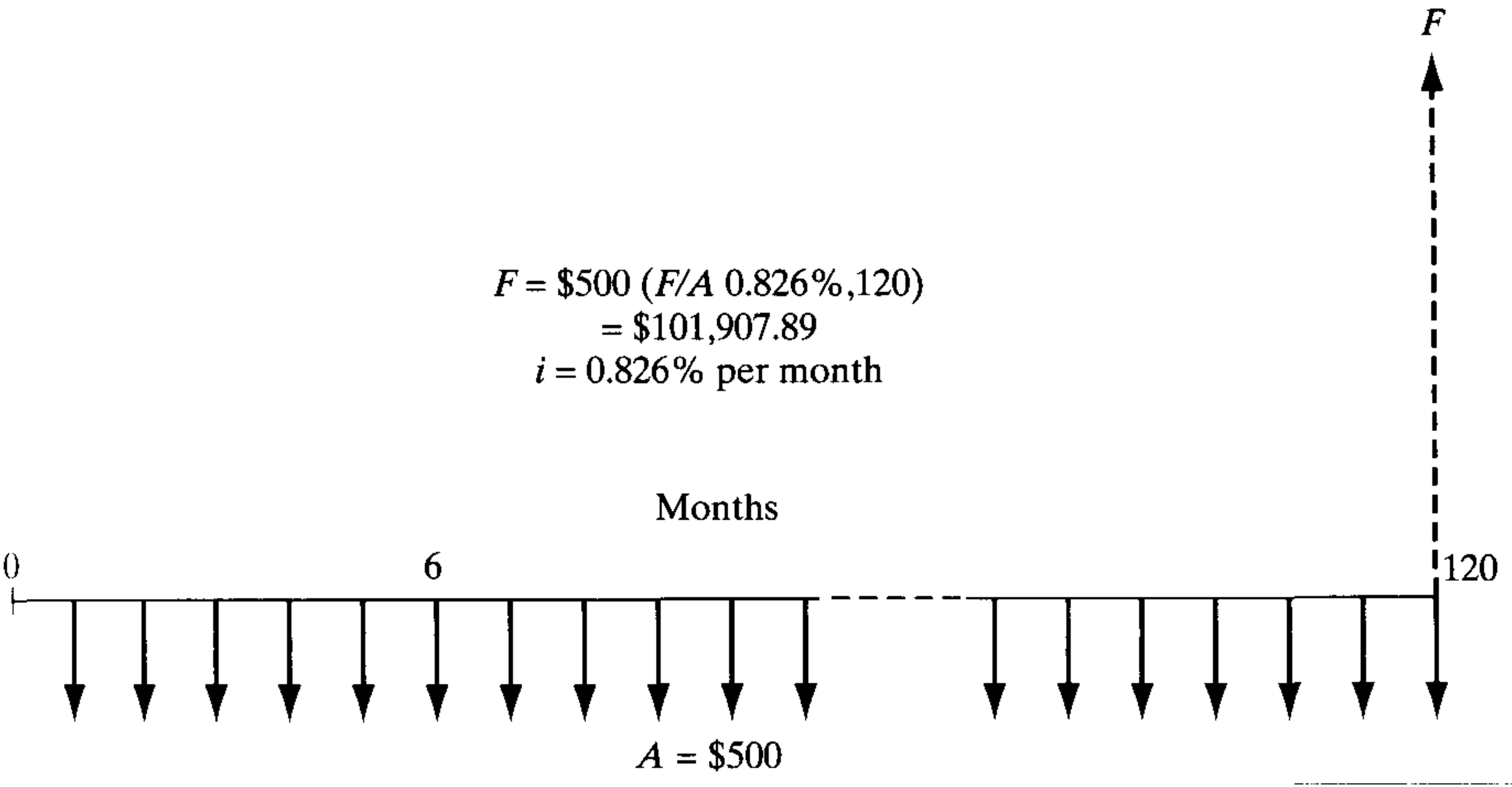
- Compute the balance at the end of 10 years



Example [4]

- The quarterly interest rate = 10%/4 = 2.5%
- Since the payments are made monthly, then we need to find out the effective monthly interest rate (i_m) that yields the same quarterly compounding
- $(1+i_m)^3 - 1 = i_q \rightarrow i_m = 0.826\%$
- Since the payments are made on monthly basis, then for 10 years we have 120 periods of 0.826% interest rate (for each month) and monthly payments of $500
- F = 500(F/A,0.826%,120) = $101,907.89

Example [4]


F
$F = \$500\ (F/A\ 0.826\%, 120)$
$= \$101{,}907.89$
$i = 0.826\%$ per month
Months
0
6
120
$A = \$500$

Example [5]

- A municipality intends to construct a water supply project after 6 years from now

- In order to make available the entire cost of the project at the beginning of project implementation, it was decided to invest in a savings account for 6 years starting from now

- The municipality is to deposit now $750,000. In addition, it will deposit $120,000 by the end of the *second* year, $230,000 by the end of the *third* year and $70,000 by the end of the *fifth* year

Example [5]

- The investment plan goes as follows: in the first year interest is compounded *continuously*, in the second year it is compounded *daily*, in the third year it is compounded *weekly*, in the fourth year it is compounded *monthly*, in the fifth year it is compounded *quarterly*, and in the last year it is compounded *semi-annually*

- If the nominal interest rate is 12%, answer the following questions:
  - Calculate the interest amount by the end of each year
  - What is the total future worth of this investment by the end of the sixth year?

Example [5]

0	$750,000
1	$0
2	$120,000
3	$230,000
4	$0
5	$70,000
6	$0

	r	**m**	**i effective**	**Beginning balance**	**Interest**	**Ending balance**
1	12%	-	12.7497%	$750,000	$95,622.64	$845,622.64
2	12%	365	12.7475%	$845,622.64	$107,795.42	$953,418.06
3	12%	52	12.7341%	$1,073,418.06	$136,690.12	$1,210,108.17
4	12%	12	12.6825%	$1,440,108.17	$182,641.76	$1,622,749.94
5	12%	4	12.5509%	$1,622,749.94	$203,669.41	$1,826,419.35
6	12%	2	12.3600%	$1,896,419.35	$234,397.43	$2,130,816.78
					FW	**$2,130,816.78**

Example [6]

- An investment plan works as follows: you deposit $1,200 now and every six months for three years. In the first year, the interest is compounded monthly, in the second year it is compounded weekly, while in the third year it is compounded daily. The nominal interest rate is 10% per year

- For this scenario, compute the future worth

Example [6]

1st year:

pp = 6 months

$m = 6$

$r = 5\%$

$$i_1 = \left(1 + \frac{5\%}{6}\right)^6 - 1 = 5.1053\%$$

2nd year:

pp = 6 mnths

$m = 26$

$r = 5\%$

$$i_2 = \left(1 + \frac{5\%}{26}\right)^{26} - 1 = 5.1221\%$$

3rd year:

pp = 6 months

$m = \frac{365}{2} = 182.5$

$r = 5\%$

$$i_3 = \left(1 + \frac{5\%}{182.5}\right)^{182.5} - 1 = 5.1264\%$$

Example [6]

$$F = 1{,}200 + 1{,}200(F/P, i_3, 1) + 1{,}200(F/P, i_3, 2) +$$
$$\left[1{,}200(F/P, i_2, 1) + 1{,}200(F/P, i_2, 2)\right]\left[(F/P, i_3, 2)\right] +$$
$$\left[1{,}200(F/P, i_1, 2) + 1{,}200(F/P, i_1, 1)\right]\left[(F/P, i_2, 2)\right]\left[(F/P, i_3, 2)\right]$$

$$F = \$9{,}806.67$$