ch 2.2
ch 2.3

Lecture 5 Fixed point continue... + Newton method

(1)

Fixed point: كيف نستعملها في التقريب

Ex: Find a suitable function $g(x)$ for $f(x) = x^3 + 4x^4 - 10$ s.t. $f(x) = g(x) - x = 0$ choose $P_0 = 1.5$

(Find P_3)

Sl: هناك طرق عدة لإيجاد $g(x)$ ولكنه ليس جميعها تكون ناجحة. وأحد الطرق الناجحة:

$f(x) = 0 \Rightarrow x^3 + 4x^2 - 10 = 0$

$\Rightarrow x^2(x+4) - 10 = 0 \Rightarrow x^2 = \frac{10}{x+4} \Rightarrow x = \sqrt{\frac{10}{x+4}}$

$\Rightarrow g(x) = \sqrt{10/(x+4)}$. Now apply fixed point.

$g(p) = p \Rightarrow$ iteration نرتبها $\Rightarrow g(P_{n-1}) = P_n$

$\Rightarrow P_n = \sqrt{10/(P_{n-1}+4)}$ s.t. $P_0 = 1.5$

$\Rightarrow P_1 = g(P_0) = \sqrt{10/(P_0+4)} = \sqrt{10/(1.5+4)}$

$\Rightarrow P_1 = 1.348399.7$

Now $P_2 = g(P_1) = \sqrt{10/(P_1+4)}$

$\Rightarrow P_2 = \sqrt{10/(1.348399+4)} = 1.36737$

$P_3 = g(P_2) = \sqrt{10/(P_2+4)}$

$\Rightarrow P_3 = \sqrt{10/(1.36737+4)} = \boxed{1.364956}$

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Absolute error for fixed point theorem:

Thm.: (Fixed point): Let ① $g \in C[a,b]$

② $g(x) \in [a,b] \ \forall \ x \in [a,b]$.

③ $g(x)$ is differentiable on (a,b) with $|g'(x)| \le K$, $0 < K < 1$

$\Rightarrow$ the sequence $\{P_n\}_{n=1}^{\infty}$ with $P_n = g(P_{n-1})$ is convergent to the unique fixed point for $g(x)$ $\forall \ P_0 \in [a,b]$. and

(i) $|P - P_n| \le \dfrac{K^n}{1-K} |P_1 - P_0|$ or

(ii) $|P - P_n| \le K^n \max\{b - P_0, P_0 - a\}$.

Ex: Let $g(x) = \dfrac{x^2 - 1}{3}$ on $[-1,1]$.

ⓐ Approximate $|P - P_3|$ by using two ways (use fixed point iteration method)

ⓑ Find the number of iterations necessary to solve $g(x) = x$ by using fixed point to get an accuracy 10^{-3}.

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①

Sl.: $P_0 = \dfrac{-1+1}{2} = 0$. $P_1 = g(P_0) = \dfrac{0^2 - 1}{3} = \dfrac{-1}{3}$

$K = \dfrac{2}{3}$ (معلومة سابقا)

Method (1): $|P - P_3| \le \dfrac{\left(\frac{2}{3}\right)^3}{1 - \frac{2}{3}} \left| -\frac{1}{3} - 0 \right| \le \dfrac{\frac{8}{27}}{\frac{1}{3}} \cdot \frac{1}{3} = \boxed{\dfrac{8}{27}}$

Method (2): $|P - P_3| \le \left(\frac{2}{3}\right)^3 \max\{1-0, 0-(-1)\}$

$= \frac{8}{27}$.

(3)

(2)

$|P - P_n| \le K^n \max\{b - P_0, P_0 - a\}$

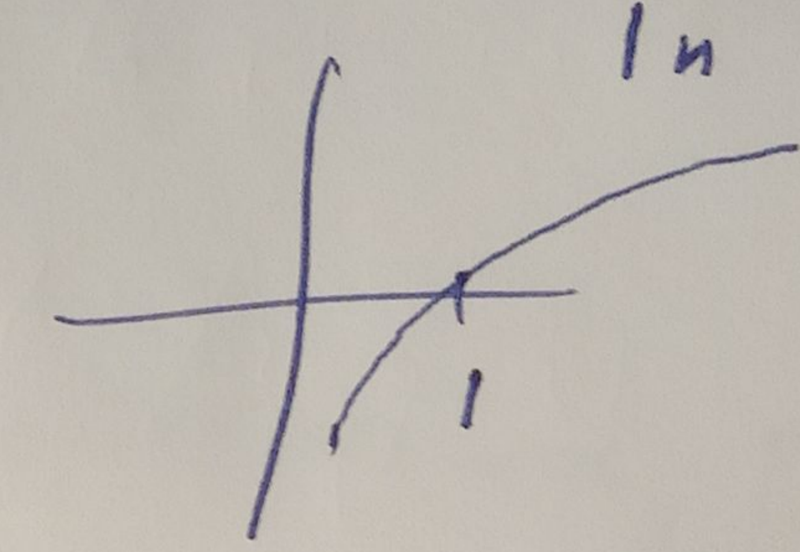
$= \left(\frac{2}{3}\right)^n \max\{1, 1\} \le 10^{-3}$

$\Rightarrow \left(\frac{2}{3}\right)^n (1) \le 10^{-3}$

$\Rightarrow n \ln \frac{2}{3} \le \ln 10^{-3}$

$\Rightarrow n \ge \dfrac{\ln 10^{-3}}{\ln \frac{2}{3}} \Rightarrow n \ge 17.03 \Rightarrow \boxed{n = 18}$

ملاحظة: عكسنا الإشارة لماذا؟



Ch 2.3: Newton's Raphson Method

Let $f \in C^2[a,b]$, x_0 is an approximation to p with $f(p) = 0$, but $f'(x_0) \ne 0$ and assume that $|P - x_0|$ is small. $\Rightarrow$ we apply first Taylor series about $x_0 \Rightarrow f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(c)\dfrac{(x-x_0)^2}{2!}$

$x_0 < c < x$

$\Rightarrow f(p) = f(x_0) + f'(x_0)(p - x_0) + f''(c)\dfrac{(p-x_0)^2}{2!}$

$\Rightarrow 0 = f(x_0) + f'(x_0)(p - x_0)$

$\Rightarrow p = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

In this stage we can define Newton's-Raphson method as $P_n = g(P_{n-1})$ where $g(P_{n-1}) = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})}$ (4)

$$\Rightarrow \boxed{P_n = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})}} \leftarrow \text{Newton's-Raphson method.}$$

Ex: Use Newton's method to solve $f(x) = \cos(x) - x$, $P_0 = \frac{\pi}{4}$

Find P_3

sl: $f'(x) = -\sin x - 1 \Rightarrow P_n = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})}$

$$\Rightarrow P_n = P_{n-1} - \frac{\cos P_{n-1} - P_{n-1}}{-\sin P_{n-1} - 1}$$

3.14 180 3.14

$$\Rightarrow P_1 = P_0 - \frac{\cos P_0 - P_0}{-\sin P_0 - 1} = \frac{\pi}{4} - \frac{\cos\frac{\pi}{4} - \frac{\pi}{4}}{-\sin\frac{\pi}{4} - 1}$$

$$\Rightarrow P_1 = 0.7395361$$

$$P_2 = P_1 - \frac{\cos P_1 - P_1}{-\sin P_1 - 1} = 0.73908$$

$$P_3 = 0.739085$$

"secant method" $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$

$$f'(P_{n-1}) = \lim_{x \to P_{n-1}} \frac{f(x) - f(P_{n-1})}{x - P_{n-1}}$$ (1)

using $x = P_{n-2} \Rightarrow f'(P_{n-1}) \approx \frac{f(P_{n-2}) - f(P_{n-1})}{P_{n-2} - P_{n-1}}$

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Since Newton's $P_n = P_{n-1} - \dfrac{f(P_{n-1})}{f'(P_{n-1})}$ —— (2)

substitute (1) in (2) $\Rightarrow$

$$\boxed{P_n = P_{n-1} - \frac{f(P_{n-1})(P_{n-2} - P_{n-1})}{f(P_{n-2}) - f(P_{n-1})}}$$ Secant method

Ex: Using secant method to solve $f(x) = \cos(x) - x$

Choose $P_0 = 0.5$, $P_1 = \frac{\pi}{4}$

sl: $P_2 = P_1 - \dfrac{f(P_1)(P_0 - P_1)}{f(P_0) - f(P_1)}$

$$P_2 = \frac{\pi}{4} - \frac{\left(\cos\frac{\pi}{4} - \frac{\pi}{4}\right)\left(0.5 - \frac{\pi}{4}\right)}{(\cos 0.5 - 0.5) - \left(\cos\frac{\pi}{4} - \frac{\pi}{4}\right)}$$

$$\Rightarrow P_2 = 0.785398 - \frac{(0.70710 - 0.785398)(0.5 - 0.785398)}{(0.87758 - 0.5) - (0.70710 - 0.785398)}$$

$= 0.7363841$

$$P_3 = P_2 - \frac{f(P_2)(P_1 - P_0)}{f(P_1) - f(P_2)} = \boxed{0.7390581}$$