Section $\boxed{2.6}$ Lecture $\boxed{7}$ Zero's of polynomial and Muller's method

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Thm (Fundamental Thm. of calculus)

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x), \quad \int_a^x f'(t)\,dt = f(x) - f(a).$$

In general $\int f'(x)\,dx = f(x) + c.$

Thm (Fundamental Thm of algebra).

If $P(x)$ is a polynomial of degree $n \geq 1$ with real or complex coefficients $\Rightarrow$ $P(x) = 0$ has @ least one (possibly complex) root. #

As consequence, we have

Corollary: If $P(x)$ is a polynomial of degree $n \geq 1$ with real or complex coefficients, $\Rightarrow$ $\exists$ unique constants $x_1, x_2, \dots, x_k$ (possibly complex) and unique possitive integers $m_1, m_2, \dots, m_k$ such that $\sum_{i=1}^{k} m_i = n$ and $\ni$

$$P(x) = a_n (x - x_1)^{m_1} (x - x_2)^{m_2} \cdots (x - x_k)^{m_k}.$$

#

Horner's method: Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

If $b_n = a_n$, $b_K = a_K + b_{K+1} x_0$ for $K = n-1, n-2, \dots, 1, 0$

$\Rightarrow b_0 = P(x_0)$. Moreover, if $Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \dots + b_2 x + b_1$

$\Rightarrow P(x) = (x - x_0)\, Q(x) + b_0$.

Ex: Let $P(x) = x^3 + 4x^2 - 3x - 2$

① Use Horner's Method to find $P(2)$, $P(1)$.

② use " " " Solve $P(x) = 0$.

Sol: ① $Q(x) = x^2 + 6x + 9$

$\Rightarrow P(2) = b_0 = P(x_0) = \boxed{16}$ $\boxed{x_0 = 2}$

x^3	x^2	x	x^0
1	4	-3	-2
	2	12	18
1	6	9	16

$P(1)$: $Q(x) = x^2 + 5x + 2$

$\Rightarrow P(1) = b_0 = P(x_0) = P(1) = \boxed{0}$ $\boxed{x_0 = 1}$

x^3	x^2	x	x^0
1	4	-3	-2
	1	5	2
1	5	2	0

$\Rightarrow$ We conclude that $(x-1)$ is a factor of P.

② From ① $P(x) = (x-1)(x^2 + 5x + 2)$

$P(x) = 0 \Rightarrow (x-1) = 0$ or $x^2 + 5x + 2$

$\Rightarrow x_1 = 1$ or $x_{2,3} = \dfrac{-5 \mp \sqrt{25 - 8}}{2} = \dfrac{-5 \pm \sqrt{17}}{2}$

$(x-2)\,Q(x) + b_0 = P(x) = (x-2)(x^2 + 6x + 9) + 16$ ① ~ نشتق:

$\Rightarrow P'(x) = (x-2)\,Q'(x) + Q(x) \Rightarrow$ In general if

$f(x) = (x - x_0)\,Q(x) + b_0 \Rightarrow f'(x_0) = Q(x_0)$. #

Muller's method: $f(x_3) \simeq P(x_3) = 0$ (3)

If we have $(x_0, f(x_0)), (x_1, f(x_1))$ and $(x_2, f(x_2))$

We want to find a quadratic function

$P(x) = a(x-x_2)^2 + b(x-x_2) + c$ with

$P(x_0) = f(x_0),\ P(x_1) = f(x_1)$ and $P(x_2) = f(x_2)$.

Method: $f(x_0) = P(x_0) = a(x_0-x_2)^2 + b(x_0-x_2) + c$ — (1)

$f(x_1) = P(x_1) = a(x_1-x_2)^2 + b(x_1-x_2) + c$ — (2)

$f(x_2) = P(x_2) = c$

Next, we want to find x_3 s.t. $f(x_3) \simeq P(x_3) = 0$.

$P(x_3) = a(x_3-x_2)^2 + b(x_3-x_2) + c = 0$

$\Rightarrow x_3 - x_2 = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\Rightarrow x_3 = \boxed{\dfrac{-b \pm \sqrt{b^2-4ac}}{2a} + x_2}$

Ex: Let $f(x) = d\cos(x) + 2\sin x$ with $(\overset{x_0}{0}, f(0))$, $(\underset{x_1}{\frac{\pi}{2}}, 2)$, and $(\underset{x_3}{\pi}, 1)$.

(1) Use Muller's method to find a quadratic function $P(x)$.

(2) Use " " " approximate x_3 s.t.

$f(x_0) = 0$.

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S1: $f(\pi) = d\cos(\pi) + 2\sin\pi \Rightarrow 1 = -d + 0 \Rightarrow \boxed{d = -1}$

$\Rightarrow f(x) = -\cos(x) + 2\sin x.$

Let $P_{(x)} = a(x - x_2)^2 + b(x - x_2) + c$

$P_{(x)} = a(x - \pi)^2 + b(x - x_2) + c$

$1 = f(\pi) = P(\pi) \Rightarrow \boxed{c = 1}$

$2 = f(\frac{\pi}{2}) = P(\frac{\pi}{2}) = a(\frac{\pi}{2} - \pi)^2 + b(\frac{\pi}{2} - \pi) + 1$

$\Rightarrow 2 = \frac{\pi^2}{4}a - \frac{\pi}{2}b + 1 \Rightarrow$

$\frac{\pi^2}{4}a - \frac{\pi}{2}b = 1$ ——— ①

$-1 = f(0) = P(0) = a(0 - \pi)^2 + b(0 - \pi) + 1$

$\Rightarrow -1 = \pi^2 a - \pi b + 1$

$\Rightarrow \pi^2 a - \pi b = -2$ ——— ②

Solving ① and ②, we get

$b = \frac{-6}{\pi}, \quad a = \frac{-8}{\pi^2}$

$\Rightarrow$ The quadratic function that we search is

$P_{(x)} = -8/\pi^2 (x - \pi)^2 + -6/\pi(x - \pi) + 1$

② $x_3 = x_2 + \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow x_3 = 3.58, 0.33$. #