Composites Involving Exponential Functions

Find the domain and range for each of the functions in Exercises 21–24.

21. $f(x) = \dfrac{1}{2 + e^x}$

22. $g(t) = \cos(e^{-t})$

23. $g(t) = \sqrt{1 + 3^{-t}}$

24. $f(x) = \dfrac{3}{1 - e^{2x}}$

Applications

T In Exercises 25–28, use graphs to find approximate solutions.

25. $2^x = 5$

26. $e^x = 4$

27. $3^x - 0.5 = 0$

28. $3 - 2^{-x} = 0$

T In Exercises 29–36, use an exponential model and a graphing calculator to estimate the answer in each problem.

29. Population growth The population of Knoxville is 500,000 and is increasing at the rate of 3.75% each year. Approximately when will the population reach 1 million?

30. Population growth The population of Silver Run in the year 1890 was 6250. Assume the population increased at a rate of 2.75% per year.

a. Estimate the population in 1915 and 1940.

b. Approximately when did the population reach 50,000?

31. Radioactive decay The half-life of phosphorus-32 is about 14 days. There are 6.6 grams present initially.

a. Express the amount of phosphorus-32 remaining as a function of time t.

b. When will there be 1 gram remaining?

32. If Jean invests \$2300 in a retirement account with a 6% interest rate compounded annually, how long will it take until Jean's account has a balance of \$4150?

33. Doubling your money Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded annually.

34. Tripling your money Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded continuously.

35. Cholera bacteria Suppose that a colony of bacteria starts with 1 bacterium and doubles in number every half hour. How many bacteria will the colony contain at the end of 24 hr?

36. Eliminating a disease Suppose that in any given year the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take

a. to reduce the number of cases to 1000?

b. to eliminate the disease; that is, to reduce the number of cases to less than 1?

1.6 Inverse Functions and Logarithms

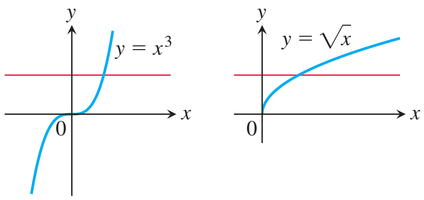
A function that undoes, or inverts, the effect of a function f is called the *inverse* of f. Many common functions, though not all, are paired with an inverse. In this section we present the natural logarithmic function $y = \ln x$ as the inverse of the exponential function $y = e^x$, and we also give examples of several inverse trigonometric functions.

One-to-One Functions

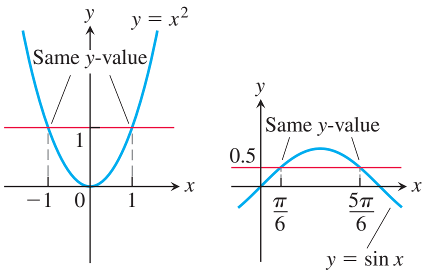
A function is a rule that assigns a value from its range to each element in its domain. Some functions assign the same range value to more than one element in the domain. The function $f(x) = x^2$ assigns the same value, 1, to both of the numbers -1 and $+1$; the sines of $\pi/3$ and $2\pi/3$ are both $\sqrt{3}/2$. Other functions assume each value in their range no more than once. The square roots and cubes of different numbers are always different. A function that has distinct values at distinct elements in its domain is called one-to-one. These functions take on any one value in their range exactly once.

DEFINITION A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

EXAMPLE 1 Some functions are one-to-one on their entire natural domain. Other functions are not one-to-one on their entire domain, but by restricting the function to a smaller domain we can create a function that is one-to-one. The original and restricted functions are not the same functions, because they have different domains. However, the



(a) One-to-one: Graph meets each horizontal line at most once.



(b) Not one-to-one: Graph meets one or more horizontal lines more than once.

FIGURE 1.58 (a) $y = x^3$ and $y = \sqrt{x}$ are one-to-one on their domains $(-\infty, \infty)$ and $[0, \infty)$. (b) $y = x^2$ and $y = \sin x$ are not one-to-one on their domains $(-\infty, \infty)$.

Caution Do not confuse the inverse function f^{-1} with the reciprocal function $1/f$.

two functions have the same values on the smaller domain, so the original function is an extension of the restricted function from its smaller domain to the larger domain.

(a) $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$.

(b) $g(x) = \sin x$ is *not* one-to-one on the interval $[0, \pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$. In fact, for each element x_1 in the subinterval $[0, \pi/2)$ there is a corresponding element x_2 in the subinterval $(\pi/2, \pi]$ satisfying $\sin x_1 = \sin x_2$, so distinct elements in the domain are assigned to the same value in the range. The sine function *is* one-to-one on $[0, \pi/2]$, however, because it is an increasing function on $[0, \pi/2]$ giving distinct outputs for distinct inputs. ■

The graph of a one-to-one function $y = f(x)$ can intersect a given horizontal line at most once. If the function intersects the line more than once, it assumes the same y-value for at least two different x-values and is therefore not one-to-one (Figure 1.58).

The Horizontal Line Test for One-to-One Functions

A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

Inverse Functions

Since each output of a one-to-one function comes from just one input, the effect of the function can be inverted to send an output back to the input from which it came.

DEFINITION Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(b) = a \quad \text{if} \quad f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

The symbol f^{-1} for the inverse of f is read "f inverse." The "-1" in f^{-1} is *not* an exponent; $f^{-1}(x)$ does not mean $1/f(x)$. Notice that the domains and ranges of f and f^{-1} are interchanged.

EXAMPLE 2 Suppose a one-to-one function $y = f(x)$ is given by a table of values

x	1	2	3	4	5	6	7	8
$f(x)$	3	4.5	7	10.5	15	20.5	27	34.5

A table for the values of $x = f^{-1}(y)$ can then be obtained by simply interchanging the values in the columns (or rows) of the table for f:

y	3	4.5	7	10.5	15	20.5	27	34.5
$f^{-1}(y)$	1	2	3	4	5	6	7	8

■

If we apply f to send an input x to the output $f(x)$ and follow by applying f^{-1} to $f(x)$, we get right back to x, just where we started. Similarly, if we take some number y in the range of f, apply f^{-1} to it, and then apply f to the resulting value $f^{-1}(y)$, we get back the value y with which we began. Composing a function and its inverse has the same effect as doing nothing.

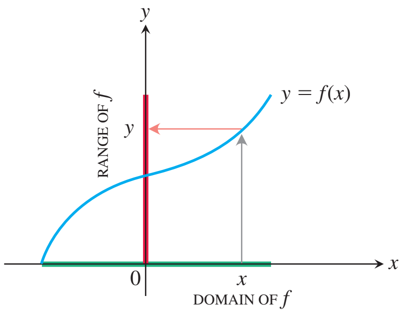
$$(f^{-1} \circ f)(x) = x, \qquad \text{for all } x \text{ in the domain of } f$$
$$(f \circ f^{-1})(y) = y, \qquad \text{for all } y \text{ in the domain of } f^{-1} \text{ (or range of } f\text{)}$$

Only a one-to-one function can have an inverse. The reason is that if $f(x_1) = y$ and $f(x_2) = y$ for two distinct inputs x_1 and x_2, then there is no way to assign a value to $f^{-1}(y)$ that satisfies both $f^{-1}(f(x_1)) = x_1$ and $f^{-1}(f(x_2)) = x_2$.

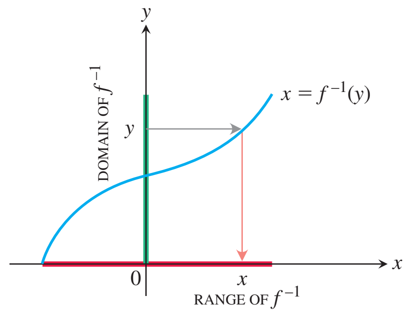
A function that is increasing on an interval satisfies the inequality $f(x_2) > f(x_1)$ when $x_2 > x_1$, so it is one-to-one and has an inverse. Decreasing functions also have an inverse. Functions that are neither increasing nor decreasing may still be one-to-one and have an inverse, as with the function $f(x) = 1/x$ for $x \neq 0$ and $f(0) = 0$, defined on $(-\infty, \infty)$ and passing the horizontal line test.

Finding Inverses

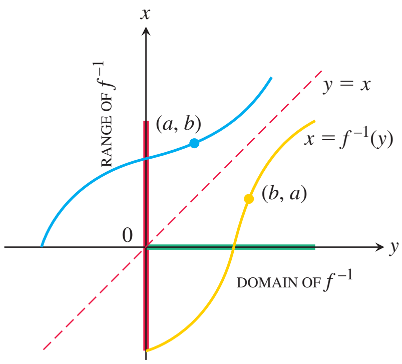
The graphs of a function and its inverse are closely related. To read the value of a function from its graph, we start at a point x on the x-axis, go vertically to the graph, and then move horizontally to the y-axis to read the value of y. The inverse function can be read from the graph by reversing this process. Start with a point y on the y-axis, go horizontally to the graph of $y = f(x)$, and then move vertically to the x-axis to read the value of $x = f^{-1}(y)$ (Figure 1.59).



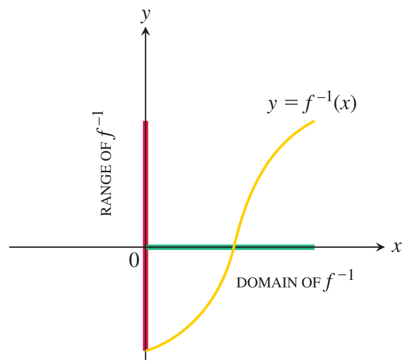
(a) To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.



(b) The graph of f^{-1} is the graph of f, but with x and y interchanged. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.



(c) To draw the graph of f^{-1} in the more usual way, we reflect the system across the line $y = x$.



(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

FIGURE 1.59 The graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ about the line $y = x$.

We want to set up the graph of f^{-1} so that its input values lie along the x-axis, as is usually done for functions, rather than on the y-axis. To achieve this we interchange the x- and y-axes by reflecting across the 45° line $y = x$. After this reflection we have a new graph that represents f^{-1}. The value of $f^{-1}(x)$ can now be read from the graph in the usual way, by starting with a point x on the x-axis, going vertically to the graph, and then horizontally to the y-axis to get the value of $f^{-1}(x)$. Figure 1.59 indicates the relationship between the graphs of f and f^{-1}. The graphs are interchanged by reflection through the line $y = x$.

The process of passing from f to f^{-1} can be summarized as a two-step procedure.

1. Solve the equation $y = f(x)$ for x. This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y.
2. Interchange x and y, obtaining a formula $y = f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable.

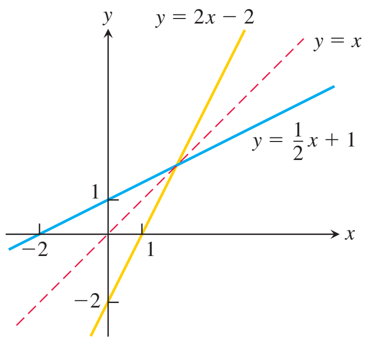


FIGURE 1.60 Graphing $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$ (Example 3).

EXAMPLE 3 Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

Solution

1. *Solve for x in terms of y*: $y = \frac{1}{2}x + 1$

$$2y = x + 2$$
$$x = 2y - 2.$$

The graph is a straight line satisfying the horizontal line test (Fig. 1.60).

2. *Interchange x and y*: $y = 2x - 2$.

The inverse of the function $f(x) = (1/2)x + 1$ is the function $f^{-1}(x) = 2x - 2$. (See Figure 1.60.) To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$

EXAMPLE 4 Find the inverse of the function $y = x^2, x \geq 0$, expressed as a function of x.

Solution For $x \geq 0$, the graph satisfies the horizontal line test, so the function is one-to-one and has an inverse. To find the inverse, we first solve for x in terms of y:

$$y = x^2$$
$$\sqrt{y} = \sqrt{x^2} = |x| = x \qquad |x| = x \text{ because } x \geq 0$$

We then interchange x and y, obtaining

$$y = \sqrt{x}.$$

The inverse of the function $y = x^2, x \geq 0$, is the function $y = \sqrt{x}$ (Figure 1.61).

Notice that the function $y = x^2, x \geq 0$, with domain *restricted* to the nonnegative real numbers, *is* one-to-one (Figure 1.61) and has an inverse. On the other hand, the function $y = x^2$, with no domain restrictions, *is not* one-to-one (Figure 1.58b) and therefore has no inverse.

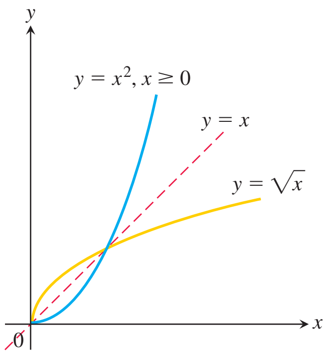
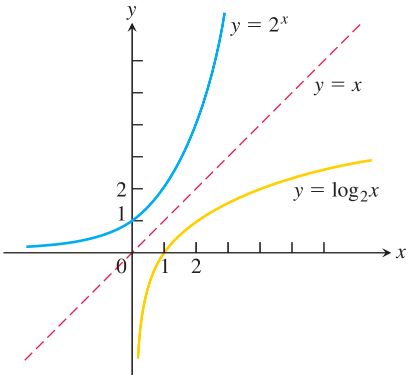


FIGURE 1.61 The functions $y = \sqrt{x}$ and $y = x^2, x \geq 0$, are inverses of one another (Example 4).

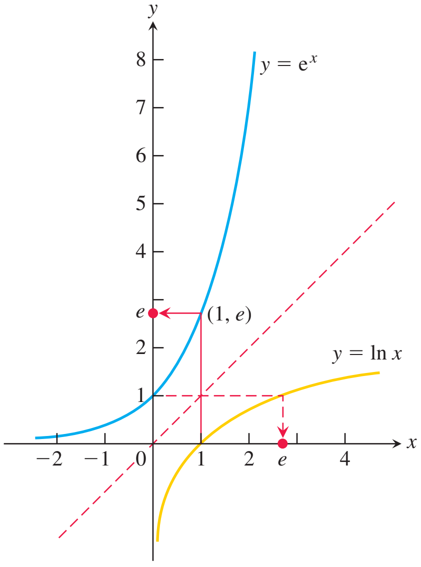
Logarithmic Functions

If a is any positive real number other than 1, the base a exponential function $f(x) = a^x$ is one-to-one. It therefore has an inverse. Its inverse is called the *logarithm function with base a*.

DEFINITION The **logarithm function with base *a***, $y = \log_a x$, is the inverse of the base a exponential function $y = a^x (a > 0, a \neq 1)$.



(a)



(b)

FIGURE 1.62 (a) The graph of 2^x and its inverse, $\log_2 x$. (b) The graph of e^x and its inverse, $\ln x$.

HISTORICAL BIOGRAPHY*
John Napier
(1550–1617)

The domain of $\log_a x$ is $(0, \infty)$, the range of a^x. The range of $\log_a x$ is $(-\infty, \infty)$, the domain of a^x.

Figure 1.23 in Section 1.1 shows the graphs of four logarithmic functions with $a > 1$. Figure 1.62a shows the graph of $y = \log_2 x$. The graph of $y = a^x, a > 1$, increases rapidly for $x > 0$, so its inverse, $y = \log_a x$, increases slowly for $x > 1$.

Because we have no technique yet for solving the equation $y = a^x$ for x in terms of y, we do not have an explicit formula for computing the logarithm at a given value of x. Nevertheless, we can obtain the graph of $y = \log_a x$ by reflecting the graph of the exponential $y = a^x$ across the line $y = x$. Figure 1.62 shows the graphs for $a = 2$ and $a = e$.

Logarithms with base 2 are commonly used in computer science. Logarithms with base e and base 10 are so important in applications that many calculators have special keys for them. They also have their own special notation and names:

$$\log_e x \quad \text{is written as} \quad \ln x.$$
$$\log_{10} x \quad \text{is written as} \quad \log x.$$

The function $y = \ln x$ is called the **natural logarithm function**, and $y = \log x$ is often called the **common logarithm function**. For the natural logarithm,

$$\ln x = y \quad \Leftrightarrow \quad e^y = x.$$

In particular, if we set $x = e$, we obtain

$$\ln e = 1$$

because $e^1 = e$.

Properties of Logarithms

Logarithms, invented by John Napier, were the single most important improvement in arithmetic calculation before the modern electronic computer. What made them so useful is that the properties of logarithms reduce multiplication of positive numbers to addition of their logarithms, division of positive numbers to subtraction of their logarithms, and exponentiation of a number to multiplying its logarithm by the exponent.

We summarize these properties for the natural logarithm as a series of rules that we prove in Chapter 3. Although here we state the Power Rule for all real powers r, the case when r is an irrational number cannot be dealt with properly until Chapter 4. We also establish the validity of the rules for logarithmic functions with any base a in Chapter 7.

THEOREM 1—Algebraic Properties of the Natural Logarithm For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln bx = \ln b + \ln x$
2. *Quotient Rule*: $\ln \frac{b}{x} = \ln b - \ln x$
3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ — Rule 2 with $b = 1$
4. *Power Rule*: $\ln x^r = r \ln x$

*To learn more about the historical figures mentioned in the text and the development of many major elements and topics of calculus, visit **www.aw.com/thomas**.

EXAMPLE 5 Here we use the properties in Theorem 1 to simplify three expressions.

(a) $\ln 4 + \ln \sin x = \ln(4 \sin x)$ Product Rule

(b) $\ln \dfrac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$ Quotient Rule

(c) $\ln \dfrac{1}{8} = -\ln 8$ Reciprocal Rule

$= -\ln 2^3 = -3 \ln 2$ Power Rule ■

Because a^x and $\log_a x$ are inverses, composing them in either order gives the identity function.

> **Inverse Properties for a^x and $\log_a x$**
>
> **1.** Base a: $a^{\log_a x} = x, \quad \log_a a^x = x, \quad a > 0, a \neq 1, x > 0$
>
> **2.** Base e: $e^{\ln x} = x, \quad \ln e^x = x, \quad x > 0$

Substituting a^x for x in the equation $x = e^{\ln x}$ enables us to rewrite a^x as a power of e:

$$\begin{aligned} a^x &= e^{\ln(a^x)} && \text{Substitute } a^x \text{ for } x \text{ in } x = e^{\ln x}. \\ &= e^{x \ln a} && \text{Power Rule for logs} \\ &= e^{(\ln a)x}. && \text{Exponent rearranged} \end{aligned}$$

Thus, the exponential function a^x is the same as e^{kx} for $k = \ln a$.

> Every exponential function is a power of the natural exponential function.
>
> $$a^x = e^{x \ln a}$$
>
> That is, a^x is the same as e^x raised to the power $\ln a$: $a^x = e^{kx}$ for $k = \ln a$.

For example,

$$2^x = e^{(\ln 2)x} = e^{x \ln 2}, \quad \text{and} \quad 5^{-3x} = e^{(\ln 5)(-3x)} = e^{-3x \ln 5}.$$

Returning once more to the properties of a^x and $\log_a x$, we have

$$\begin{aligned} \ln x &= \ln(a^{\log_a x}) && \text{Inverse Property for } a^x \text{ and } \log_a x \\ &= (\log_a x)(\ln a). && \text{Power Rule for logarithms, with } r = \log_a x \end{aligned}$$

Rewriting this equation as $\log_a x = (\ln x)/(\ln a)$ shows that every logarithmic function is a constant multiple of the natural logarithm $\ln x$. This allows us to extend the algebraic properties for $\ln x$ to $\log_a x$. For instance, $\log_a bx = \log_a b + \log_a x$.

> **Change of Base Formula**
>
> Every logarithmic function is a constant multiple of the natural logarithm.
>
> $$\log_a x = \frac{\ln x}{\ln a} \qquad (a > 0, a \neq 1)$$

Applications

In Section 1.5 we looked at examples of exponential growth and decay problems. Here we use properties of logarithms to answer more questions concerning such problems.

EXAMPLE 6 If \$1000 is invested in an account that earns 5.25% interest compounded annually, how long will it take the account to reach \$2500?

Solution From Example 1, Section 1.5, with $P = 1000$ and $r = 0.0525$, the amount in the account at any time t in years is $1000(1.0525)^t$, so to find the time t when the account reaches \$2500 we need to solve the equation

$$1000(1.0525)^t = 2500.$$

Thus we have

$$\begin{aligned}(1.0525)^t &= 2.5 && \text{Divide by 1000.}\\ \ln(1.0525)^t &= \ln 2.5 && \text{Take logarithms of both sides.}\\ t\ln 1.0525 &= \ln 2.5 && \text{Power Rule}\\ t &= \frac{\ln 2.5}{\ln 1.0525} \approx 17.9 && \text{Values obtained by calculator}\end{aligned}$$

The amount in the account will reach \$2500 in 18 years, when the annual interest payment is deposited for that year. ■

EXAMPLE 7 The **half-life** of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay. It is a notable fact that the half-life is a constant that does not depend on the number of radioactive nuclei initially present in the sample, but only on the radioactive substance.

To see why, let y_0 be the number of radioactive nuclei initially present in the sample. Then the number y present at any later time t will be $y = y_0e^{-kt}$. We seek the value of t at which the number of radioactive nuclei present equals half the original number:

$$\begin{aligned}y_0e^{-kt} &= \frac{1}{2}y_0\\ e^{-kt} &= \frac{1}{2}\\ -kt &= \ln\frac{1}{2} = -\ln 2 && \text{Reciprocal Rule for logarithms}\\ t &= \frac{\ln 2}{k}. && (1)\end{aligned}$$

This value of t is the half-life of the element. It depends only on the value of k; the number y_0 does not have any effect.

The effective radioactive lifetime of polonium-210 is so short that we measure it in days rather than years. The number of radioactive atoms remaining after t days in a sample that starts with y_0 radioactive atoms is

$$y = y_0e^{-5\times10^{-3}t}.$$

The element's half-life is

$$\begin{aligned}\text{Half-life} &= \frac{\ln 2}{k} && \text{Eq. (1)}\\ &= \frac{\ln 2}{5\times10^{-3}} && \text{The } k \text{ from polonium's decay equation}\\ &\approx 139 \text{ days.}\end{aligned}$$

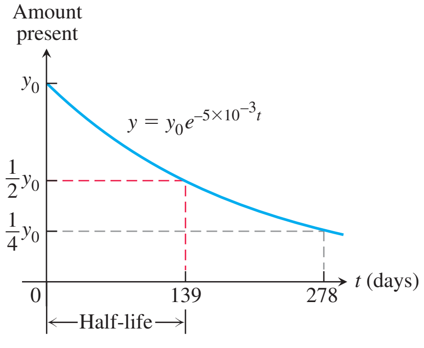


FIGURE 1.63 Amount of polonium-210 present at time t, where y_0 represents the number of radioactive atoms initially present (Example 7).

This means that after 139 days, $1/2$ of y_0 radioactive atoms remain; after another 139 days (or 278 days altogether) half of those remain, or $1/4$ of y_0 radioactive atoms remain, and so on (see Figure 1.63). ■

Inverse Trigonometric Functions

The six basic trigonometric functions of a general radian angle x were reviewed in Section 1.3. These functions are not one-to-one (their values repeat periodically). However, we can restrict their domains to intervals on which they are one-to-one. The sine function

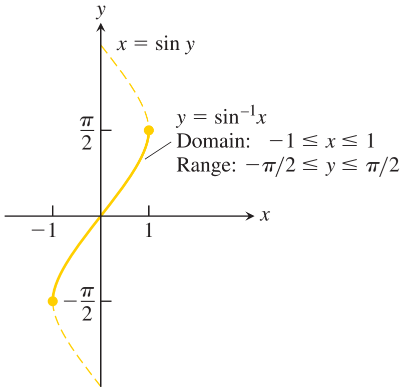
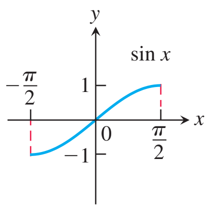


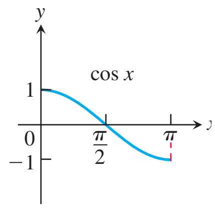
FIGURE 1.64 The graph of $y = \sin^{-1} x$.

increases from -1 at $x = -\pi/2$ to $+1$ at $x = \pi/2$. By restricting its domain to the interval $[-\pi/2, \pi/2]$ we make it one-to-one, so that it has an inverse $\sin^{-1} x$ (Figure 1.64). Similar domain restrictions can be applied to all six trigonometric functions.

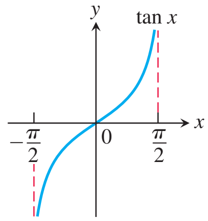
Domain restrictions that make the trigonometric functions one-to-one



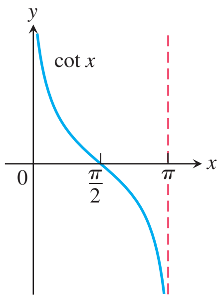
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



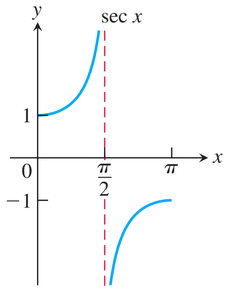
$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



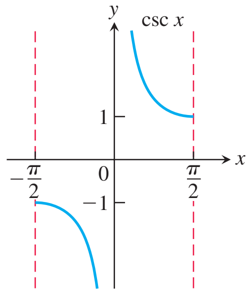
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$



$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

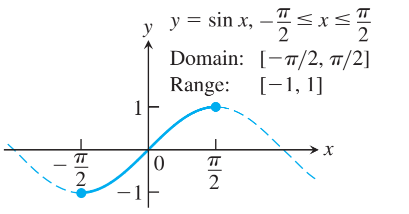
Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$
\begin{aligned}
y &= \sin^{-1} x \quad \text{or} \quad y = \arcsin x \\
y &= \cos^{-1} x \quad \text{or} \quad y = \arccos x \\
y &= \tan^{-1} x \quad \text{or} \quad y = \arctan x \\
y &= \cot^{-1} x \quad \text{or} \quad y = \operatorname{arccot} x \\
y &= \sec^{-1} x \quad \text{or} \quad y = \operatorname{arcsec} x \\
y &= \csc^{-1} x \quad \text{or} \quad y = \operatorname{arccsc} x
\end{aligned}
$$

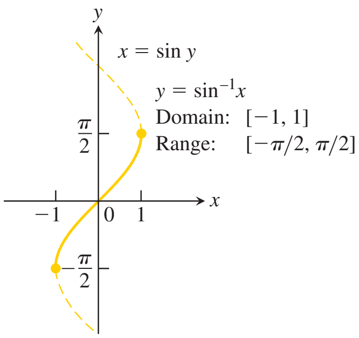
These equations are read "y equals the arcsine of x" or "y equals arcsin x" and so on.

Caution The -1 in the expressions for the inverse means "inverse." It does *not* mean reciprocal. For example, the *reciprocal* of $\sin x$ is $(\sin x)^{-1} = 1/\sin x = \csc x$.

The graphs of the six inverse trigonometric functions are obtained by reflecting the graphs of the restricted trigonometric functions through the line $y = x$. Figure 1.65b shows the graph of $y = \sin^{-1} x$ and Figure 1.66 shows the graphs of all six functions. We now take a closer look at two of these functions.



(a)



(b)

FIGURE 1.65 The graphs of (a) $y = \sin x, -\pi/2 \le x \le \pi/2$, and (b) its inverse, $y = \sin^{-1} x$. The graph of $\sin^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.

The Arcsine and Arccosine Functions

We define the arcsine and arccosine as functions whose values are angles (measured in radians) that belong to restricted domains of the sine and cosine functions.

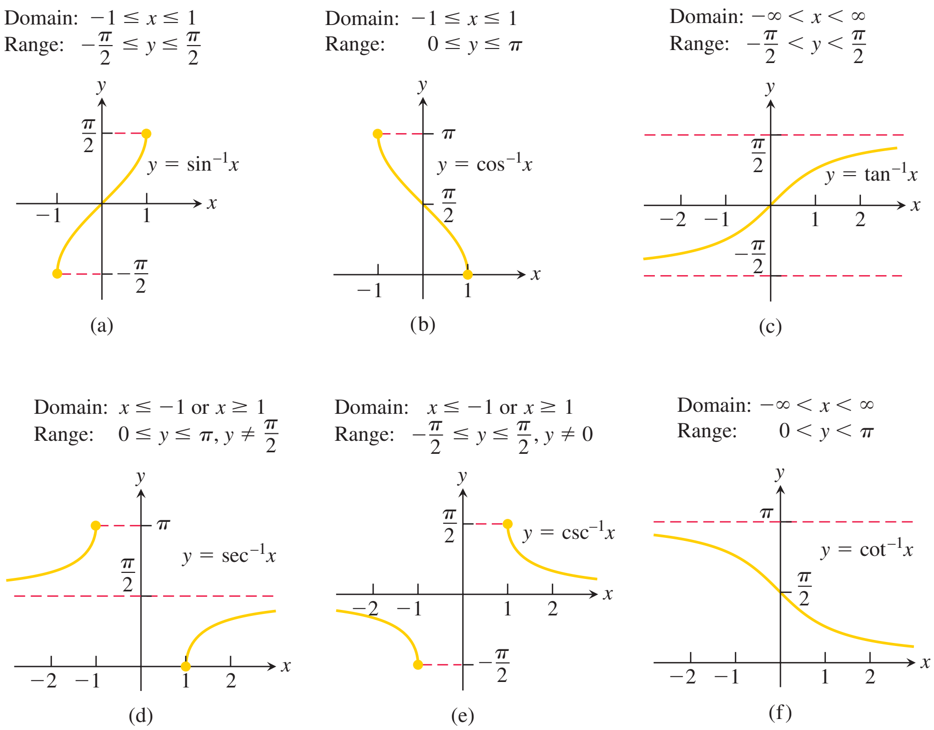


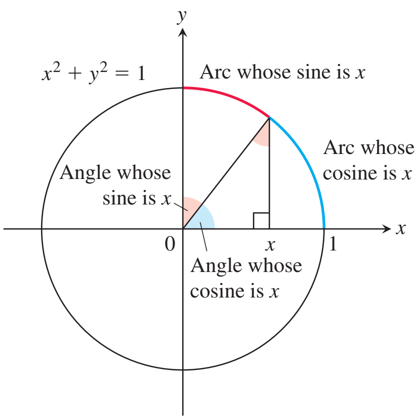
FIGURE 1.66 Graphs of the six basic inverse trigonometric functions.

DEFINITION

$\mathbf{y = \sin^{-1}x}$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$\mathbf{y = \cos^{-1}x}$ is the number in $[0, \pi]$ for which $\cos y = x$.

The "Arc" in Arcsine and Arccosine
For a unit circle and radian angles, the arc length equation $s = r\theta$ becomes $s = \theta$, so central angles and the arcs they subtend have the same measure. If $x = \sin y$, then, in addition to being the angle whose sine is x, y is also the length of arc on the unit circle that subtends an angle whose sine is x. So we call y "the arc whose sine is x."



The graph of $y = \sin^{-1}x$ (Figure 1.65b) is symmetric about the origin (it lies along the graph of $x = \sin y$). The arcsine is therefore an odd function:

$$\sin^{-1}(-x) = -\sin^{-1}x. \tag{2}$$

The graph of $y = \cos^{-1}x$ (Figure 1.67b) has no such symmetry.

EXAMPLE 8 Evaluate **(a)** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and **(b)** $\cos^{-1}\left(-\frac{1}{2}\right)$.

Solution

(a) We see that

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

because $\sin(\pi/3) = \sqrt{3}/2$ and $\pi/3$ belongs to the range $[-\pi/2, \pi/2]$ of the arcsine function. See Figure 1.68a.

(b) We have

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

because $\cos(2\pi/3) = -1/2$ and $2\pi/3$ belongs to the range $[0, \pi]$ of the arccosine function. See Figure 1.68b. ■

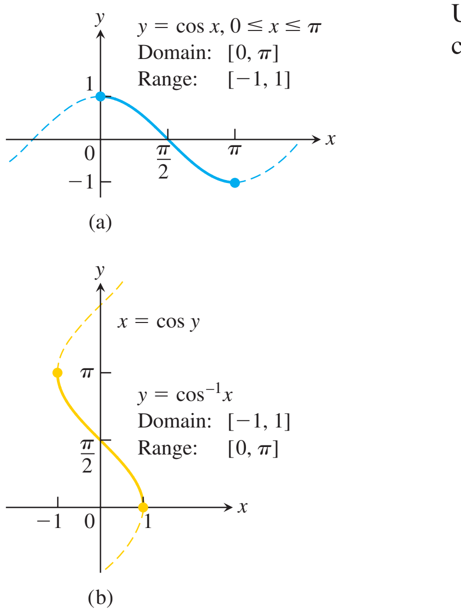


FIGURE 1.67 The graphs of (a) $y = \cos x$, $0 \leq x \leq \pi$, and (b) its inverse, $y = \cos^{-1} x$. The graph of $\cos^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \cos y$.

Using the same procedure illustrated in Example 8, we can create the following table of common values for the arcsine and arccosine functions.

x	$\sin^{-1} x$	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/3$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3$
$-\sqrt{2}/2$	$-\pi/4$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/6$

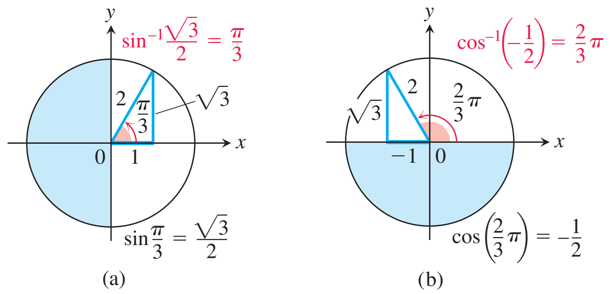


FIGURE 1.68 Values of the arcsine and arccosine functions (Example 8).

EXAMPLE 9 During a 240 mi airplane flight from Chicago to St. Louis, after flying 180 mi the navigator determines that the plane is 12 mi off course, as shown in Figure 1.69. Find the angle a for a course parallel to the original correct course, the angle b, and the drift correction angle $c = a + b$.

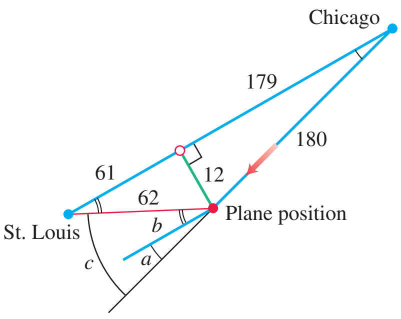


FIGURE 1.69 Diagram for drift correction (Example 9), with distances surrounded to the nearest mile (drawing not to scale).

Solution From the Pythagorean theorem and given information, we compute an approximate hypothetical flight distance of 179 mi, had the plane been flying along the original correct course (see Figure 1.69). Knowing the flight distance from Chicago to St. Louis, we next calculate the remaining leg of the original course to be 61 mi. Applying the Pythagorean theorem again then gives an approximate distance of 62 mi from the position of the plane to St. Louis. Finally, from Figure 1.69, we see that $180 \sin a = 12$ and $62 \sin b = 12$, so

$$a = \sin^{-1}\frac{12}{180} \approx 0.067 \text{ radian} \approx 3.8°$$

$$b = \sin^{-1}\frac{12}{62} \approx 0.195 \text{ radian} \approx 11.2°$$

$$c = a + b \approx 15°.$$

Identities Involving Arcsine and Arccosine

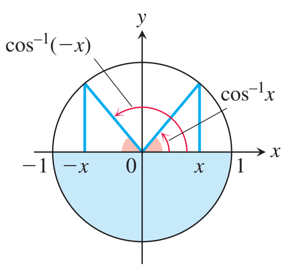


FIGURE 1.70 $\cos^{-1} x$ and $\cos^{-1}(-x)$ are supplementary angles (so their sum is π).

As we can see from Figure 1.70, the arccosine of x satisfies the identity

$$\cos^{-1} x + \cos^{-1}(-x) = \pi, \tag{3}$$

or

$$\cos^{-1}(-x) = \pi - \cos^{-1} x. \tag{4}$$

Also, we can see from the triangle in Figure 1.71 that for $x > 0$,

$$\sin^{-1} x + \cos^{-1} x = \pi/2. \tag{5}$$

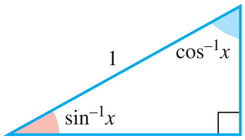


FIGURE 1.71 $\sin^{-1}x$ and $\cos^{-1}x$ are complementary angles (so their sum is $\pi/2$).

Equation (5) holds for the other values of x in $[-1, 1]$ as well, but we cannot conclude this from the triangle in Figure 1.71. It is, however, a consequence of Equations (2) and (4) (Exercise 76).

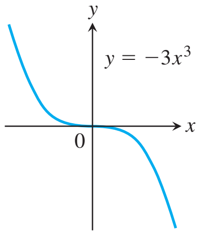
The arctangent, arccotangent, arcsecant, and arccosecant functions are defined in Section 3.9. There we develop additional properties of the inverse trigonometric functions in a calculus setting using the identities discussed here.

Exercises 1.6

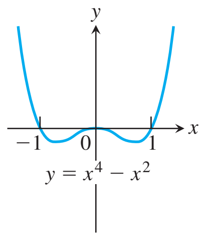
Identifying One-to-One Functions Graphically

Which of the functions graphed in Exercises 1–6 are one-to-one, and which are not?

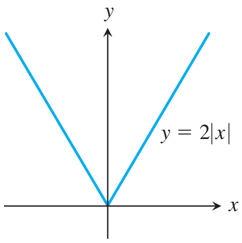
1.



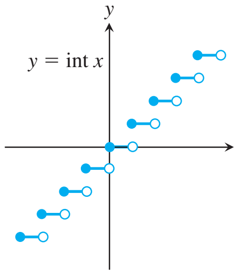
2.



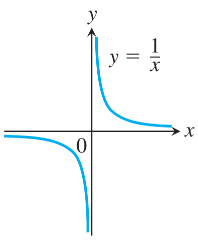
3.



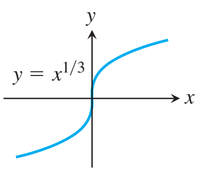
4.



5.



6.



In Exercises 7–10, determine from its graph if the function is one-to-one.

7. $f(x) = \begin{cases} 3 - x, & x < 0 \\ 3, & x \geq 0 \end{cases}$

8. $f(x) = \begin{cases} 2x + 6, & x \leq -3 \\ x + 4, & x > -3 \end{cases}$

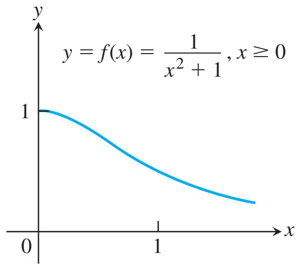
9. $f(x) = \begin{cases} 1 - \dfrac{x}{2}, & x \leq 0 \\ \dfrac{x}{x + 2}, & x > 0 \end{cases}$

10. $f(x) = \begin{cases} 2 - x^2, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

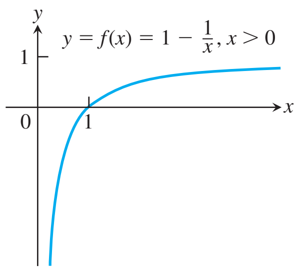
Graphing Inverse Functions

Each of Exercises 11–16 shows the graph of a function $y = f(x)$. Copy the graph and draw in the line $y = x$. Then use symmetry with respect to the line $y = x$ to add the graph of f^{-1} to your sketch. (It is not necessary to find a formula for f^{-1}.) Identify the domain and range of f^{-1}.

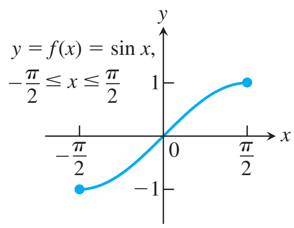
11.



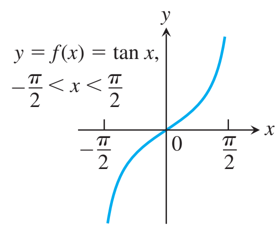
12.



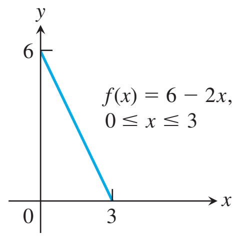
13.



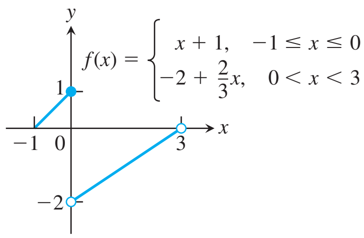
14.



15.



16.



17. a. Graph the function $f(x) = \sqrt{1 - x^2}$, $0 \leq x \leq 1$. What symmetry does the graph have?

b. Show that f is its own inverse. (Remember that $\sqrt{x^2} = x$ if $x \geq 0$.)

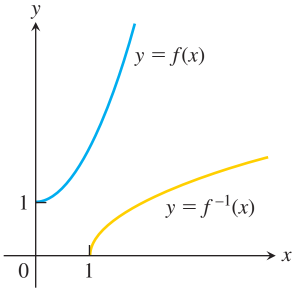
18. a. Graph the function $f(x) = 1/x$. What symmetry does the graph have?

b. Show that f is its own inverse.

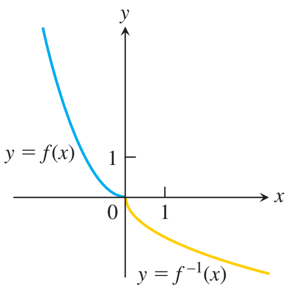
Formulas for Inverse Functions

Each of Exercises 19–24 gives a formula for a function $y = f(x)$ and shows the graphs of f and f^{-1}. Find a formula for f^{-1} in each case.

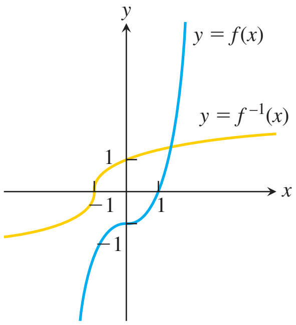
19. $f(x) = x^2 + 1, \quad x \geq 0$



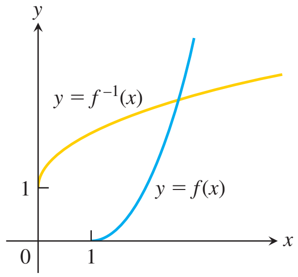
20. $f(x) = x^2, \quad x \leq 0$



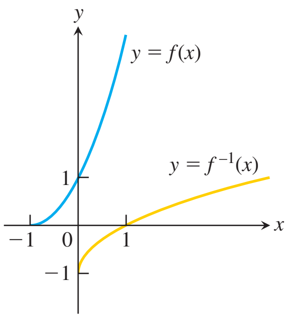
21. $f(x) = x^3 - 1$



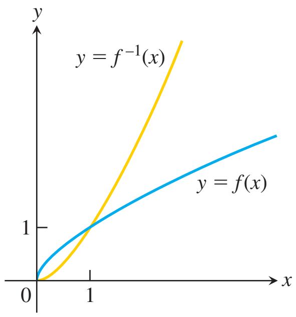
22. $f(x) = x^2 - 2x + 1, \quad x \geq 1$



23. $f(x) = (x + 1)^2, \quad x \geq -1$



24. $f(x) = x^{2/3}, \quad x \geq 0$



Each of Exercises 25–36 gives a formula for a function $y = f(x)$. In each case, find $f^{-1}(x)$ and identify the domain and range of f^{-1}. As a check, show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

25. $f(x) = x^5$

26. $f(x) = x^4, \quad x \geq 0$

27. $f(x) = x^3 + 1$

28. $f(x) = (1/2)x - 7/2$

29. $f(x) = 1/x^2, \quad x > 0$

30. $f(x) = 1/x^3, \quad x \neq 0$

31. $f(x) = \dfrac{x + 3}{x - 2}$

32. $f(x) = \dfrac{\sqrt{x}}{\sqrt{x} - 3}$

33. $f(x) = x^2 - 2x, \quad x \leq 1$
(*Hint:* Complete the square.)

34. $f(x) = (2x^3 + 1)^{1/5}$

35. $f(x) = \dfrac{x + b}{x - 2}, \quad b > -2$ and constant

36. $f(x) = x^2 - 2bx, \quad b > 0$ and constant, $x \leq b$

Inverses of Lines

37. a. Find the inverse of the function $f(x) = mx$, where m is a constant different from zero.

b. What can you conclude about the inverse of a function $y = f(x)$ whose graph is a line through the origin with a nonzero slope m?

38. Show that the graph of the inverse of $f(x) = mx + b$, where m and b are constants and $m \neq 0$, is a line with slope $1/m$ and y-intercept $-b/m$.

39. a. Find the inverse of $f(x) = x + 1$. Graph f and its inverse together. Add the line $y = x$ to your sketch, drawing it with dashes or dots for contrast.

b. Find the inverse of $f(x) = x + b$ (b constant). How is the graph of f^{-1} related to the graph of f?

c. What can you conclude about the inverses of functions whose graphs are lines parallel to the line $y = x$?

40. a. Find the inverse of $f(x) = -x + 1$. Graph the line $y = -x + 1$ together with the line $y = x$. At what angle do the lines intersect?

b. Find the inverse of $f(x) = -x + b$ (b constant). What angle does the line $y = -x + b$ make with the line $y = x$?

c. What can you conclude about the inverses of functions whose graphs are lines perpendicular to the line $y = x$?

Logarithms and Exponentials

41. Express the following logarithms in terms of ln 2 and ln 3.

a. $\ln 0.75$ **b.** $\ln (4/9)$
c. $\ln (1/2)$ **d.** $\ln \sqrt[3]{9}$
e. $\ln 3\sqrt{2}$ **f.** $\ln \sqrt{13.5}$

42. Express the following logarithms in terms of ln 5 and ln 7.

a. $\ln (1/125)$ **b.** $\ln 9.8$
c. $\ln 7\sqrt{7}$ **d.** $\ln 1225$
e. $\ln 0.056$ **f.** $(\ln 35 + \ln (1/7))/(\ln 25)$

Use the properties of logarithms to write the expressions in Exercises 43 and 44 as a single term.

43. a. $\ln \sin \theta - \ln \left(\dfrac{\sin \theta}{5}\right)$ **b.** $\ln (3x^2 - 9x) + \ln \left(\dfrac{1}{3x}\right)$

c. $\dfrac{1}{2}\ln (4t^4) - \ln b$

44. a. $\ln \sec \theta + \ln \cos \theta$ **b.** $\ln (8x + 4) - 2 \ln c$

c. $3 \ln \sqrt[3]{t^2 - 1} - \ln (t + 1)$

Find simpler expressions for the quantities in Exercises 45–48.

45. a. $e^{\ln 7.2}$ **b.** $e^{-\ln x^2}$ **c.** $e^{\ln x - \ln y}$

46. a. $e^{\ln (x^2 + y^2)}$ **b.** $e^{-\ln 0.3}$ **c.** $e^{\ln \pi x - \ln 2}$

47. a. $2 \ln \sqrt{e}$ **b.** $\ln (\ln e^e)$ **c.** $\ln (e^{-x^2 - y^2})$

48. a. $\ln (e^{\sec \theta})$ **b.** $\ln (e^{(e^x)})$ **c.** $\ln (e^{2 \ln x})$

In Exercises 49–54, solve for y in terms of t or x, as appropriate.

49. $\ln y = 2t + 4$

50. $\ln y = -t + 5$

51. $\ln (y - b) = 5t$

52. $\ln (c - 2y) = t$

53. $\ln (y - 1) - \ln 2 = x + \ln x$

54. $\ln (y^2 - 1) - \ln (y + 1) = \ln (\sin x)$

In Exercises 55 and 56, solve for k.

55. **a.** $e^{2k} = 4$ **b.** $100e^{10k} = 200$ **c.** $e^{k/1000} = a$

56. **a.** $e^{5k} = \frac{1}{4}$ **b.** $80e^{k} = 1$ **c.** $e^{(\ln 0.8)k} = 0.8$

In Exercises 57–60, solve for t.

57. **a.** $e^{-0.3t} = 27$ **b.** $e^{kt} = \frac{1}{2}$ **c.** $e^{(\ln 0.2)t} = 0.4$

58. **a.** $e^{-0.01t} = 1000$ **b.** $e^{kt} = \frac{1}{10}$ **c.** $e^{(\ln 2)t} = \frac{1}{2}$

59. $e^{\sqrt{t}} = x^2$ **60.** $e^{(x^2)}e^{(2x+1)} = e^t$

Simplify the expressions in Exercises 61–64.

61. **a.** $5^{\log_5 7}$ **b.** $8^{\log_8 \sqrt{2}}$ **c.** $1.3^{\log_{1.3} 75}$

d. $\log_4 16$ **e.** $\log_3 \sqrt{3}$ **f.** $\log_4\left(\frac{1}{4}\right)$

62. **a.** $2^{\log_2 3}$ **b.** $10^{\log_{10}(1/2)}$ **c.** $\pi^{\log_\pi 7}$

d. $\log_{11} 121$ **e.** $\log_{121} 11$ **f.** $\log_3\left(\frac{1}{9}\right)$

63. **a.** $2^{\log_4 x}$ **b.** $9^{\log_3 x}$ **c.** $\log_2(e^{(\ln 2)(\sin x)})$

64. **a.** $25^{\log_5(3x^2)}$ **b.** $\log_e(e^x)$ **c.** $\log_4(2^{e^x \sin x})$

Express the ratios in Exercises 65 and 66 as ratios of natural logarithms and simplify.

65. **a.** $\dfrac{\log_2 x}{\log_3 x}$ **b.** $\dfrac{\log_2 x}{\log_8 x}$ **c.** $\dfrac{\log_x a}{\log_{x^2} a}$

66. **a.** $\dfrac{\log_9 x}{\log_3 x}$ **b.** $\dfrac{\log_{\sqrt{10}} x}{\log_{\sqrt{2}} x}$ **c.** $\dfrac{\log_a b}{\log_b a}$

Arcsine and Arccosine

In Exercises 67–70, find the exact value of each expression.

67. **a.** $\sin^{-1}\left(\frac{-1}{2}\right)$ **b.** $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ **c.** $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

68. **a.** $\cos^{-1}\left(\frac{1}{2}\right)$ **b.** $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ **c.** $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

69. **a.** $\arccos(-1)$ **b.** $\arccos(0)$

70. **a.** $\arcsin(-1)$ **b.** $\arcsin\left(-\frac{1}{\sqrt{2}}\right)$

Theory and Examples

71. If $f(x)$ is one-to-one, can anything be said about $g(x) = -f(x)$? Is it also one-to-one? Give reasons for your answer.

72. If $f(x)$ is one-to-one and $f(x)$ is never zero, can anything be said about $h(x) = 1/f(x)$? Is it also one-to-one? Give reasons for your answer.

73. Suppose that the range of g lies in the domain of f so that the composite $f \circ g$ is defined. If f and g are one-to-one, can anything be said about $f \circ g$? Give reasons for your answer.

74. If a composite $f \circ g$ is one-to-one, must g be one-to-one? Give reasons for your answer.

75. Find a formula for the inverse function f^{-1} and verify that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

a. $f(x) = \dfrac{100}{1 + 2^{-x}}$ **b.** $f(x) = \dfrac{50}{1 + 1.1^{-x}}$

76. The identity $\sin^{-1}x + \cos^{-1}x = \pi/2$ Figure 1.71 establishes the identity for $0 < x < 1$. To establish it for the rest of $[-1, 1]$, verify by direct calculation that it holds for $x = 1$, 0, and -1. Then, for values of x in $(-1, 0)$, let $x = -a$, $a > 0$, and apply Eqs. (3) and (5) to the sum $\sin^{-1}(-a) + \cos^{-1}(-a)$.

77. Start with the graph of $y = \ln x$. Find an equation of the graph that results from

a. shifting down 3 units.

b. shifting right 1 unit.

c. shifting left 1, up 3 units.

d. shifting down 4, right 2 units.

e. reflecting about the y-axis.

f. reflecting about the line $y = x$.

78. Start with the graph of $y = \ln x$. Find an equation of the graph that results from

a. vertical stretching by a factor of 2.

b. horizontal stretching by a factor of 3.

c. vertical compression by a factor of 4.

d. horizontal compression by a factor of 2.

T **79.** The equation $x^2 = 2^x$ has three solutions: $x = 2$, $x = 4$, and one other. Estimate the third solution as accurately as you can by graphing.

T **80.** Could $x^{\ln 2}$ possibly be the same as $2^{\ln x}$ for $x > 0$? Graph the two functions and explain what you see.

81. Radioactive decay The half-life of a certain radioactive substance is 12 hours. There are 8 grams present initially.

a. Express the amount of substance remaining as a function of time t.

b. When will there be 1 gram remaining?

82. Doubling your money Determine how much time is required for a $500 investment to double in value if interest is earned at the rate of 4.75% compounded annually.

83. Population growth The population of Glenbrook is 375,000 and is increasing at the rate of 2.25% per year. Predict when the population will be 1 million.

84. Radon-222 The decay equation for radon-222 gas is known to be $y = y_0 e^{-0.18t}$, with t in days. About how long will it take the radon in a sealed sample of air to fall to 90% of its original value?

Chapter 1 Questions to Guide Your Review

1. What is a function? What is its domain? Its range? What is an arrow diagram for a function? Give examples.
2. What is the graph of a real-valued function of a real variable? What is the vertical line test?
3. What is a piecewise-defined function? Give examples.
4. What are the important types of functions frequently encountered in calculus? Give an example of each type.
5. What is meant by an increasing function? A decreasing function? Give an example of each.
6. What is an even function? An odd function? What symmetry properties do the graphs of such functions have? What advantage can we take of this? Give an example of a function that is neither even nor odd.
7. If f and g are real-valued functions, how are the domains of $f + g, f - g, fg$, and f/g related to the domains of f and g? Give examples.
8. When is it possible to compose one function with another? Give examples of composites and their values at various points. Does the order in which functions are composed ever matter?
9. How do you change the equation $y = f(x)$ to shift its graph vertically up or down by $|k|$ units? Horizontally to the left or right? Give examples.
10. How do you change the equation $y = f(x)$ to compress or stretch the graph by a factor $c > 1$? Reflect the graph across a coordinate axis? Give examples.
11. What is radian measure? How do you convert from radians to degrees? Degrees to radians?
12. Graph the six basic trigonometric functions. What symmetries do the graphs have?
13. What is a periodic function? Give examples. What are the periods of the six basic trigonometric functions?
14. Starting with the identity $\sin^2\theta + \cos^2\theta = 1$ and the formulas for $\cos(A + B)$ and $\sin(A + B)$, show how a variety of other trigonometric identities may be derived.
15. How does the formula for the general sine function $f(x) = A\sin((2\pi/B)(x - C)) + D$ relate to the shifting, stretching, compressing, and reflection of its graph? Give examples. Graph the general sine curve and identify the constants A, B, C, and D.
16. Name three issues that arise when functions are graphed using a calculator or computer with graphing software. Give examples.
17. What is an exponential function? Give examples. What laws of exponents does it obey? How does it differ from a simple power function like $f(x) = x^n$? What kind of real-world phenomena are modeled by exponential functions?
18. What is the number e, and how is it defined? What are the domain and range of $f(x) = e^x$? What does its graph look like? How do the values of e^x relate to x^2, x^3, and so on?
19. What functions have inverses? How do you know if two functions f and g are inverses of one another? Give examples of functions that are (are not) inverses of one another.
20. How are the domains, ranges, and graphs of functions and their inverses related? Give an example.
21. What procedure can you sometimes use to express the inverse of a function of x as a function of x?
22. What is a logarithmic function? What properties does it satisfy? What is the natural logarithm function? What are the domain and range of $y = \ln x$? What does its graph look like?
23. How is the graph of $\log_a x$ related to the graph of $\ln x$? What truth is in the statement that there is really only one exponential function and one logarithmic function?
24. How are the inverse trigonometric functions defined? How can you sometimes use right triangles to find values of these functions? Give examples.

Chapter 1 Practice Exercises

Functions and Graphs

1. Express the area and circumference of a circle as functions of the circle's radius. Then express the area as a function of the circumference.
2. Express the radius of a sphere as a function of the sphere's surface area. Then express the surface area as a function of the volume.
3. A point P in the first quadrant lies on the parabola $y = x^2$. Express the coordinates of P as functions of the angle of inclination of the line joining P to the origin.
4. A hot-air balloon rising straight up from a level field is tracked by a range finder located 500 ft from the point of liftoff. Express the balloon's height as a function of the angle the line from the range finder to the balloon makes with the ground.

In Exercises 5–8, determine whether the graph of the function is symmetric about the y-axis, the origin, or neither.

5. $y = x^{1/5}$
6. $y = x^{2/5}$
7. $y = x^2 - 2x - 1$
8. $y = e^{-x^2}$

In Exercises 9–16, determine whether the function is even, odd, or neither.

9. $y = x^2 + 1$

10. $y = x^5 - x^3 - x$

11. $y = 1 - \cos x$

12. $y = \sec x \tan x$

13. $y = \dfrac{x^4 + 1}{x^3 - 2x}$

14. $y = x - \sin x$

15. $y = x + \cos x$

16. $y = x \cos x$

17. Suppose that f and g are both odd functions defined on the entire real line. Which of the following (where defined) are even? odd?

a. fg **b.** f^3 **c.** $f(\sin x)$ **d.** $g(\sec x)$ **e.** $|g|$

18. If $f(a - x) = f(a + x)$, show that $g(x) = f(x + a)$ is an even function.

In Exercises 19–28, find the **(a)** domain and **(b)** range.

19. $y = |x| - 2$

20. $y = -2 + \sqrt{1 - x}$

21. $y = \sqrt{16 - x^2}$

22. $y = 3^{2-x} + 1$

23. $y = 2e^{-x} - 3$

24. $y = \tan(2x - \pi)$

25. $y = 2\sin(3x + \pi) - 1$

26. $y = x^{2/5}$

27. $y = \ln(x - 3) + 1$

28. $y = -1 + \sqrt[3]{2 - x}$

29. State whether each function is increasing, decreasing, or neither.

a. Volume of a sphere as a function of its radius

b. Greatest integer function

c. Height above Earth's sea level as a function of atmospheric pressure (assumed nonzero)

d. Kinetic energy as a function of a particle's velocity

30. Find the largest interval on which the given function is increasing.

a. $f(x) = |x - 2| + 1$

b. $f(x) = (x + 1)^4$

c. $g(x) = (3x - 1)^{1/3}$

d. $R(x) = \sqrt{2x - 1}$

Piecewise-Defined Functions

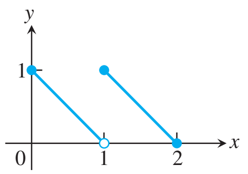
In Exercises 31 and 32, find the **(a)** domain and **(b)** range.

31. $y = \begin{cases} \sqrt{-x}, & -4 \le x \le 0 \\ \sqrt{x}, & 0 < x \le 4 \end{cases}$

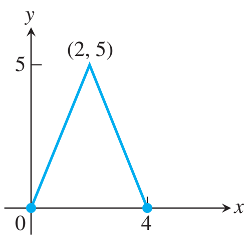
32. $y = \begin{cases} -x - 2, & -2 \le x \le -1 \\ x, & -1 < x \le 1 \\ -x + 2, & 1 < x \le 2 \end{cases}$

In Exercises 33 and 34, write a piecewise formula for the function.

33.



34.



Composition of Functions

In Exercises 35 and 36, find

a. $(f \circ g)(-1)$.

b. $(g \circ f)(2)$.

c. $(f \circ f)(x)$.

d. $(g \circ g)(x)$.

35. $f(x) = \dfrac{1}{x}$, $\quad g(x) = \dfrac{1}{\sqrt{x + 2}}$

36. $f(x) = 2 - x$, $\quad g(x) = \sqrt[3]{x + 1}$

In Exercises 37 and 38, **(a)** write formulas for $f \circ g$ and $g \circ f$ and find the **(b)** domain and **(c)** range of each.

37. $f(x) = 2 - x^2$, $\quad g(x) = \sqrt{x + 2}$

38. $f(x) = \sqrt{x}$, $\quad g(x) = \sqrt{1 - x}$

For Exercises 39 and 40, sketch the graphs of f and $f \circ f$.

39. $f(x) = \begin{cases} -x - 2, & -4 \le x \le -1 \\ -1, & -1 < x \le 1 \\ x - 2, & 1 < x \le 2 \end{cases}$

40. $f(x) = \begin{cases} x + 1, & -2 \le x < 0 \\ x - 1, & 0 \le x \le 2 \end{cases}$

Composition with absolute values In Exercises 41–48, graph f_1 and f_2 together. Then describe how applying the absolute value function in f_2 affects the graph of f_1.

	$f_1(x)$	$f_2(x)$
41.	x	$\lvert x\rvert$
42.	x^2	$\lvert x\rvert^2$
43.	x^3	$\lvert x^3\rvert$
44.	$x^2 + x$	$\lvert x^2 + x\rvert$
45.	$4 - x^2$	$\lvert 4 - x^2\rvert$
46.	$\dfrac{1}{x}$	$\dfrac{1}{\lvert x\rvert}$
47.	$\sqrt{x}$	$\sqrt{\lvert x\rvert}$
48.	$\sin x$	$\sin \lvert x\rvert$

Shifting and Scaling Graphs

49. Suppose the graph of g is given. Write equations for the graphs that are obtained from the graph of g by shifting, scaling, or reflecting, as indicated.

a. Up $\frac{1}{2}$ unit, right 3

b. Down 2 units, left $\frac{2}{3}$

c. Reflect about the y-axis

d. Reflect about the x-axis

e. Stretch vertically by a factor of 5

f. Compress horizontally by a factor of 5

50. Describe how each graph is obtained from the graph of $y = f(x)$.

a. $y = f(x - 5)$

b. $y = f(4x)$

c. $y = f(-3x)$

d. $y = f(2x + 1)$

e. $y = f\left(\dfrac{x}{3}\right) - 4$

f. $y = -3f(x) + \dfrac{1}{4}$

In Exercises 51–54, graph each function, not by plotting points, but by starting with the graph of one of the standard functions presented in Figures 1.15–1.17, and applying an appropriate transformation.

51. $y = -\sqrt{1 + \frac{x}{2}}$

52. $y = 1 - \frac{x}{3}$

53. $y = \frac{1}{2x^2} + 1$

54. $y = (-5x)^{1/3}$

Trigonometry

In Exercises 55–58, sketch the graph of the given function. What is the period of the function?

55. $y = \cos 2x$

56. $y = \sin \frac{x}{2}$

57. $y = \sin \pi x$

58. $y = \cos \frac{\pi x}{2}$

59. Sketch the graph $y = 2\cos\left(x - \frac{\pi}{3}\right)$.

60. Sketch the graph $y = 1 + \sin\left(x + \frac{\pi}{4}\right)$.

In Exercises 61–64, *ABC* is a right triangle with the right angle at *C*. The sides opposite angles *A*, *B*, and *C* are *a*, *b*, and *c*, respectively.

61. a. Find a and b if $c = 2, B = \pi/3$.

b. Find a and c if $b = 2, B = \pi/3$.

62. a. Express a in terms of A and c.

b. Express a in terms of A and b.

63. a. Express a in terms of B and b.

b. Express c in terms of A and a.

64. a. Express $\sin A$ in terms of a and c.

b. Express $\sin A$ in terms of b and c.

65. Height of a pole Two wires stretch from the top *T* of a vertical pole to points *B* and *C* on the ground, where *C* is 10 m closer to the base of the pole than is *B*. If wire *BT* makes an angle of 35° with the horizontal and wire *CT* makes an angle of 50° with the horizontal, how high is the pole?

66. Height of a weather balloon Observers at positions *A* and *B* 2 km apart simultaneously measure the angle of elevation of a weather balloon to be 40° and 70°, respectively. If the balloon is directly above a point on the line segment between *A* and *B*, find the height of the balloon.

T **67. a.** Graph the function $f(x) = \sin x + \cos(x/2)$.

b. What appears to be the period of this function?

c. Confirm your finding in part (b) algebraically.

T **68. a.** Graph $f(x) = \sin(1/x)$.

b. What are the domain and range of f?

c. Is f periodic? Give reasons for your answer.

Transcendental Functions

In Exercises 69–72, find the domain of each function.

69. a. $f(x) = 1 + e^{-\sin x}$ **b.** $g(x) = e^x + \ln \sqrt{x}$

70. a. $f(x) = e^{1/x^2}$ **b.** $g(x) = \ln|4 - x^2|$

71. a. $h(x) = \sin^{-1}\left(\frac{x}{3}\right)$ **b.** $f(x) = \cos^{-1}(\sqrt{x} - 1)$

72. a. $h(x) = \ln(\cos^{-1} x)$ **b.** $f(x) = \sqrt{\pi - \sin^{-1} x}$

73. If $f(x) = \ln x$ and $g(x) = 4 - x^2$, find the functions $f \circ g, g \circ f, f \circ f, g \circ g$, and their domains.

74. Determine whether f is even, odd, or neither.

a. $f(x) = e^{-x^2}$ **b.** $f(x) = 1 + \sin^{-1}(-x)$

c. $f(x) = |e^x|$ **d.** $f(x) = e^{\ln|x|+1}$

T **75.** Graph $\ln x$, $\ln 2x$, $\ln 4x$, $\ln 8x$, and $\ln 16x$ (as many as you can) together for $0 < x \le 10$. What is going on? Explain.

T **76.** Graph $y = \ln(x^2 + c)$ for $c = -4, -2, 0, 3$, and 5. How does the graph change when c changes?

T **77.** Graph $y = \ln|\sin x|$ in the window $0 \le x \le 22, -2 \le y \le 0$. Explain what you see. How could you change the formula to turn the arches upside down?

T **78.** Graph the three functions $y = x^a$, $y = a^x$, and $y = \log_a x$ together on the same screen for $a = 2$, 10, and 20. For large values of x, which of these functions has the largest values and which has the smallest values?

Theory and Examples

In Exercises 79 and 80, find the domain and range of each composite function. Then graph the composites on separate screens. Do the graphs make sense in each case? Give reasons for your answers and comment on any differences you see.

79. a. $y = \sin^{-1}(\sin x)$ **b.** $y = \sin(\sin^{-1} x)$

80. a. $y = \cos^{-1}(\cos x)$ **b.** $y = \cos(\cos^{-1} x)$

81. Use a graph to decide whether f is one-to-one.

a. $f(x) = x^3 - \frac{x}{2}$ **b.** $f(x) = x^3 + \frac{x}{2}$

T **82.** Use a graph to find to 3 decimal places the values of x for which $e^x > 10{,}000{,}000$.

83. a. Show that $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses of one another.

T **b.** Graph f and g over an x-interval large enough to show the graphs intersecting at $(1, 1)$ and $(-1, -1)$. Be sure the picture shows the required symmetry in the line $y = x$.

84. a. Show that $h(x) = x^3/4$ and $k(x) = (4x)^{1/3}$ are inverses of one another.

T **b.** Graph h and k over an x-interval large enough to show the graphs intersecting at $(2, 2)$ and $(-2, -2)$. Be sure the picture shows the required symmetry in the line $y = x$.

Chapter 1 Additional and Advanced Exercises

Functions and Graphs

1. Are there two functions f and g such that $f \circ g = g \circ f$? Give reasons for your answer.

2. Are there two functions f and g with the following property? The graphs of f and g are not straight lines but the graph of $f \circ g$ is a straight line. Give reasons for your answer.

3. If $f(x)$ is odd, can anything be said of $g(x) = f(x) - 2$? What if f is even instead? Give reasons for your answer.

4. If $g(x)$ is an odd function defined for all values of x, can anything be said about $g(0)$? Give reasons for your answer.

5. Graph the equation $|x| + |y| = 1 + x$.

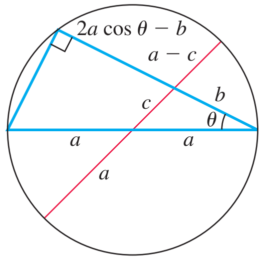
6. Graph the equation $y + |y| = x + |x|$.

Derivations and Proofs

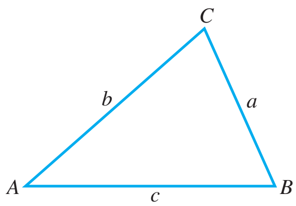
7. Prove the following identities.

a. $\dfrac{1 - \cos x}{\sin x} = \dfrac{\sin x}{1 + \cos x}$ **b.** $\dfrac{1 - \cos x}{1 + \cos x} = \tan^2 \dfrac{x}{2}$

8. Explain the following "proof without words" of the law of cosines. (*Source:* Kung, Sidney H., "Proof Without Words: The Law of Cosines," *Mathematics Magazine*, Vol. 63, no. 5, Dec. 1990, p. 342.)



9. Show that the area of triangle ABC is given by $(1/2)ab \sin C = (1/2)bc \sin A = (1/2)ca \sin B$.



10. Show that the area of triangle ABC is given by $\sqrt{s(s - a)(s - b)(s - c)}$ where $s = (a + b + c)/2$ is the semiperimeter of the triangle.

11. Show that if f is both even and odd, then $f(x) = 0$ for every x in the domain of f.

12. a. Even-odd decompositions Let f be a function whose domain is symmetric about the origin, that is, $-x$ belongs to the domain whenever x does. Show that f is the sum of an even function and an odd function:

$$f(x) = E(x) + O(x),$$

where E is an even function and O is an odd function. (*Hint:* Let $E(x) = (f(x) + f(-x))/2$. Show that $E(-x) = E(x)$, so that E is even. Then show that $O(x) = f(x) - E(x)$ is odd.)

b. Uniqueness Show that there is only one way to write f as the sum of an even and an odd function. (*Hint:* One way is given in part (a). If also $f(x) = E_1(x) + O_1(x)$ where E_1 is even and O_1 is odd, show that $E - E_1 = O_1 - O$. Then use Exercise 11 to show that $E = E_1$ and $O = O_1$.)

Effects of Parameters on Graphs

T **13.** What happens to the graph of $y = ax^2 + bx + c$ as

a. a changes while b and c remain fixed?

b. b changes (a and c fixed, $a \neq 0$)?

c. c changes (a and b fixed, $a \neq 0$)?

T **14.** What happens to the graph of $y = a(x + b)^3 + c$ as

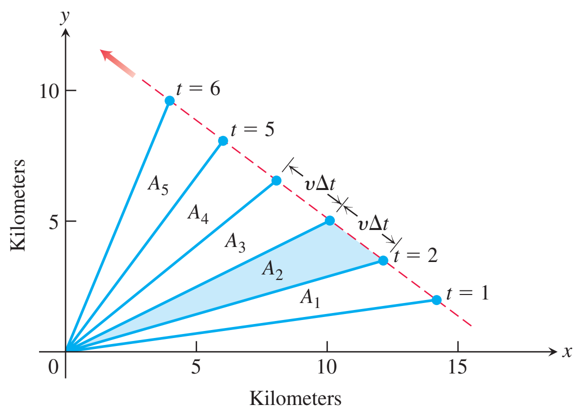
a. a changes while b and c remain fixed?

b. b changes (a and c fixed, $a \neq 0$)?

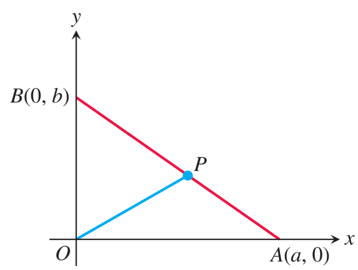
c. c changes (a and b fixed, $a \neq 0$)?

Geometry

15. An object's center of mass moves at a constant velocity v along a straight line past the origin. The accompanying figure shows the coordinate system and the line of motion. The dots show positions that are 1 sec apart. Why are the areas $A_1, A_2, \ldots, A_5$ in the figure all equal? As in Kepler's equal area law (see Section 13.6), the line that joins the object's center of mass to the origin sweeps out equal areas in equal times.



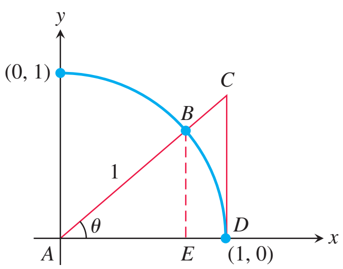
16. a. Find the slope of the line from the origin to the midpoint P of side AB in the triangle in the accompanying figure ($a, b > 0$).



b. When is OP perpendicular to AB?

17. Consider the quarter-circle of radius 1 and right triangles ABE and ACD given in the accompanying figure. Use standard area formulas to conclude that

$$\frac{1}{2}\sin\theta\cos\theta < \frac{\theta}{2} < \frac{1}{2}\frac{\sin\theta}{\cos\theta}.$$



18. Let $f(x) = ax + b$ and $g(x) = cx + d$. What condition must be satisfied by the constants a, b, c, d in order that $(f \circ g)(x) = (g \circ f)(x)$ for every value of x?

Theory and Examples

19. Domain and range Suppose that $a \neq 0$, $b \neq 1$, and $b > 0$. Determine the domain and range of the function.

a. $y = a(b^{c-x}) + d$ **b.** $y = a\log_b(x - c) + d$

20. Inverse functions Let

$$f(x) = \frac{ax + b}{cx + d}, \quad c \neq 0, \quad ad - bc \neq 0.$$

a. Give a convincing argument that f is one-to-one.

b. Find a formula for the inverse of f.

21. Depreciation Smith Hauling purchased an 18-wheel truck for \$100,000. The truck depreciates at the constant rate of \$10,000 per year for 10 years.

a. Write an expression that gives the value y after x years.

b. When is the value of the truck \$55,000?

22. Drug absorption A drug is administered intravenously for pain. The function

$$f(t) = 90 - 52\ln(1 + t), \quad 0 \le t \le 4$$

gives the number of units of the drug remaining in the body after t hours.

a. What was the initial number of units of the drug administered?

b. How much is present after 2 hours?

c. Draw the graph of f.

23. Finding investment time If Juanita invests \$1500 in a retirement account that earns 8% compounded annually, how long will it take this single payment to grow to \$5000?

24. The rule of 70 If you use the approximation $\ln 2 \approx 0.70$ (in place of $0.69314\ldots$), you can derive a rule of thumb that says, "To estimate how many years it will take an amount of money to double when invested at r percent compounded continuously, divide r into 70." For instance, an amount of money invested at 5% will double in about $70/5 = 14$ years. If you want it to double in 10 years instead, you have to invest it at $70/10 = 7\%$. Show how the rule of 70 is derived. (A similar "rule of 72" uses 72 instead of 70, because 72 has more integer factors.)

25. For what $x > 0$ does $x^{(x^x)} = (x^x)^x$? Give reasons for your answer.

T **26. a.** If $(\ln x)/x = (\ln 2)/2$, must $x = 2$?

b. If $(\ln x)/x = -2\ln 2$, must $x = 1/2$?

Give reasons for your answers.

27. The quotient $(\log_4 x)/(\log_2 x)$ has a constant value. What value? Give reasons for your answer.

T **28. $\log_x(2)$ vs. $\log_2(x)$** How does $f(x) = \log_x(2)$ compare with $g(x) = \log_2(x)$? Here is one way to find out.

a. Use the equation $\log_a b = (\ln b)/(\ln a)$ to express $f(x)$ and $g(x)$ in terms of natural logarithms.

b. Graph f and g together. Comment on the behavior of f in relation to the signs and values of g.

Chapter 1 Technology Application Projects

An Overview of Mathematica

An overview of *Mathematica* sufficient to complete the *Mathematica* modules appearing on the Web site.

Mathematica/Maple Module:

Modeling Change: Springs, Driving Safety, Radioactivity, Trees, Fish, and Mammals

Construct and interpret mathematical models, analyze and improve them, and make predictions using them.

2

Limits and Continuity

OVERVIEW Mathematicians of the seventeenth century were keenly interested in the study of motion for objects on or near the earth and the motion of planets and stars. This study involved both the speed of the object and its direction of motion at any instant, and they knew the direction at a given instant was along a line tangent to the path of motion. The concept of a limit is fundamental to finding the velocity of a moving object and the tangent to a curve. In this chapter we develop the limit, first intuitively and then formally. We use limits to describe the way a function varies. Some functions vary *continuously*; small changes in x produce only small changes in $f(x)$. Other functions can have values that jump, vary erratically, or tend to increase or decrease without bound. The notion of limit gives a precise way to distinguish between these behaviors.

2.1 Rates of Change and Tangents to Curves

Calculus is a tool that helps us understand how a change in one quantity is related to a change in another. How does the speed of a falling object change as a function of time? How does the level of water in a barrel change as a function of the amount of liquid poured into it? We see change occurring in nearly everything we observe in the world and universe, and powerful modern instruments help us see more and more. In this section we introduce the ideas of average and instantaneous rates of change, and show that they are closely related to the slope of a curve at a point P on the curve. We give precise developments of these important concepts in the next chapter, but for now we use an informal approach so you will see how they lead naturally to the main idea of this chapter, the *limit*. The idea of a limit plays a foundational role throughout calculus.

Average and Instantaneous Speed

HISTORICAL BIOGRAPHY*
Galileo Galilei
(1564–1642)

In the late sixteenth century, Galileo discovered that a solid object dropped from rest (not moving) near the surface of the earth and allowed to fall freely will fall a distance proportional to the square of the time it has been falling. This type of motion is called **free fall**. It assumes negligible air resistance to slow the object down, and that gravity is the only force acting on the falling object. If y denotes the distance fallen in feet after t seconds, then Galileo's law is

$$y = 16t^2,$$

where 16 is the (approximate) constant of proportionality. (If y is measured in meters, the constant is 4.9.)

A moving object's **average speed** during an interval of time is found by dividing the distance covered by the time elapsed. The unit of measure is length per unit time: kilometers per hour, feet (or meters) per second, or whatever is appropriate to the problem at hand.

*To learn more about the historical figures mentioned in the text and the development of many major elements and topics of calculus, visit **www.aw.com/thomas**.