chapter "2"

Determinants

المحددات $A_{n\times n}$

$\det(A)$, $|A|$, $\Leftarrow$ scalar.

case 1 $\longrightarrow$ 1×1

$A = [a] \rightarrow \det(A) = a$

ex $A = [5] \rightarrow \det(A) = 5$, $A^{-1} = \left[\frac{1}{5}\right]$

case 2 $\longrightarrow$ 2×2

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightarrow \det(A) = a_{11}a_{22} - a_{21}a_{12}$

ex $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix} \rightarrow \det(A) = (1\cdot-5) - (2\cdot3)$
$= -5 - 6 = -11$

det : $+$, $-$, 0

ex $\begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix} = 10 - 4 = 6$.

ex $A = \begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix}$, find $|A| = 10 - 4 = 6$.

case 3 $\rightarrow$ 3×3 $\longrightarrow$ فأكثر

let $A = (a_{ij})$, be an $n\times n$ matrix, and

let ① M_{ij} $\underline{(n-1)\times(n-1)}$: the matrix obtained by deleting the row (i) and column (j) [contain a_{ij}]

② Minor of a_{ij} = determinant of M_{ij}

③ cofactor $A_{ij} = (-1)^{i+j}\det(M_{ij})$

ex

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \rightarrow M_{13} = \begin{bmatrix} 2 & 5 \\ 3 & 6 \end{bmatrix}$$

minor of $a_{13} = \det(M_{13}) = 12 - 15 = -3$

cofactor of $A_{13} = (-1)^{1+3} \det(M_{13})$

$= 1 \times -3 = -3$

ex If $A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}_{3\times3}$

find $\det(A)$:

$\boxed{\text{first row}}$ $\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$$= 2(-1)^{1+1}\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} + 5(-1)^{1+2}\begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} + 4(-1)^{1+3}\begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix}$$

[illegible]

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$(-1)^{i+j} \longrightarrow 1 \longrightarrow +$

$(-1)^{i+j} \longrightarrow -1 \longrightarrow -$

solve again: first row:

$$\det(A) = 2\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} - 5\begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} + 4\begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix}$$

$$= 2(-2) - 5(8) + 4(7) = -16$$

use second column

$$A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$$

$$\det = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$$

find $\det(A)$:

$$\det(A) = -5\begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}$$

$$= -5(8) + 1(-8) - 4(-8) = -16$$

← لا تتغير بتغير الطريقة.

ex

$$\begin{vmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{vmatrix} = ??$$

نختار الصف أو العمود الذي يحتوي على أكبر عدد من الاصفار

sol:

first column

$$\det(A) = 0 + 0 + 0 - 2\begin{vmatrix} 2 & 3 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 3 \end{vmatrix} = -2 * -6 = 12$$

Find $\begin{vmatrix} 2 & 3 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 3 \end{vmatrix} = 0 + 0 + 3\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 3(10 - 12) = -6$

Diagonal matrix : مصفوفة قطرية

$$\begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix} , \quad \underline{\underline{ex}}\ A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

$$\det(A) = 2 * 3 * 5 * 7$$

Upper triangular matrix

$$\begin{bmatrix} a_{11} & - & - & - \\ & a_{22} & - & - \\ & & \ddots & - \\ 0 & & & a_{nn} \end{bmatrix} , \quad \underline{\underline{ex}}\ A = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 7 & 1 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\Rightarrow \det(A) = -1 * 7 * 8 = 56$$

Lower triangular matrix

مصفوفة مثلثية سفلية.

$$\begin{bmatrix} a_{11} & & & 0 \\ - & a_{22} & & \\ - & - & \ddots & \\ - & - & - & a_{nn} \end{bmatrix} , \quad \underline{\underline{ex}}: A = \begin{bmatrix} 5 & 0 \\ 4 & 10 \end{bmatrix}$$

$$\longrightarrow \det(A) = 5 * 10 = 50$$

<u>**Theorem:**</u> If A is a $n \times n$ triangular (lower or upper) matrix, then the determinante of A equals the product of the diagonal elements of A.

Thm A is $n \times n$ matrix

① If A has a row or column consisting entirely of zeros then $\det(A) = 0$.

ex $A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 0 & 5 \\ 3 & 0 & 6 \end{bmatrix} \to \det(A) = 0$

② If A has two identical rows or columns, then $\det(A) = 0$.

ex $A = \begin{vmatrix} 1 & 1 & 5 \\ 2 & 2 & 7 \\ 0 & 0 & 6 \end{vmatrix} = 0$

ex $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 7 & 1 \\ 5 & 10 & 15 \end{vmatrix} = 0$

$*5$

Thm

$\det(A) = 0 \iff A$ is singular

$\det(A) \neq 0 \iff A$ is nonsingular (invertible)

ex Determine if A is singular??

$$A = \begin{bmatrix} 5 & 10 \\ 2 & 4 \end{bmatrix}$$

sol: $\det(A) = 20 - 20 = 0$

$\det(A) = 0 \rightarrow A$ is singular

ex

Find all values of (c) that would make the following matrix singular.

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{bmatrix}$$

Sol: A singular $\rightarrow \det(A) = 0$

$$\det(A) = 1\begin{vmatrix} 9 & c \\ c & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & c \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 9 \\ 1 & c \end{vmatrix}$$

$$= 1(27 - c^2) - (3 - c) + (c - 9)$$

$$= 27 - c^2 - 3 + c + c - 9 = -c^2 + 2c + 15$$

$$\det(A) = -c^2 + 2c + 15 = 0$$

$$c^2 - 2c - 15 = 0$$

$$(c-5)(c+3) = 0 \Rightarrow c = 5, -3$$

صيغة اخرى للسؤال ← بالعكس nonsingular

find all values of (C) that would make the matrix invertible (nonsingular)

$\boxed{\det(A) \neq 0} \Leftarrow$ اسهل نعكس الحل

$$\neq \;\Leftarrow\; =$$

$$-c^2 + 2c + 15 \neq 0$$

$$c^2 - 2c - 15 \neq 0$$

$$(c-5)(c+3) \neq 0$$

$$c \neq 5, -3$$

$$c \in \mathbb{R} - \{-3, 5\}$$