Section 1.4 Matrix Algebra:

Definition. $\underset{n\times n}{A}$ is symmetric if $A^T = A$

Rules for Transpose: Let $\underset{n\times m}{A}$ and $\alpha \in \mathbb{R}$

$\boxed{1}$ $(A^T)^T = A$

(2) $(\alpha A)^T = \alpha A^T$

$\boxed{3}$ $(A+B)^T = A^T + B^T$

$\boxed{4}$ $(AB)^T = B^T A^T$

Example: Let A be $n\times n$ matrix show that $A^T A$ is a symmetric matrix.

solution: $(A^T A)^T = A^T (A^T)^T$
$= A^T A$

Example: Let A be $n\times n$ matrix show that AA^T is a symmetric matrix

Sol $(AA^T)^T = (A^T)^T A^T$
$= A A^T$

Algebraic Rules:

Let α, β be any scalars and A, B, C are matrices

1. $A + B = B + A$
2. $(A+B)+C = A+(B+C)$
3. $(AB)C = A(BC)$
4. $A(B+C) = AB + AC$
5. $(A+B)C = AC + BC$
6. $(\alpha\beta)A = \alpha(\beta A)$
7. $\alpha AB = (\alpha A)B = A(\alpha B)$
8. $(\alpha+\beta)A = \alpha A + \beta A$
9. $\alpha(A+B) = \alpha A + \alpha B$
10. $A^k = \underbrace{AA \cdots A}_{k\text{-times}}$

example

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$AAA = A^2A$

$A^3 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$ ---- $A^n = \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}$

The Identity Matrix:

It is $n \times n$ matrix : on the main diagonal 1's
else where 0's

example:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If $B_{m \times n}$ then:

$$\underset{m \times m}{I_m} \underset{m \times n}{B} = B \quad \text{and} \quad \underset{m \times n}{B} \underset{n \times n}{I_n} = B$$

Example $B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & -1 \end{bmatrix}_{2 \times 3}$

$$I_2 B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & -1 \end{bmatrix} = B$$

$$B I_3 = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & -1 \end{bmatrix} = B$$

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Notes

① $AB \neq BA$ in general

② $AB = 0 \Rightarrow A = 0$ or $B = 0$: False

example: let $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \neq 0$

$B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \neq 0$

$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

③ If $\underset{2\times 2}{A}$ and $A \neq 0$ Where A is symmetric

then $A^2 \neq 0$: True

Pf $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} a^2+b^2 & \text{---} \\ \text{---} & b^2+c^2 \end{bmatrix}$

one of these entries $\neq 0$ thus the main diagonal of A^2 is not zero for at Least one entry.

[4] Find A, B where $A \neq 0$, $B \neq 0$ when $AB = 0$

Solution: $A = B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

[5] Find A, B, C nonzero matrices with

$AC = BC$ and $A \neq B$

Solution: $A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

If C^{-1} exists

we can't

6 $A + A^T$ where A is $n \times n$ is Symmetric

Pf $(A + A^T)^T = A^T + (A^T)^T$

$= A^T + A$

$= A + A^T$

But $A - A^T$ is not Symmetric

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$B_{n \times n}$ is Symmetric

AB is also Symmetric

$\Rightarrow \boxed{AB = BA}$

Matrix Inversion:

Definition An $n\times n$ matrix A is said to be nonsingular or invertable if there exists a matrix B such that $AB = BA = I$

or $= A^{-1}$ such that $AA^{-1} = A^{-1}A = I$

$B = A^{-1}$ is called the multiplicative inverse of A

Note The multiplicative inverse of A is unique

Let $AB = BA = I$

and $AC = CA = I$

Then: $B = BI = B(AC) = (BA)C = IC = C$

Example: The matrices $A = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix}$

are inverses of each other:

$$AB = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix} = I_2$$

$$BA = \begin{bmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} = I_2$$

example $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has no inverse.

Pf

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \overset{?}{=} I_2$$

$$\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can't find a matrix B with $BA = I$

Definition An $n \times n$ matrix A is said to be singular if it does not have a multiplicative inverse.

Rules

$\boxed{1}$ $(A^{-1})^{-1} = A$

$\boxed{2}$ $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$

$\boxed{3}$ $(AB)^{-1} = B^{-1} A^{-1}$

$\boxed{4}$ $(A^{-1})^T = (A^T)^{-1}$

P.f (4): $I = (AA^{-1})^T = (A^{-1})^T A^T$

$I = (A^{-1}A)^T = A^T (A^{-1})^T$ $\Rightarrow (A^T)^{-1} = (A^{-1})^T$

Notes

1. If A and B are $n \times n$ invertable matrices then AB is also invertable and $(AB)^{-1} = B^{-1}A^{-1}$

2. If $A, B, C, \ldots$ are $n \times n$ invertable matrices then ABC is invertable and

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

3. Let A and B be $n \times n$ and $AB = A$ with $B \neq I$ then A is singular

Pf: If not then A^{-1} exists

$$AB = A$$
$$A^{-1}AB = A^{-1}A$$
$$IB = I$$
$$B = I$$

which is a contradiction.

So A must be singular

$\boxed{4}$ If A is $n \times n$ and A is invertable

then for $m = 1, 2, \ldots$

A^m is invertable and $(A^m)^{-1} = (A^{-1})^m$

$\boxed{5}$ If A and B are singular $n \times n$ matrices

then $A+B$ is also singular ; False

ex $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow$ singular

$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow$ singular

$A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow$ nonsingular $= I_2$

$\boxed{6}$ $\underset{n\times n}{A}\,\underset{n\times n}{C} = \underset{n\times n}{B}C$ then $A = B$ False

$\boxed{7}$ $\underset{n\times n}{A}\,\underset{n\times n}{C} = \underset{n\times n}{B}\,\underset{n\times n}{C}$ and C is invertable then $A = B$ True

$\boxed{8}$ For the Homogeneous Linear system:

$Ax = 0$

If A^{-1} exists then : $(A^{-1}A)x = A^{-1}0$

$Ix = 0$

$x = 0$

that is the system has unique solution $x = 0$

Thm For the system: $AX = b$ of n Linear equations we have unique solution if and only if A is invertable.

$$2X = 5$$
$$\left(\tfrac{1}{2}\right)2X = \tfrac{1}{2}(5)$$
$$X = \tfrac{5}{2}$$

If:
$$AX = b$$
$$A^{-1}AX = A^{-1}b$$
$$IX = A^{-1}b$$
$$\boxed{X = A^{-1}b}$$

Finding the inverse of a matrix A:

Method (1): $[A \mid I] \overset{e.r.o}{\sim} [I \mid A^{-1}]$

ex

$A = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$ find A^{-1}:

solution
$$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 3 & 1 & 0 & 1 \end{array}\right]$$
$$\left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & -5 & \frac{-3}{2} & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{10} & \frac{-1}{5} \end{array}\right]$$
$$\left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{10} & \frac{2}{5} \\ 0 & 1 & \frac{3}{10} & -\frac{1}{5} \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix}$$

ex(2) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ find A^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -2 & 5 \\ 0 & 1 & 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow$$

$$A^{-1} = \begin{bmatrix} 0 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

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Example $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ be a Linear system

solution $A x = b$ we find A^{-1}

$$A^{-1} A x = A^{-1} b$$

$$x = \begin{bmatrix} 0 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\Rightarrow$ The solution is unique and is $\{(-2, -4, 0)\}$.