1.4 Matrix Algebra

Theorem: (Algebraic Rules)

A, B, C : matrices

α, β : scalar

1. $A+B = B+A$
2. $(A+B)+C = A+(B+C)$
3. $(AB)C = A(BC)$
4. $A(B+C) = AB + AC$
5. $(A+B)C = AC + BC$
6. $(\alpha\beta)A = \alpha(\beta A)$
7. $\alpha(AB) = (\alpha A)B = A(\alpha B)$
8. $(\alpha+\beta)A = \alpha A + \beta A.$
9. $\alpha(A+B) = \alpha A + \beta A$

ex $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

find: $A(B+C)$, $A(BC)$, ----

power (A n×n matrix) → square matrix
مصفوفة مربعة

$$A^k = \underbrace{A \cdot A \cdot A \cdots A}_{k\text{-times}}$$

ex let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find A^n:

sol:

$A^2 = A \cdot A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$= 2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 2A = 2^1 A$

$A^3 = \underbrace{A \cdot A} \cdot A = A^2 \cdot A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} = 4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 4A = 2^2 A$

$A^4 = A^2 \cdot A^2 = A^3 \cdot A^1 = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} = 8\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 8A = 2^3 A$

$\Rightarrow A^n = 2^{n-1} A$

Identity matrix (I)

Def:

The $\underline{n \times n}$ identity matrix is the matrix $I = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$$I_1 = [1]$$

$$I_2 = I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ex

$10 * 1 = 10$

$1 * 10 = 10$

$$\boxed{\text{note} \quad \underset{n\times n}{A}\,\underset{n\times n}{I} = \underset{n\times n}{I}\,\underset{n\times n}{A} = \underset{n\times n}{A}}$$

ex $A = \begin{bmatrix} 3 & 1 & 4 \\ 5 & 6 & 7 \\ 8 & 1 & 9 \end{bmatrix}$

sol:

$$AI = \begin{bmatrix} 3 & 1 & 4 \\ 5 & 6 & 7 \\ 8 & 1 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 4 \\ 5 & 6 & 7 \\ 8 & 1 & 9 \end{bmatrix} \rightarrow A$$

[illegible]

3×3

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 4 \\ 5 & 6 & 7 \\ 8 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 4 \\ 5 & 6 & 7 \\ 8 & 1 & 9 \end{bmatrix} \rightarrow A$$

Matrix Inversion

$8 * \frac{1}{8} = 1$

$\frac{1}{8} * 8 = 1$

$A * \underbrace{A^{-1}}_{\text{inverse of } A} = \underbrace{A^{-1}}_{\text{inverse of } A} * A = I$

inverse of A

النظير الضربي لـ A

$$\boxed{A * A^{-1} = A^{-1} * A = I}$$

Def: An $n \times n$ matrix A is said to be nonsingular (غير منفردة) or invertible If there exists a matrix B such that $AB = BA = I$.

The matrix B is said to be multiplicative inverse of A.

ex show that $A = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} \frac{-1}{10} & \frac{2}{5} \\ \frac{3}{10} & \frac{-1}{5} \end{bmatrix}$ are inverses each other.

sol: $AB = I \Rightarrow AB = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{-1}{10} & \frac{2}{5} \\ \frac{3}{10} & \frac{-1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$BA = \begin{bmatrix} \frac{-1}{10} & \frac{2}{5} \\ \frac{3}{10} & \frac{-1}{5} \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$AB = BA = I$

$AA^{-1} = A^{-1}A = I$

$\Rightarrow \boxed{B = A^{-1}} \rightarrow$ B is inverse of A

Def:

An $n \times n$ matrix is said to be singular if it doesnt have a multiplicative inverse.

منفردة

ex $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Thm

① $(A^{-1})^{-1} = A$

② $(A^T)^{-1} = (A^{-1})^T$

~~③ $(A+B)^{-1} = A^{-1} + B^{-1}$~~

③ A, B nonsingular $n \times n$ matrices then AB is nonsingular and

$$(AB)^{-1} = B^{-1} A^{-1}$$

Def:

An $\boxed{n \times n}$ matrix A is said to be symmetric if $A^T = A$.

متماثلة

ex $A = \begin{bmatrix} 1 & 8 & 10 \\ 8 & 5 & -1 \\ 10 & -1 & 7 \end{bmatrix}$ $\rightarrow$ main diagonal

قطر رئيسي

3×3

$A^T = \begin{bmatrix} 1 & 8 & 10 \\ 8 & 5 & -1 \\ 10 & -1 & 7 \end{bmatrix}$ so $A = A^T \rightarrow A$ is symmetric ex T/F

If A, B symmetric, then (AB) symmetric.

sol: A sym $\longrightarrow A = A^T$

B sym. $\longrightarrow B = B^T$

(AB) is it sym ?? $\longrightarrow$ find $(AB)^T$

$(AB)^T = B^T A^T = BA \neq AB \to$ not symmetric.

F

If A, B commute

$AB = BA$

T

ex If A symmetric

$A^T A$ sym. T/F

sol

$(A^T A)^T = A^T A^{TT} = A^T A$, T