2-3 Cramers Rule

A : $n \times n$ matrix

The Adjoint of a matrix ;

cofactor $(-1)^{1+2} \det(M_{12})$

$$adj(A) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^T$$

$$\boxed{A^{-1} = \frac{1}{\det(A)} * adj(A)} \quad , \det A \neq 0$$

case 1 (2×2) matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \longrightarrow adj(A) = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

ex $A = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}$ find ① $adj(A)$ ② A^{-1}

sol: ① $adj(A) = \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix}$

$\det(A) = 3 - {}^-10 = 13$

$$A^{-1} = \frac{1}{\det(A)} \cdot adj(A) = \frac{1}{13} \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 3/13 & 2/13 \\ -5/13 & 1/13 \end{bmatrix}$$

case 2 (3x3) matrix

ex $A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$: find (1) adj(A)
(2) A^{-1}.

Sol:

$$\mathrm{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^{T}$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} = 1 * 2 = 2$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = -1 * 7 = -7$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = 1 * 4 = 4$$

$$A_{21} = (-1)^{1+2} \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1 * -1 = 1$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 1 * 4 = 4$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -1 * 3 = -3$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = -2$$

$$A_{32} = (-) \begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} = 2$$

$$A_{33} = (+) \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1$$

$$\text{adj}(A) = \begin{bmatrix} 2 & -7 & 4 \\ 1 & 4 & -3 \\ -2 & 2 & 1 \end{bmatrix}^T$$

$$\text{adj}(A) = \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$$

Find $\det(A) = \det \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} = 5$ (واجب)

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A) = \frac{1}{5} \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 2/5 & 1/5 & -2/5 \\ -7/5 & 4/5 & 2/5 \\ 4/5 & -3/5 & 1/5 \end{bmatrix}$$

⊛ use A^{-1} to solve a system.

$\Rightarrow \underbrace{AX}_{\substack{X \downarrow \\ \text{solution} \\ \text{(unknown matrix)}}} = b \quad \Longrightarrow \underbrace{A^{-1}A}_{I} X = A^{-1} b$

قانون
مغنط

$\Rightarrow IX = A^{-1}b \Rightarrow \boxed{X = A^{-1}b}$

ex solve the system using inverse

$$2x_1 + x_2 + 2x_3 = 1$$
$$3x_1 + 2x_2 + 2x_3 = 2$$
$$x_1 + 2x_2 + 3x_3 = 3$$

sol:

the solution is $X = A^{-1} b$

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \longrightarrow \text{find } A^{-1} = \begin{bmatrix} 2/5 & 1/5 & -2/5 \\ -7/5 & 4/5 & 2/5 \\ 1/5 & -3/5 & 1/5 \end{bmatrix}$$

من السابق

A^{-1} is unique

$$X = A^{-1} b$$

$$= \underset{3\times 3}{\begin{bmatrix} 2/5 & 1/5 & -2/5 \\ -7/5 & 4/5 & 2/5 \\ 1/5 & -3/5 & 1/5 \end{bmatrix}} \cdot \underset{3\times 1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

$$= \begin{bmatrix} -2/5 \\ 7/5 \\ 1/5 \end{bmatrix} \begin{matrix} \rightarrow x_1 \\ \rightarrow x_2 \\ \rightarrow x_3 \end{matrix}$$

the solution: $\left(\frac{-2}{5}, \frac{7}{5}, \frac{1}{5}\right)$

this system has a unique solution,

⊗ an $n\times n$ system and A is invertible $\longrightarrow$ the system has unique sol.

system

$\swarrow$ ([illegible]) $(m\times n)$ Gaussian Elimination (REF) (chapter 1)

$\downarrow$ Inverse $(n\times n)$ A invertible (nonsingular)

$\searrow$ Cramers. $(n\times n)$ A: invertible.

A: is the coefficient matrix

Cramer's Rule

let A be an $n\times n$ nonsingular matrix, $b\in\mathbb{R}^n$, let A_i: the matrix obtained by replacing the i^{th} column of A by (b)

$$x_i=\frac{\det(A_i)}{\det(A)}, \quad i=1,2,\dots,n.$$

(unique sol. if we use cramer)

ex Use Cramers Rule to solve the system.

$$x_1 + 2x_2 + x_3 = 5$$

$$2x_1 + 2x_2 + x_3 = 6$$

$$x_1 + 2x_2 + 3x_3 = 9 .$$

sol: $AX = b$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 9 \end{bmatrix}$$

$\rightarrow \det(A) = -4$ (واجب)

$$A_1 = \begin{bmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{bmatrix} \longrightarrow \det(A_1) = -4$$

$$A_2 = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{bmatrix} \longrightarrow \det(A_2) = -4$$

$$A_3 = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{bmatrix} \longrightarrow \det(A_3) = -8$$

$$\Rightarrow x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-4}{-4} = 1$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{-4}{-4} = 1$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{-8}{-4} = 2$$

the solution: $(1, \cancel{1}, 2)$.

(unique sol.).

واجب حل هذا السؤال بطريقتين

عن هذه الطريقة.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 2 & 2 & 1 & 6 \\ 1 & 2 & 3 & 9 \end{array}\right]$$

inverse.

ex find the matrix A such that:

$$3I + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

sol: $3I$ نتخلص من

$$3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{\text{نريد التخلص منه}} A = \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}$$

نريد التخلص منه
عن طريق ضربه ب
النظير الضربي

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{I} A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}$$

now $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \rightarrow$ find B^{-1}

$\text{adj}(B) = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$, $\det(B) = -2$

$$B^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

$$\rightarrow A = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ -9/2 & 1/2 \end{bmatrix}$$