Chapter 2: Determinants

2.1 The Determinant of a Matrix $A_{n\times n} = \det(A) = |A|$

$$\boxed{A = [a] \Rightarrow \det(A) = a}$$ 1x1 matrix

$$\boxed{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \det(A) = ad - bc}$$ 2x2 matrix

ex $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$ find $\det(A)$ or $\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix}$

sol $\det(A) = \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = 3\times 1 - 4\times 2$

$= 3 - 8$

$= -5$

Note $\det(A) = |A|$ is a real number.

ex $A = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \Rightarrow \det(A) = 0\times 3 - 2\times 1 = -2$

ex $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \Rightarrow \det(A) = 2\times 6 - 3\times 4 = 0$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Notes ① The minor M_{ij} of a_{ij} are:

$$M_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}, \quad M_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}, \quad M_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

② The Cofactor A_{ij} of a_{ij} is

$$A_{ij} = (-1)^{i+j} \det(M_{ij})$$

$$A_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix},$$

example

$$A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$$

using the first row :

The ~~fo~~ minor M_{11} of a_{11} is $M_{11} = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}$

The minor M_{12} of a_{12} is $M_{12} = \begin{bmatrix} 3 & 2 \\ 5 & 6 \end{bmatrix}$

The minor M_{13} of a_{13} is $M_{13} = \begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix}$

The Cofactor A_{11} of $a_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} = 6 - 8 = -2$

" " A_{12} of $a_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} = -(18-10) = -8$

" " A_{13} of $a_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} = 12 - 5 = 7$

$\det(A) = (2)(-2) + 5(-8) + 4(7) = -4 - 40 + 28 = -16$

$$A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Using the 2nd row

$$M_{21} = \begin{bmatrix} 5 & 4 \\ 4 & 6 \end{bmatrix}, \quad M_{22} = \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix}, \quad M_{23} = \begin{bmatrix} 2 & 5 \\ 5 & 4 \end{bmatrix}$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 5 & 4 \\ 4 & 6 \end{vmatrix} = -(30 - 16) = -14$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} = 12 - 20 = -18$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 5 \\ 5 & 4 \end{vmatrix} = -(8 - 25) = 17$$

$$\det(A) = (3)(-14) + 1(-18) + 2(17) = -16$$

Using the 3rd column:

$$\begin{bmatrix} & & + \\ & & - \\ & & + \end{bmatrix}$$

$$M_{13} = \begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix}, \quad M_{23} = \begin{bmatrix} 2 & 5 \\ 5 & 4 \end{bmatrix}, \quad M_{33} = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} = 12 - 5 = 7$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 5 \\ 5 & 4 \end{vmatrix} = -(8 - 25) = 17$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} = 2 - 15 = -13$$

$$\det(A) = 4(7) + 2(17) + 6(-13)$$
$$= 28 + 34 - 78 = -16$$

ex

$$A = \begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

using first column:

$$M_{41} = \begin{bmatrix} 3 & 3 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

$$A_{41} = (-1)^{4+1} \begin{vmatrix} 3 & 3 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 3 \end{vmatrix} = (-1)^{4+1} \left[(-1)^{3+3}(3) \begin{vmatrix} 3 & 3 \\ 4 & 5 \end{vmatrix} \right]$$

$$= (-1)(3)(15 - 12)$$

$$= (-3)(3)$$

$$= -9$$

$$\det(A) = 2 A_{41} = 2(-9) = -18$$

Rules

$$\det(A) = a_{11} A_{11} + a_{12} A_{12} + \cdots + a_{1n} A_{1n} \qquad \text{row } 1$$

$$\det(A) = a_{21} A_{21} + a_{22} A_{22} + \cdots + a_{2n} A_{2n} \qquad \text{row } 2$$

$\vdots$

or

$$\det(A) = a_{11} A_{11} + a_{21} A_{21} + \cdots + a_{n1} A_{n1} \qquad \text{Column } 1$$

$$\det(A) = a_{12} A_{12} + a_{22} A_{22} + \cdots + a_{n2} A_{n2} \qquad \text{Column } 2$$

$\vdots$

Where $A_{ij} = (-1)^{i+j} \det(M_{ij})$

Cases

1. $$A = \begin{bmatrix} \text{main diagonal} & 0 \\ 0 & \end{bmatrix}$$ then A is called diagonal matrix

2. $$A = \begin{bmatrix} \text{main diagonal} & X \\ 0 & \end{bmatrix}$$ then A is called upper triangular matrix

3. $$A = \begin{bmatrix} \text{m.d.} & 0 \\ X & \end{bmatrix}$$ then A is called Lower triangular matrix

for case 1, 2 and 3 det(A) = product of the main diagonal entries.

ex

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{vmatrix} = (2)(3)(-1) = -6$$

$$\begin{vmatrix} 2 & 0 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & -1 \end{vmatrix} = (2)(5)(-1) = -10$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 7 & 3 \end{vmatrix} = (1)(0)(3) = 0$$

Thm Let A be an $n\times n$ matrix

1. If A has a row (or column) consisting entirely of zeros, then $\det(A) = 0$

2. If A has two identical rows (or two identical columns) then $\det(A) = 0$

3. $\det(A) = \det(A^T)$

4. $\det(A) = 0$ $\boxed{\text{if and only if}}$ A is singular

example
$$\begin{vmatrix} 2 & 4 & 0 \\ -1 & 0 & 0 \\ 5 & 3 & 0 \end{vmatrix} = 0 \quad , \quad \begin{vmatrix} 1 & 5 & -1 \\ 0 & 2 & 7 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 4 & 1 & 4 \\ 7 & 5 & 7 \\ -2 & \frac{1}{2} & -2 \end{vmatrix} = 0 \quad , \quad \begin{vmatrix} 0 & 2 & -1 \\ 7 & 1 & 5 \\ 0 & 2 & -1 \end{vmatrix} = 0$$

example If $\det(A) = 12$
find $\det(A^T)$

solution: $\det(A^T) = \det(A) = 12$

example Show that $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ is invertable

solution: $\det(A) = (2)(3) - 1(5) = 6-5 = 1$

$\det(A) \neq 0$, then A is nonsingular
A^{-1} exists, A is invertable.

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Find all values of λ for which

$$A = \begin{bmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{bmatrix}$$ is singular

Sol

$$\det(A) = (2-\lambda)(3-\lambda) - 4(3)$$
$$= 6 + \lambda^2 - 5\lambda - 12$$
$$= \lambda^2 - 5\lambda - 6$$

$\det(A) = 0 \Rightarrow$

$$\lambda^2 - 5\lambda - 6 = 0$$
$$(\lambda - 6)(\lambda + 1) = 0 \Rightarrow \boxed{\lambda = 6 \text{ or } \lambda = -1}$$

Section 2.2 Properties of Determinants:

1 Row operation I:

Two rows (columns) of A are interchanged to obtain A^*

$$\det(A^*) = -\det(A)$$

example
$$\begin{vmatrix} a & b & c \\ d & e & f \\ L & m & g \end{vmatrix} = \theta$$

then
$$\begin{vmatrix} c & b & a \\ f & e & d \\ g & m & L \end{vmatrix} = -\theta$$

and
$$\begin{vmatrix} d & e & f \\ a & b & c \\ L & m & g \end{vmatrix} = -\theta$$

ex
$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix} = -60$$

then
$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ -3 & 6 & 4 \end{vmatrix} = -(-60) = 60$$

and
$$\begin{vmatrix} 2 & 1 & 3 \\ 6 & -3 & 4 \\ 4 & 2 & 1 \end{vmatrix} = 60$$

2 Row Operation II.

A row of A is multiplied by a nonzero constant α and obtain A^* (also column)

$$\det(\tilde{A}) = \alpha \det(A)$$

example
$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix} = -60$$

Then
$$\begin{vmatrix} 2 & 1 & 3 \\ 8 & 4 & 2 \\ 6 & -3 & 4 \end{vmatrix} = (-60)(2) = -120$$