Section 2.2 Properties of Determinants:

1. Row operation I:

Two rows (columns) of A are interchanged to obtain A^*

$$\det(A^*) = -\det(A)$$

example
$$\begin{vmatrix} a & b & c \\ d & e & f \\ L & m & g \end{vmatrix} = \theta$$

then
$$\begin{vmatrix} c & b & a \\ f & e & d \\ g & m & L \end{vmatrix} = -\theta$$

and
$$\begin{vmatrix} d & e & f \\ a & b & c \\ L & m & g \end{vmatrix} = -\theta$$

ex
$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix} = -60$$

then
$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ -3 & 6 & 4 \end{vmatrix} = -(-60) = 60$$

and
$$\begin{vmatrix} 2 & 1 & 3 \\ 6 & -3 & 4 \\ 4 & 2 & 1 \end{vmatrix} = 60$$

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Row Operation II

A row of A is multiplied by a nonzero constant α and obtain A^* (also column)

$$\det(\tilde{A}) = \alpha \det(A)$$

example
$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix} = -60$$

Then
$$\begin{vmatrix} 2 & 1 & 3 \\ 8 & 4 & 2 \\ 6 & -3 & 4 \end{vmatrix} = (-60)(2) = -120$$

Note $\det(\alpha \underset{n\times n}{A}) = \alpha^n \det(A)$

example $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{bmatrix}$

If $\det(A) = -60$

find $\det(\frac{1}{2}A)$

solution
$$\det(\tfrac{1}{2}A) = \left(\tfrac{1}{2}\right)^3 \det(A)$$
$$= \frac{1}{8}(-60)$$
$$= -\frac{60}{8}$$

Row Operation III

If A^* is obtained by A when:

a multiple of one row is added to another row

then $\det(A^*) = \det(A)$.

example 1 Find $\det A$ where $A = \begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix}$

Solution let $\det(A) = a$

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix} = a$$ then by r.o. III

$$\begin{vmatrix} 2 & 1 & 3 \\ 0 & 0 & -5 \\ 0 & -6 & -5 \end{vmatrix} = a$$ then by r.o. I

$$\begin{vmatrix} 2 & 1 & 3 \\ 0 & -6 & -5 \\ 0 & 0 & -5 \end{vmatrix} = -a$$

Thus: $-a = (2)(-6)(-5)$

$-a = 60$

$\boxed{a = -60}$

i.e $\det(A) = -60$

ex (2) Find $\begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & -2 \\ 1 & 4 & 0 \end{vmatrix} = |A|$, Find $\begin{vmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = |B|$

sol $\underbrace{(2)(-1)^2 \begin{vmatrix} 2 & -2 \\ 4 & 0 \end{vmatrix}}_{A_{11}} + \underbrace{(-1)(-1)^3 \begin{vmatrix} -1 & -2 \\ 1 & 0 \end{vmatrix}}_{A_{12}} + \underbrace{3(-1)^4 \begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix}}_{A_{13}}$

$= 0$

or using Row operations:

$$|A| = \begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & -2 \\ 1 & 4 & 0 \end{vmatrix} = a$$

$$\begin{array}{c} \\ (2) \\ (-2) \end{array}\begin{vmatrix} 2 & -1 & 3 \\ -2 & 4 & -4 \\ -2 & -8 & 0 \end{vmatrix} = (2)(-2)\,a$$

$$\begin{vmatrix} 2 & -1 & 3 \\ 0 & 3 & -1 \\ 0 & -9 & 3 \end{vmatrix} = (2)(-2)\,a$$

$$\begin{vmatrix} 2 & -1 & 3 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{vmatrix} = (2)(-2)(a)$$

$$0 = (2)(-2)(a)$$

$a = 0$.

$$|B| = \begin{vmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = b$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 2 & 1 & 2 \end{vmatrix} = -b$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -7 \\ 0 & -3 & -4 \end{vmatrix} = -b$$

$$\begin{array}{c} \\ 3R_2 \\ -4R_3 \end{array}\begin{vmatrix} 1 & 2 & 3 \\ 0 & -12 & -21 \\ 0 & 12 & 16 \end{vmatrix} = (3)(-4)(-b)$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -12 & -21 \\ 0 & 0 & -5 \end{vmatrix} = (3)(-4)(-b)$$

$$(1)(-12)(-5) = (3)(-4)(-b)$$

$\boxed{5 = b}$

Example ② Use elimination method to evaluate $\det(A)$

Where $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \end{bmatrix}$

Solution

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \end{vmatrix} = a$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \end{vmatrix} = -a$$

$\boxed{2R_1 + R_3}$

$\boxed{-R_1 + R_4}$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 1 & -3 & -4 \end{vmatrix} = -a$$

$\boxed{-R_2 + R_4}$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & -5 & -7 \end{vmatrix} = -a$$

$\boxed{R_3 + R_4}$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & -2 \end{vmatrix} = -a$$

$(1)(1)(5)(-2) = -a$

$\Rightarrow \boxed{a = 10}$

$\det(A) = 10$

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Rules

① $\det(AB) = \det(BA) = \det(A)\det(B)$

② $\det(A^T) = \det(A)$

③ $\det(A^{-1}) = \dfrac{1}{\det(A)}$

④ $\det(\alpha A) = \alpha^n \det(A)$

⑤ $\det(\alpha A^{-1}) = \alpha^n \det(A^{-1}) = \alpha^n \dfrac{1}{\det(A)}$

⑥ $\det((\alpha A)^{-1}) = \det\left(\frac{1}{\alpha} A^{-1}\right) = \left(\frac{1}{\alpha}\right)^n \det(A^{-1})$

$$= \frac{1}{\alpha^n \det(A)}$$

⑦ AB is nonsingular iff A and B are nonsingular.

ex Let A and B be two matrices of order 4×4

with $\det(A) = 2$, $\det(B) = -3$

Find

1. $\det(A^T) = \det(A) = 2$
2. $\det(AB) = \det(A)\det(B) = (2)(-3) = -6$
3. $\det(A^{-1}B) = \det(A^{-1})\det(B) = \frac{1}{2}(-3) = \frac{-3}{2}$
4. $\det(\frac{1}{2}B^{-1}) = (\frac{1}{2})^4 \det(B^{-1}) = (\frac{1}{2})^4 \frac{1}{-3} = \cdots$
5. $\det((AB)^{-1}) = \dfrac{1}{\det(AB)} = \dfrac{1}{-6}$
6. $\det(5A) = 5^4 \det(A) = 5^4(2) = \cdots$
7. $\det((5A)^{-1}) = \dfrac{1}{\det(5A)} = \dfrac{1}{5^4 \det(A)} = \dfrac{1}{5^4(2)}$

Ex Find c for which A is singular where

$$A = \begin{array}{ccc} + & - & + \\ \end{array}$$
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{bmatrix}$$

Sol

$$\det(A) = \overset{A_{11}}{(1)(-1)^2 \begin{vmatrix} 9 & c \\ c & 3 \end{vmatrix}} + \overset{A_{12}}{1(-1)^3 \begin{vmatrix} 1 & c \\ 1 & 3 \end{vmatrix}} + \overset{A_{13}}{1(-1)^4 \begin{vmatrix} 1 & 9 \\ 1 & c \end{vmatrix}}$$

$$= 27 - c^2 - (3-c) + (c-9)$$

$$= 27 - c^2 + 2c - 12$$

$$= -c^2 + 2c + 15$$

$$0 = -c^2 + 2c + 15$$

$$0 = c^2 - 2c - 15$$

$$0 = (c-5)(c+3)$$

$$c = 5, -3$$

If $c \in \{-3, 5\} \Rightarrow A$ is singular and $\det(A) = 0$

If $c \in \mathbb{R} \setminus \{-3, 5\} \Rightarrow A$ is nonsingular and $\det(A) \neq 0$

Note

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 8 & c-1 \\ 0 & c-1 & 2 \end{vmatrix} = (1)(-1)^2 \begin{vmatrix} 8 & c-1 \\ c-1 & 2 \end{vmatrix}$$

$$= 16 - (c-1)^2$$

$$= 0$$

Thus: $16 = (c-1)^2$

$$\Rightarrow c-1 = \mp 4 \begin{cases} c-1 = -4 \Rightarrow c = -3 \\ c-1 = 4 \Rightarrow c = 5 \end{cases}$$

Sec 2.3 **Additional Topics**

Let A be $n \times n$ matrix

then the cofactor $A_{ij} = (-1)^{i+j} \det(M_{ij})$

then : Adjoint of A is

$$\operatorname{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & & A_{nn} \end{bmatrix}^T$$

If $\det(A) \neq 0$ then :

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

Note

for 2×2 matrix A we have

$$\operatorname{adj}(A) = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

example : $A = \begin{bmatrix} 1 & -2 \\ 0 & -3 \end{bmatrix} \Rightarrow \operatorname{adj}(A) = \begin{bmatrix} -3 & 2 \\ 0 & 1 \end{bmatrix}$

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example

Let $A = \begin{bmatrix} 2 & -1 \\ 8 & 3 \end{bmatrix}$ find ① $adj(A)$

② A^{-1}

Sol ① $adj(A) = \begin{bmatrix} 3 & 1 \\ -8 & 2 \end{bmatrix}$

② $A^{-1} = \dfrac{1}{det(A)} \, adj(A)$

$= \dfrac{1}{(2)(3) - (-1)(8)} \begin{bmatrix} 3 & 1 \\ -8 & 2 \end{bmatrix}$

$= \dfrac{1}{14} \begin{bmatrix} 3 & 1 \\ -8 & 2 \end{bmatrix}$

$= \begin{bmatrix} \frac{3}{14} & \frac{1}{14} \\ \frac{-8}{14} & \frac{2}{14} \end{bmatrix}$

If A^{-1} exist, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\Rightarrow A^{-1} = \dfrac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

det(A)

Example

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

find ① det (A)

② adj (A)

③ A^{-1}

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Solution

$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(-1)^{1+1}\begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} & -(-1)^{1+2}\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} & +(-1)^{1+3}\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} \\ -(-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & +(-1)^{2+2}\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} & -(-1)^{2+3}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \\ +(-1)^{3+1}\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} & -(-1)^{3+2}\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} & +(-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & -7 & 4 \\ 1 & 4 & -3 \\ -2 & 2 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}_{3\times 3}$$

$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$= (2)(2) + (1)(-7) + (2)(4) = 5$

$\det(A) = (2)(2) + (1)(\bar{7}) + (2)(4) = 5$

$$\operatorname{adj}(A) = \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$$

$$\det(A) = 5$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & \frac{-2}{5} \\ \frac{-7}{5} & \frac{4}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{-3}{5} & \frac{1}{5} \end{bmatrix}$$

Note $A A^{-1} = I$ and

$A^{-1} A = I$

H.W ①

$$[A \mid I] \sim [A^{-1} \mid I]$$

② find $\det(A)$ using elemination.

③ solve:

$$\overset{A}{\begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Note

For $AX = b$ where A is $n \times n$

The Linear system has unique solution

if and only if A is nonsingular

To solve $n \times n$ systems $AX = b$ we can use

① Gaussian Elemination r.e.f

② Jordan - Gauss r.r.e.f

③ Finding A^{-1} and $x = A^{-1}b$ (Inverse method)

③ $x_i = \dfrac{\det(A_i)}{\det(A)}$, $i = 1, 2, \ldots, n$ (Gramer's Rule)

$\Downarrow$

If $\det(A) \neq 0$

i.e If A is invertable

If A is nonsingular

and in this case the solution is unique.

If The system is $n \times m$ with $n \neq m$ or if $|A| = 0$

then we must use method ① or ② only.