Chapter 3: Vector Spaces

3.1 Definition and Examples

Definition: A vector space is a set V with two operations:

1. Scalar multiplication:
   If $x \in V$ and α is a scalar, then $\alpha x \in V$
2. addition:
   If $x, y \in V$ then $x + y \in V$

such that the following axioms are satisfied:

A1) $x + y = y + x$ for any $x, y \in V$

A2) $(x + y) + z = x + (y + z)$

A3) There exists an element 0 in V such that $x + 0 = x$ for each $x \in V$

A4) For each $x \in V$, there exists an element $-x$ in V such that $x + -x = 0$

A5) $\alpha(x + y) = \alpha x + \alpha y$ for each scalar α and $x, y \in V$

A6) $(\alpha + \beta)x = \alpha x + \beta x$ for any scalar α, β and any $x \in V$

A7) $(\alpha\beta)x = \alpha(\beta x)$ for any scalar α, β and $x \in V$

A8) $1 \cdot x = x$ for all $x \in V$.

Example ① The set $W = \{(a,1) : a \in \mathbb{R}\}$ is not a vector space under usual addition and usual scalar multiplication.

To illustrate: let $x = (2,1) \in W$
$y = (3,1) \in W$

but $x+y = (2+3, 1+1)$
$= (5, 2) \notin W$

also $3x = 3(2,1) = (6,3) \notin W$

Example ② $W = \{(a,b) : a,b \in \mathbb{R}\}$

$\alpha(a,b) = (\alpha a, \alpha b)$

$(a_1, b_1) \oplus (a_2, b_2) = (a_1 \oplus a_2, 0)$

is not a vector space.

To illustrate: A_3 does not satisfied

$(2,4) \oplus \boxed{(0,0)} = (2,0) \neq (2,4)$

what ever the vector $(0,0)$ was defined.

H.W Q13 page 116 --- (m.c)

Q11 page 116 --- $\alpha(x,y) = (\alpha x, y)$

Theorem If V a vector space and $X \in V$, then:

(i) $\underset{\text{scalar}}{0}X = \underset{\text{vector}}{0}$

(ii) $X + y = 0 \Rightarrow y = -X$

the additive inverse of X is unique.

(iii) $(-1)X = -X$

Example of Vector spaces!

① Euclidean Vector space R^n

$R^2 = \{(a,b),\ a,b \in \mathbb{R}\}$

$\mathbb{R}^2 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix},\ a,b \in \mathbb{R} \right\}$

$\mathbb{R}^3 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix},\ a,b,c \in \mathbb{R} \right\}$

show that $\mathbb{R}^2 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix},\ a,b \in \mathbb{R} \right\}$ with usual addition and scaler multiplication is a vector space.

sol ① $X = (x_1, x_2)$, $y = (y_1, y_2)$ $\Rightarrow$ $X + y = (x_1 + y_1, x_2 + y_2) \in \mathbb{R}^2$

$\alpha X = (\alpha x_1, \alpha x_2) \in \mathbb{R}^2$

A_1: $X + y = (x_1 + y_1, x_2 + y_2)$

$y + X = (y_1 + x_1, y_2 + x_2) = (x_1 + y_1, x_2 + y_2) = X + y$

⋮

A_3: $X + 0 = (x_1, x_2) + (0, 0) = (x_1, x_2) = X$

⋮

A_8: $1X = 1(x_1, x_2) = (1x_1, 1x_2) = (x_1, x_2) = X$

$\boxed{2}$ $\mathbb{R}^{n\times m} = \{ A_{n\times m}$ with real entries $\}$ is a Vector space.

example

$$\mathbb{R}^{2\times 3} = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} ; a, b, c, d, e, f \in \mathbb{R} \right\}$$

(1) $\begin{array}{l} A \in \mathbb{R}^{2\times 3} \\ B \in \mathbb{R}^{2\times 3} \end{array} \Rightarrow A + B \in \mathbb{R}^{2\times 3}$

(2) $\alpha A \in \mathbb{R}^{2\times 3}$

<u>now</u> A_1 : $A + B = B + A$

A_2 : $A + (B + C) = (A + B) + C$

A_3 : $A + \underset{2\times 3}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} = A$

$\vdots$

A_6 : $\alpha(A + B) = \alpha A + \alpha B$

<u>Note</u> $\mathbb{R}^{n} = \mathbb{R}^{n\times 1}$

3.2 Subspaces

Definition: If $S \neq \phi$ and $S \subseteq V$ and S satisfies:

(1) $\alpha x \in S$ if $x \in S$ for any α

(2) $x + y \in S$ if $x, y \in S$

Then S is called a subspace of the Vector space V

ex Show that $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} ; x_2 = 2x_1 \right\}$ is a subspace of $\mathbb{R}^2$

$\left\{ \begin{bmatrix} c \\ 2c \end{bmatrix}, c \in \mathbb{R} \right\}$

Sol Let $x = \begin{bmatrix} a \\ 2a \end{bmatrix}$, $y = \begin{bmatrix} b \\ 2b \end{bmatrix}$, $x, y \in S$

(1) $\alpha x = \alpha \begin{bmatrix} a \\ 2a \end{bmatrix} = \begin{bmatrix} \alpha a \\ 2\alpha a \end{bmatrix} = \begin{bmatrix} (\alpha a) \\ 2(\alpha a) \end{bmatrix} \in S$

(2) $x + y = \begin{bmatrix} a \\ 2a \end{bmatrix} + \begin{bmatrix} b \\ 2b \end{bmatrix} = \begin{bmatrix} a+b \\ 2a+2b \end{bmatrix} = \begin{bmatrix} a+b \\ 2(a+b) \end{bmatrix} \in S$

So, S is subspace of $\mathbb{R}^2$

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ex Let $S = \{ (a,b,c)^T ; a=b \}$ is a subspace of $\mathbb{R}^3$, illustrate?

Sol $S \neq \phi$ since $(1,1,0)^T \in S$

$x = (a_1, a_1, b_1)$
$y = (a_2, a_2, b_2)$

$\alpha x = (\alpha a_1, \alpha a_1, \alpha b_1) \in S$

$x+y = (a_1+a_2, a_1+a_2, b_1+b_2) \in S$

ex Let $S = \left\{ \begin{bmatrix} x \\ 1 \end{bmatrix}, x \in \mathbb{R} \right\}$ is not a subspace.

illustration: $\alpha \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha x \\ \alpha \end{bmatrix} \notin S$ if $\alpha \neq 1$

also $\begin{bmatrix} x \\ 1 \end{bmatrix} + \begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} x+y \\ 2 \end{bmatrix} \notin S$

ex Let $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} ; b = -c \right\}$ is a subspace of $\mathbb{R}^{2\times 2}$

illustration $\begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix} \in S$, $S \neq \phi$

$A = \begin{bmatrix} a_1 & a_2 \\ -a_2 & a_3 \end{bmatrix}$, $B = \begin{bmatrix} b_1 & b_2 \\ -b_2 & b_3 \end{bmatrix}$ $A, B \in S$

$\alpha A = \begin{bmatrix} \alpha a_1 & \alpha a_2 \\ -\alpha a_2 & \alpha a_3 \end{bmatrix} \in S$

$A+B = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ -a_2+-b_2 & a_3+b_3 \end{bmatrix} = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ -(a_2+b_2) & a_3+b_3 \end{bmatrix} \in S$

Examples of Subspaces of $\mathbb{R}^{2\times 2}$

1) The set of all symmetric 2×2 matrices

Pf $\quad A^T = A$

$\quad B^T = B$

$$(A+B)^T = A^T + B^T = (A+B) \quad \text{symm} \checkmark$$

$$(\alpha A)^T = \alpha A^T = \alpha A \quad \text{sym} \checkmark$$

2) The set of all diagonal 2×2 matrices.

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$B = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$$

$$A+B = \begin{bmatrix} a+c & 0 \\ 0 & b+d \end{bmatrix} \quad \text{diag} \checkmark$$

$$\alpha A = \begin{bmatrix} \alpha a & 0 \\ 0 & \alpha b \end{bmatrix} \quad \text{diag} \checkmark$$

1.

The Null space of a Matrix

Def If A is $m\times n$ matrix, The null space of A is the set of all solutions of the homogeneous system $AX=0$

$$N(A)=\{X\in\mathbb{R}^n \mid AX=0\}$$

Note: $N(A)$ is a subspace of $\mathbb{R}^n$

Pf $(0,\dots,0)^T\in N(A)$ so $N(A)$ is not empty

Let $X\in N(A)\Rightarrow AX=0$
$\quad y\in N(A)\Rightarrow Ay=0$

(1) αX : $A(\alpha X)=$
$\alpha(AX)=0$
$\Rightarrow \alpha X\in N(A)$

(2) $X+y$: $A(X+y)=$
$AX+Ay=$
$0+0=0$
$\Rightarrow X+y\in N(A)$

$\therefore$ $N(A)$ is a subspace of $\mathbb{R}^n$

Example: Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$

Determine $N(A)$

Sol: Solve the system $AX = 0$ (Gauss-Jordan)

$$\begin{array}{c} (-2) \\ \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right] \sim$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$x_3 = \alpha$

$x_4 = \beta$

$x_2 + 2x_3 - x_4 = 0 \Rightarrow x_2 = -2\alpha + \beta$

$x_1 - x_3 + x_4 = 0 \Rightarrow x_1 = \alpha - \beta$

$$X = \begin{bmatrix} \alpha - \beta \\ -2\alpha + \beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$N(A) = \left\{ \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \; ; \; \alpha, \beta \in \mathbb{R} \right\}$$

The span of a Set of Vectors

Def Let $V_1, V_2, \dots, V_n$ be vectors in a vector space V.
A sum of the form $\alpha_1 V_1 + \alpha_2 V_2 + \dots + \alpha_n V_n$ where $\alpha_1, \dots, \alpha_n$ are scalars is called a Linear Combination of $V_1, \dots, V_n$.

The set of all Linear combinations of $V_1, \dots, V_n$ is called the span of $V_1, \dots, V_n$ denoted by span $(V_1, \dots, V_n)$

ex
$$N(A) = \left\{ \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \ \alpha, \beta \in \mathbb{R} \right\}$$
$$= \text{Span} \left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

and if $\alpha = 1, \beta = 2$ then

$$\underbrace{1}_{=\alpha_1} \underbrace{\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{V_1} + \underbrace{2}_{=\alpha_2} \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{V_2} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$
is a Linear Combination of $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Standard Vectors:

$\boxed{1}$ $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ standard vectors of $\mathbb{R}^2$

$$\operatorname{span}(e_1, e_2) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$= \left\{ \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \ \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$

$$= \mathbb{R}^2$$

So $\operatorname{span}(e_1, e_2) = \mathbb{R}^2$

$\boxed{2}$ $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ standard vectors of $\mathbb{R}^3$

$$\operatorname{span}(e_1, e_2, e_3) = \left\{ \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \ ; \ \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}$$

$$= \left\{ \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \right\}$$

$$= \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}, \ \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}$$

$$= \mathbb{R}^3$$

So $\operatorname{span}(e_1, e_2, e_3) = \mathbb{R}^3$

ex) Find span(e_1, e_2) in $\mathbb{R}^3$

Sol span$(e_1, e_2) = \left\{ \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} : \alpha, \beta \in \mathbb{R} \right\}$

$= \left\{ \begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix}, \alpha, \beta \in \mathbb{R} \right\}$

Note span(e_1, e_2) is a subspace of $\mathbb{R}^3$

To verify: $x = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}, y = \begin{bmatrix} c \\ d \\ 0 \end{bmatrix}$

$x + y = \begin{bmatrix} a+c \\ b+d \\ 0 \end{bmatrix} \in$ Span(e_1, e_2)

$\alpha x = \begin{bmatrix} \alpha a \\ \alpha b \\ 0 \end{bmatrix} \in$ Span(e_1, e_2)

3) $e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Standard vectors of $\mathbb{R}^{2\times 2}$

Span$(e_{11}, e_{12}, e_{21}, e_{22})$

$= \{ \alpha_1 e_{11} + \alpha_2 e_{12} + \alpha_3 e_{21} + \alpha_4 e_{22} \; ; \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R} \}$

$= \left\{ \begin{bmatrix} \alpha_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \alpha_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \alpha_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \alpha_4 \end{bmatrix} ; \alpha_1, \dots, \alpha_4 \in \mathbb{R} \right\}$

$= \left\{ \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} ; \alpha_1, \dots, \alpha_4 \in \mathbb{R} \right\} = \mathbb{R}^{2\times 2}$

Note: Let $v_1, v_2, \dots, v_n$ be vectors in vector space V then $\mathrm{span}(v_1, v_2, \dots, v_n)$ is a subspace of V

Pf

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n \in \mathrm{Span}(v_1, \dots, v_n)$$
$$w = \beta_1 v_1 + \cdots + \beta_n v_n \in \mathrm{Span}(v_1, \dots, v_n)$$

$$v + w = (\alpha_1 + \beta_1) v_1 + \cdots + (\alpha_1 + \beta_n) v_n \in \mathrm{Span}(v_1, \dots, v_n)$$

take scalar a :

$$av = a(\alpha_1 v_1 + \cdots + \alpha_n v_n)$$
$$= (a\alpha_1) v_1 + \cdots + (a\alpha_n) v_n \in \mathrm{Span}(v_1, \dots, v_n)$$

Def The set $\{v_1, \dots, v_n\}$ is a spanning set for V if and only if every vector in V can be written as a linear combination of $v_1, \dots, v_n$

Example (11 page 123):

$\boxed{1}$ Is $\{e_1, e_2, e_3, (1,2,3)^T\}$ a spanning set for $\mathbb{R}^3$

Sol $S_1 = \mathrm{Span}\{e_1, e_2, e_3, (1,2,3)^T\}$

$$= \left\{ \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} ; \alpha_1, \dots \alpha_4 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \cancel{\alpha_4 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} , \alpha_1, \dots, \alpha_4 \in \mathbb{R} \right\}$$

take $\alpha_4 = 0$

$\Rightarrow S_1$ is a spanning set for $\mathbb{R}^3$

Is $\{(1,1,1)^T, (1,1,0)^T, (1,0,0)^T\}$ a spanning set for $\mathbb{R}^3$? 13

Sol $S = \text{Span}\left((1,1,1)^T, (1,1,0)^T, (1,0,0)^T\right)$

$$= \left\{ \alpha_1 \begin{bmatrix}1\\1\\1\end{bmatrix} + \alpha_2 \begin{bmatrix}1\\1\\0\end{bmatrix} + \alpha_3 \begin{bmatrix}1\\0\\0\end{bmatrix} ; \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix}\alpha_1+\alpha_2+\alpha_3\\ \alpha_1+\alpha_2\\ \alpha_1\end{bmatrix} ; \alpha_1, \dots, \alpha_3 \in \mathbb{R} \right\}$$

take arbitrary element of $\mathbb{R}^3$: $\begin{bmatrix}a\\b\\c\end{bmatrix} \in \mathbb{R}^3$

find $\alpha_1, \alpha_2, \alpha_3$ if any:

$$\begin{aligned} \alpha_1+\alpha_2+\alpha_3 &= a \\ \alpha_1+\alpha_2 &= b \\ \alpha_1 &= c \end{aligned} \implies \begin{bmatrix}1&1&1\\1&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&1&1&a\\1&1&0&b\\1&0&0&c\end{array}\right] \sim \begin{array}{c} \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad\;\; \\ \left[\begin{array}{ccc|c}1&0&0&c\\1&1&0&b\\1&1&1&a\end{array}\right]\end{array}$$

$\boxed{\alpha_1 = c}$

$\alpha_1 + \alpha_2 = b \implies \boxed{\alpha_2 = b - c}$

$\alpha_1 + \alpha_2 + \alpha_3 = a \implies \boxed{\alpha_3 = a - b}$

$$\underbrace{\begin{bmatrix}a\\b\\c\end{bmatrix} = c\begin{bmatrix}1\\1\\1\end{bmatrix}}_{\alpha_1 V_1} + \underbrace{(b-c)\begin{bmatrix}1\\1\\0\end{bmatrix}}_{\alpha_2 V_2 +} + \underbrace{(a-b)\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\alpha_3 V_3}$$

The system has unique solution and for example if $\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}1\\3\\5\end{bmatrix}$ then

take $\alpha_1 = 5$, $\alpha_2 = -2$, $\alpha_3 = -2$ and $5\begin{bmatrix}1\\1\\1\end{bmatrix} + -2\begin{bmatrix}1\\1\\0\end{bmatrix} + -2\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\3\\5\end{bmatrix}$

$S_3 = \text{Span}\{(1,0,1)^T, (0,1,0)^T\}$

$$= \left\{ \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} \alpha \\ \beta \\ \alpha \end{bmatrix}, \alpha, \beta \in \mathbb{R} \right\}$$

This is a subspace of $\mathbb{R}^3$ but $\neq \mathbb{R}^3$

Thus S_3 is not a spanning set for $\mathbb{R}^3$

False $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ where $a \neq c$ then

it can not be written as a Linear combination of $\{(1,0,1)^T, (0,1,0)^T\}$

For example $\begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix} \notin S_3$ but $\begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix} \in \mathbb{R}^3$

$\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} \in S_3$

$\{(1,2,4),\ (2,1,3),\ (4,-1,1)\}$

$$S_4 = \operatorname{Span}\{(1,2,4)^T, (2,1,3)^T, (4,-1,1)^T\}$$

$$= \left\{ \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \alpha_3 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} ,\ \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} \alpha_1 + 2\alpha_2 + 4\alpha_3 \\ 2\alpha_1 + \alpha_2 + -\alpha_3 \\ 4\alpha_1 + 3\alpha_2 + \alpha_3 \end{bmatrix} ;\ \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}$$

Let $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ be an arbitrary element.

Solve the Linear system (If any)

$$\begin{aligned} \alpha_1 + 2\alpha_2 + 4\alpha_3 &= a \\ 2\alpha_1 + \alpha_2 + -\alpha_3 &= b \\ 4\alpha_1 + 3\alpha_2 + \alpha_3 &= c \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & a \\ 2 & 1 & -1 & b \\ 4 & 3 & 1 & c \end{array}\right]$$

Use Gaussian elimination.

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & a \\ 0 & -3 & -9 & -2a+b \\ 0 & -5 & -15 & -4a+c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 4 & a \\ 0 & 1 & 3 & \frac{2a-b}{3} \\ 0 & 1 & 3 & \frac{4a-c}{5} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 4 & a \\ 0 & 1 & 3 & \frac{2a-b}{3} \\ 0 & 0 & 0 & \frac{2a+5b-3c}{15} \end{array}\right]$$

The system has solution only if $\boxed{2a+5b-3c=0}$

Also $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow 2+5-3 = 4 \neq 0 \Rightarrow$ No sol. $\Rightarrow S_4 \neq \mathbb{R}^3$