Chapter 4: Linear Transformation:

Def A mapping L from a vector space V into a vector space W is said to be a Linear transformation if

$$L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2)$$

for α, β scalar and $v_1, v_2 \in V$

Note $\boxed{1}$ $L : V \longrightarrow W$

$\boxed{*}$ $L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2)$

or

$\boxed{*|*}$
$$L(v_1 + v_2) = L(v_1) + L(v_2)$$
$$L(\alpha v_1) = \alpha L(v_1)$$

Case $\boxed{*}$ and $\boxed{*|*}$ are equivalent.

$\boxed{2}$ If $V = W$ then $L : V \longrightarrow V$ is called Linear operator

Ex 1 $L: \mathbb{R}^2 \to \mathbb{R}^2$ with

$L(x) = 3x$, Is L a Linear operator ?

Sol $v_1, v_2 \in \mathbb{R}^2$

(1) $L(v_1 + v_2) = 3(v_1 + v_2) = 3v_1 + 3v_2$

$L(v_1) + L(v_2) = 3v_1 + 3v_2$

So $L(v_1 + v_2) = L(v_1) + L(v_2)$

(2) $L(\alpha v_1) = 3(\alpha v_1) = \alpha(3v_1)$

$\alpha L(v_1) = \alpha(3v_1) = \alpha(3v_1)$

So $L(\alpha v_1) = \alpha L(v_1)$

by (1) and (2) L is Linear operator

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or

$$L(\alpha v_1 + \beta v_2) = 3(\alpha v_1 + \beta v_2)$$
$$= \alpha 3 v_1 + \beta 3 v_2$$

$\alpha L(v_1) + \beta L(v_2) = \alpha(3v_1) + \beta(3v_2)$

So $L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2)$

Thus L is a Linear operator

Note: one can use $v_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $v_2 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

(1) $L(v_1+v_2) = L\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix} = 3\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix} =$

$= \begin{bmatrix} 3x_1+3y_1 \\ 3x_2+3y_2 \end{bmatrix} = 3\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 3\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$= L(v_1) + L(v_2)$

(2) $L(\alpha v_1) = L\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix} = \begin{bmatrix} 3\alpha x_1 \\ 3\alpha x_2 \end{bmatrix} = \alpha\begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$

$= \alpha L(v_1)$

H.W $L: \mathbb{R}^2 \to \mathbb{R}^2$

$L\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$ show that L is Linear operator.

ex Show that $L: \mathbb{R}^2 \to \mathbb{R}$; $L(x) = \sqrt{x_1^2 + x_2^2}$ is not a Linear transformation

sol $L(v_1+v_2) = L\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix} = \sqrt{(x_1+y_1)^2 + (x_2+y_2)^2}$

$L(v_1) + L(v_2) = \sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2}$ $\neq$

or $x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\alpha = -2$

$L(\alpha x) = \sqrt{(-4)^2 + (-6)^2} = \sqrt{16+36} = 2\sqrt{13}$

$\alpha L(x) = -2\sqrt{(2)^2 + (3)^2} = -2\sqrt{4+9} = -2\sqrt{13}$ $\neq$

H.W $L: \mathbb{R}^2 \to \mathbb{R}^1$

$L(x) = x_1 + x_2$ for $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Show that L is a Linear transformation.

ex $L: \mathbb{R}^2 \to \mathbb{R}^3$

$L\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{bmatrix}$ is a Linear transformation L.T

Pf (1) $L(x+y) = L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right)$

$= L\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$

$= \begin{bmatrix} x_2 + y_2 \\ x_1 + y_1 \\ x_1 + y_1 + x_2 + y_2 \end{bmatrix}$ equal

Now $L(x) + L(y) = \begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{bmatrix} + \begin{bmatrix} y_2 \\ y_1 \\ y_1 + y_2 \end{bmatrix} = \begin{bmatrix} x_2 + y_2 \\ x_1 + y_1 \\ x_1 + x_2 + y_1 + y_2 \end{bmatrix}$

So: $L(x+y) = L(x) + L(y)$

(2) $L(\alpha x) = L\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix} = \begin{bmatrix} \alpha x_2 \\ \alpha x_1 \\ \alpha x_1 + \alpha x_2 \end{bmatrix}$

$\alpha L(x) = \alpha \begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} \alpha x_2 \\ \alpha x_1 \\ \alpha(x_1 + x_2) \end{bmatrix}$ are equal

So $L(\alpha x) = \alpha L(x)$

by (1) and (2) L is L.T

Define a L.T L_A on a matrix $A_{m \times n}$ by

$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$L_A(X) = AX$

ex $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 5 \end{bmatrix}$

$L : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$L_A(x) = \underset{2\times 3}{A} \underset{3\times 1}{X}$

$= \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

To show that $L_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is L.T

(1) $L(x+y) = A(x+y)$
$= Ax + Ay$
$= L(x) + L(y)$

(2) $L(\alpha x) = A(\alpha x)$
$= \alpha Ax$
$= \alpha L(x)$

Notes $L : V \longrightarrow W$

(1) $L(0_V) = 0_W$

(2) $L(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = \alpha_1 L(v_1) + \cdots + \alpha_n L(v_n)$

(3) $L(-v) = -L(v)$ (addition inverse)

Def Identity operator I

$$I : V \rightarrow V$$
$$I(v) = v \qquad \text{is a L.O}$$

PF (1) $I(v_1 + v_2) = v_1 + v_2$
$I(v_1) + I(v_2) = v_1 + v_2$

(2) $I(\alpha v_1) = \alpha v_1$
$\alpha I(v_1) = \alpha v_1$

The Image and Kernel :

Def (1) $L : V \rightarrow W$ is L.T . The Kernel of L

denoted $Ker(L)$ is

$$Ker(L) = \{ v \in V ; L(v) = 0_W \}$$

and $Ker(L)$ is a subspace of V

Def (2) $L : V \rightarrow W$ is L.T. The image of a set S

$$L(S) = \{ w \in W \mid w = L(v) \text{ for some } v \in S \}$$

(*) If $S = V \Rightarrow L(V) = \text{range of } L$

* If S is a subspace of V then $L(S)$ is a subspace of W

ex $L: \mathbb{R}^2 \to \mathbb{R}^2$ is L.O

$$L\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 e_1$$

find (1) $\ker(L)$ (2) $\text{Range}(L)$

sol $\ker(L)$:

$$L\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff x_1 = 0 \text{ and } x_2 = \alpha$$

Thus $\ker(L) = \left\{ \begin{bmatrix} 0 \\ \alpha \end{bmatrix}, \alpha \in \mathbb{R} \right\}$.

$$\text{Range}(L) = \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix}_{x_1 \in \mathbb{R}} \right\} = \left\{ x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x_1 \in \mathbb{R} \right\}$$

$$= x_1 e_1$$

ex2 :

ex 2 $L: \mathbb{R}^3 \to \mathbb{R}^2$ is L.T

$$L\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix}$$

Let $S = \text{span}(e_1, e_3)$

Find: $\text{Ker}(L)$, $L(S)$, $\text{Image}(L)$

sol $\text{Ker}(L)$: $\begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{array}{l} x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \end{array} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_3 = \alpha$

$x_2 = -\alpha$

$x_1 = \alpha$

$$\text{Ker}(L) = \left\{ \begin{bmatrix} \alpha \\ -\alpha \\ \alpha \end{bmatrix}, \alpha \in \mathbb{R} \right\}$$

$$= \left\{ \alpha \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R} \right\} = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right)$$

is a subspace of $\mathbb{R}^3$